Kalman Filtering

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- The Scalar Variable Model

Observation Model :

$$y(n) = x(n) + v_1(n)$$

Measurement noise

Desired time-varying random signal

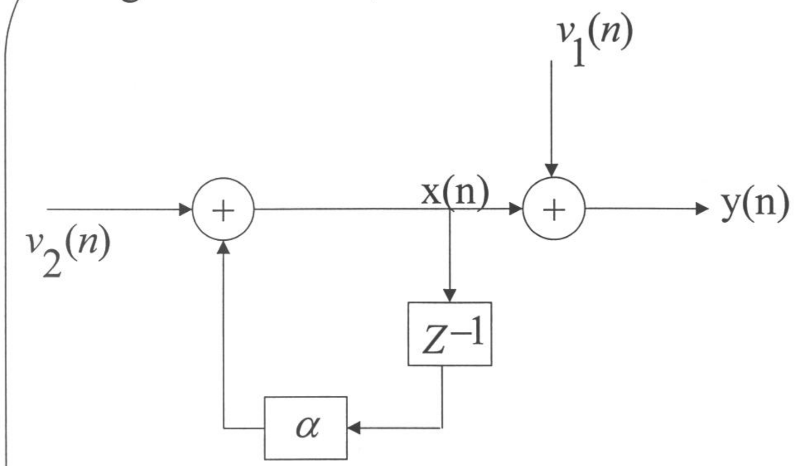
where $v_1(n)$ is a white noise process with variance $\sigma_{v_1}^2$ uncorrelated with $x(n)$.

Desired signal model : Assume that the desired signal has time evolution given by :

$$x(n) = \alpha\, x(n-1) + v_2(n)$$

where $v_2(n)$ is a white noise process different from $v_1(n)$.

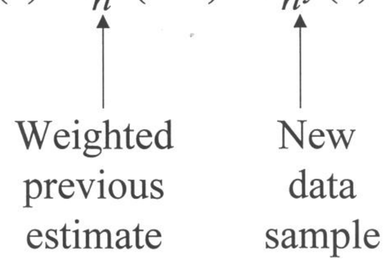
Signal and Observation Model :



Recursive Estimation Problem :

We require that the recursive estimate to have the following structure :

$$\hat{x}(n) = a_n \hat{x}(n-1) + K_n y(n)$$



Determine a_n and K_n such that

$$\text{MSE} = E[(\hat{x}(n) - x(n))^2]$$

is minimized

Define : $\varepsilon(n) = x(n) - \hat{x}(n)$ error

error variance : $\sigma^2_{\varepsilon(n)} = \text{var}[\varepsilon(n)]$

From our assumption on noise being white and uncorrelated with signal, we write :

$$E[x(n)y^*(n-l)] = R_x(l)$$

$$E[y(n)y^*(n-l)] = R_x(l) + \sigma^2_{v_2}\delta(l)$$

and $$E[y(n)v_2^*(n)] = E[(x(n) + v_2(n))v_2^*(n)] = \sigma^2_{v_2}$$

To find a_n and K_n :Apply orthogonality principle :

$$E[\varepsilon(n)y^*(n-l)]=0, \quad l=0,1,\ldots n \qquad (1)$$

$l=0$:

$$E[\varepsilon(n)(x^*(n)+v_2^*(n))]=E[\varepsilon(n)x^*(n)] + E[\varepsilon(n)v_2^*(n)]=0$$

Note : $E[\varepsilon(n)y^*(n)]=\sigma^2_{\varepsilon(n)}$

and substitute : $\begin{cases}\varepsilon(n)=x(n)-\hat{x}(n)\\ \hat{x}(n)=a_n\hat{x}(n-1)+K_n y(n)\end{cases}$

and simplifying :

$$\sigma^2_{\varepsilon(n)}-K_n\sigma^2_{v_2}=0$$

$$\boxed{K_n=\frac{\sigma^2_{\varepsilon(n)}}{\sigma^2_{v_2}}} \longleftarrow \text{real and positive} \qquad (2)$$

For $l>0$ (1) can be written as

$$E[\varepsilon(n)y^*(n-l)]$$

$$=E\Big[[x(n)-a_n\hat{x}(n-1)-K_n y(n)]y^*(n-l)\Big]$$

$$=E[\underbrace{x(n)y^*(n-l)}_{R_x(l)}]-a_n E[\hat{x}(n-1)y^*(n-l)]$$

$$-K_n E[\underbrace{y(n)y^*(n-l)}_{R_x(l)}]=0$$

Follows:

$$(1\text{-}K_n)R_x(l)-a_n E[(x(n-1)-\varepsilon(n\text{-}1))y^*(n-l)]=0$$

$$R_x(l)-\frac{a_n}{(1\text{-}K_n)}R_x(l-1)=0 \qquad l>0$$

compare this : $s(l)=b\,s(l\text{-}1)$

$$s(l)=b^l s(0) \qquad l>0$$

$$R_x(l)=R_x(0)\,d^l \qquad l>0$$

$$d=\frac{a_n}{(1\text{-}K_n)}=\alpha \qquad (3)$$

Conclusion : For a recursive optimal estimate, the correlation of signal model must be of the form :

$$R_x(l) = R_x(0)\left(\frac{a_n}{1-K_n}\right)^l \qquad l>0$$

Return to signal model :

$$x(n) = \alpha x(n-1) + v_2(n)$$

Follows: $R_x(l) = \sigma_x^2 \alpha^l \qquad l \geq 0$

where $\sigma_x^2 = R_x(0) = \dfrac{\sigma_{v_2}^2}{1-|\alpha|^2}$

Conclusion : The signal generated by the above equation and the estimate $\hat{x}$ generated ($\hat{x}(n) = a_n \hat{x}(n-1) + K_n y(n)$) satisfies the orthogonality principle and hence an optimal estimate.

Now :

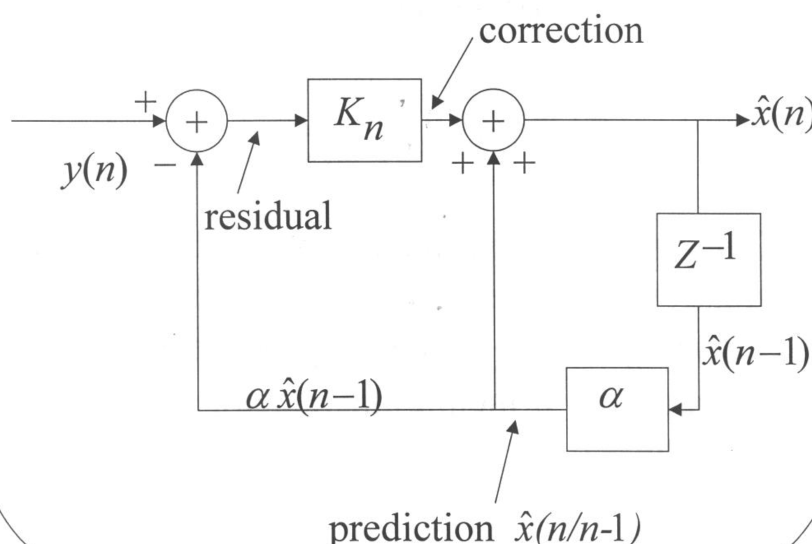
$$\hat{x}(n) = a_n \hat{x}(n-1) + K_n y(n)$$
$$= \alpha(1-K_n)\hat{x}(n-1) + K_n y(n) \qquad (4)$$

or

$$\hat{x}(n) = a_n \hat{x}(n-1) + K_n \underbrace{[y(n) - \alpha\,\hat{x}(n-1)]}_{\text{correction term}} \qquad (5)$$

or $\hat{x}(n/n-1) = \alpha\,\hat{x}(n-1)$

$$\hat{x}(n) = \hat{x}(n/n-1) + K_n[y(n) - \hat{x}(n/n-1)] \qquad (6)$$



Recursive Expansion for the Variance $\sigma^2_{\varepsilon(n)}$:

We can obtain recursive expression for $\sigma^2_{\varepsilon(n)}$:

$$\sigma^2_{\varepsilon(n)} = \frac{\sigma^2_{v_2} + |\alpha|^2 \sigma^2_{\varepsilon(n-1)}}{\sigma^2_{v_1} + \sigma^2_{v_2} + |\alpha|^2 \sigma^2_{\varepsilon(n-1)}} \sigma^2_{v_1} \qquad n > 0$$

$$\sigma^2_{\varepsilon(0)} = \frac{\sigma^2_{v_2} \sigma^2_{v_1}}{\sigma^2_{v_2} + \sigma^2_{v_1} (1 - |\alpha|^2)}$$

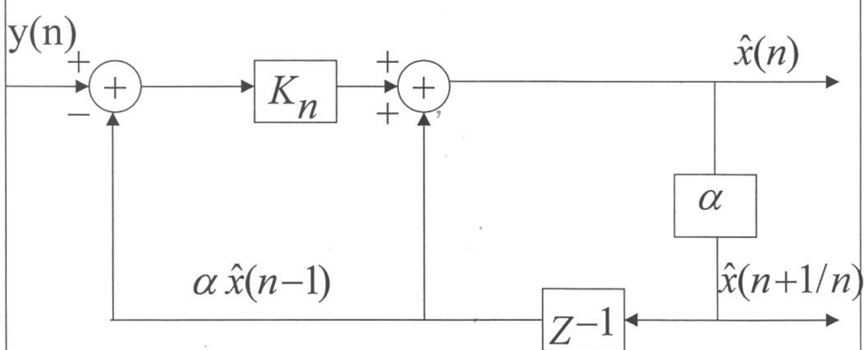
$$\sigma^2_{\varepsilon(n)} = \left(\frac{1}{\sigma^2_{v_1}} + \frac{1}{|\alpha|^2 \sigma^2_{\varepsilon(n-1)} + \sigma^2_{v_2}} \right)^{-1} \qquad n > 0$$

- Kalman Recursive Predictor

Obtain the optimal linear estimate of $x(n+1)$; that $\hat{x}(n+1/n)$, given the data and the previous estimate at n by minimizing :

$$E\left[[x(n+1)-\hat{x}(n+1/n)]^2\right]$$

Simultaneous Filtering and Prediction solution :



$$\sigma^2_{\varepsilon(n+1/n)} = \sigma^2_{v_2} + \frac{|\alpha|^2 \sigma^2_{v_1} \sigma^2_{\varepsilon(n/n-1)}}{\sigma^2_{v_1} + \sigma^2_{\varepsilon(n/n-1)}}$$

- Review Of State Variable Representation Of Dynamic Systems :

Example : Consider

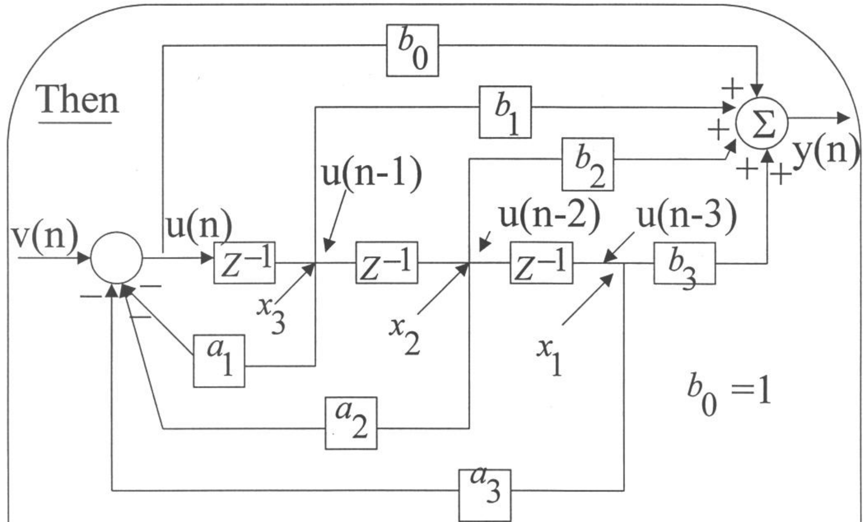
$$y(n)+a_1 y(n-1)+a_2 y(n-2)+a_3 y(n-3)$$
$$=v(n)+b_1 v(n-1)+b_2 v(n-2)+b_3 v(n-3)$$

Rewrite : Define two equivalent equations

(1) $u(n)=v(n)-a_1 u(n-1)-a_2 u(n-2)$
$-a_3 u(n-3)$

(2) $y(n)=u(n)+b_1 u(n-1)+b_2 u(n-2)$
$+b_3 u(n-3)$

Then



Define a state vector : $[x_1(n), x_2(n), x_3(n)]$

State equations :

$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = x_3(n)$$
$$x_3(n+1) = -a_3 x_1(n) - a_2 x_2(n) - a_1 x_3(n) + v(n)$$

Output equation :

$$\begin{aligned} y(n) &= b_3 x_1(n) + b_2 x_2(n) + b_1 x_3(n) + x_3(n+1) \\ &= b_3 x_1(n) + b_2 x_2(n) + b_1 x_3(n) - a_3 x_1(n) - a_2 x_2(n) \\ &\quad - a_1 x_3(n) + v(n) \end{aligned}$$

$$\underline{x}(n+1)=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix}\underline{x}+\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}v(n)$$

$$y(n)=\left[(b_3-a_3),(b_2-a_2),(b_1-a_1)\right]\underline{x}+(1)v(n)$$

More Compact Form :

$$\underline{x}(n+1)=\underset{\sim}{A}\,\underline{x}(n)+\underline{b}v(n)$$

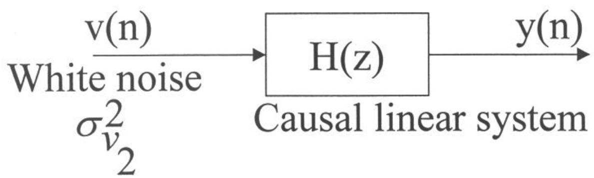
$$y(n)=\underline{C}^T\,\underline{x}(n)+dv(n)$$

For multi-input and multi-output systems :

$$\underline{x}(n+1)=\underset{\sim}{A}\,\underline{x}(n)+\underset{\sim}{B}\,\underline{v}(n)$$

$$\underline{y}(n)=\underset{\sim}{C}\,\underline{x}(n)+\underset{\sim}{D}\,\underline{v}(n)$$

- Dynamic Systems Driven By White Noise



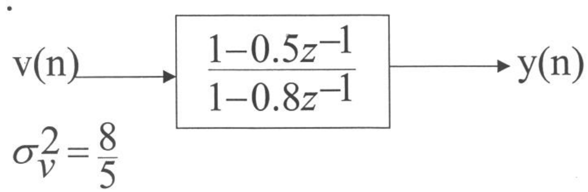
Power Spectral Density :

$$S_y(e^{jw}) = H(z)H(-z)\Big|_{z=e^{jw}} \qquad \sigma_y^2 = \left|H(e^{jw})\right|^2 \sigma_v^2$$

Example :

$$S_y(z) = \frac{8}{5} \cdot \frac{1-0.5z^{-1}}{1-0.8z^{-1}} \cdot \frac{1-0.5z}{1-0.8z}$$

Follows :

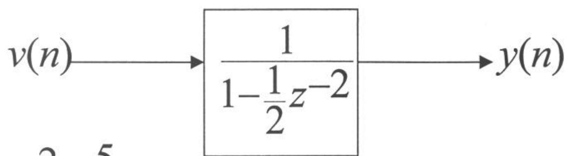


$$y(n) - 0.8y(n-1) = v(n) - 0.5v(n-1)$$

Example:

$$S_y(z)=\frac{-5z^{-2}}{(1-\frac{1}{2}z^{-2})(1-2z^{-2})}$$

$$=\frac{5}{2}\frac{1}{1-\frac{1}{2}z^{-2}}\cdot\frac{1}{1-\frac{1}{2}z^{-2}}$$



$$\sigma_v^2=\frac{5}{2}$$

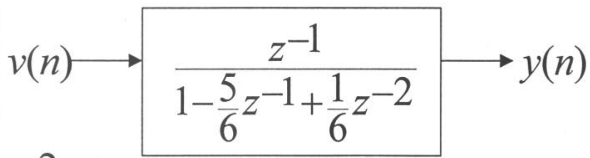
$$\therefore\ y(n)-\frac{1}{2}y(n-2)=\frac{5}{2}v(n)$$

Example :

$$S_y(e^{jw})=\frac{36}{6e^{2jw}-35e^{jw}+62-35e^{jw}+6e^{-j2w}}$$

$$S_y(z)=\frac{36}{6z^2-35z+62-35z^{-1}+6z^{-2}}$$

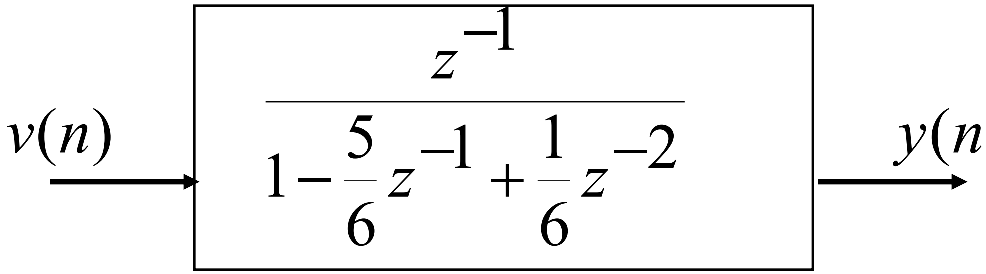
$$=\underbrace{\frac{6z^{-1}}{6-5z^{-1}+z^{-2}}}_{H(z)}\cdot\frac{6z}{6-5z+z^2}$$



$\sigma_v^2 = 1$

$$\therefore y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = v(n-1)$$

Conclusion : The output variable of a dynamic system can be modelled to have the desired power spectrum.Dynamic systems represented by difference equations correspond to some desired power spectrum associated with a random process. Dynamic systems equivalently can be modelled by state variable formulation. State variable formulation allows the inclusion of multi-variable and nonstationary conditions.



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Conclusion

The output variable of a dynamic system can be modelled to have the desired power spectrum. Dynamic systems represented by difference equations correspond to some desired power spectrum associated with a random process. Dynamic systems equivalently can be modelled by state variable formulation. State variable formulation allows the inclusion of multi - variable and nonstationary conditions.

• Recursive Estimation From the Innovation Process

$\underline{\boldsymbol{y}}_{n-1}$: The space spanned by the observations $y(1),\ldots\ldots\ldots, y(n\text{-}1)$

Suppose we have additional observation $y(n)$ at time 'n'. Suppose we want to compute the updated estimate $\hat{x}(n/\underline{\boldsymbol{y}}_n)$ of the related random variable $x(n)$.

Store the previous estimate $\hat{x}(n-1/\underline{\boldsymbol{y}}_{n-1})$ and use this somehow to obtain $\hat{x}(n/\underline{\boldsymbol{y}}_n)$ without redoing the whole estimation again.

The Innovation Process

Let $f_{n-1}(n) = y(n) - \hat{y}(n / \underline{\mathbf{y}}_{n-1})$

be the forward prediction error, where

$\hat{y}(n / \underline{\mathbf{y}}_{n-1})$ is one step prediction of $y(n)$

at time n.

Note :

$f_{n-1}(n)$ is orthogonal to all past

observations $y(1), \ldots\ldots y(n-1)$. It is also a

measure of the new information in $y(n)$

observed at n.

Define : $\alpha(n) = f_{n-1}(n) \qquad = 1,2.........$

as the innovation

$\alpha(n)$ contains the part of the observation $y(n)$ that is really new.

$\alpha(n)$ satisfies :

a. $E[\alpha(n) y^*(k)] = 0 \qquad 1 \le k \le n-1$

b. $E[\alpha(n) \alpha^*(k)] = 0 \qquad 1 \le k \le n-1$

$\alpha(n)$ is a white noise process

c. $\{y(1), y(2), \ldots y(n)\} \Leftrightarrow \{\alpha(1), \alpha(2), \ldots \alpha(n)\}$

one to one correspondence

If $\{y(1), y(2), \ldots y(n)\}$ are linearly independent, we can use the Gram-Schmidt orthogonalization process to obtain

$$\begin{pmatrix} \alpha(1) \\ \alpha(2) \\ \cdot \\ \cdot \\ \alpha(n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ a_{1,1} & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1,n-1} & a_{n-2,n-2} & \cdot & 1 \end{pmatrix} \begin{pmatrix} y(1) \\ y(2) \\ \cdot \\ y(n) \end{pmatrix}$$

Now : Define the estimation problem

$$\hat{x}(n/\underline{\boldsymbol{y}}_n) = \text{minimum mean-square estimate}$$

of $x(n)$ given the observed data $y(1)$, $y(2), \ldots\ldots. y(n)$.

or

$= \text{minimum mean-square estimate of}$ $x(n)$ given the innovations $\alpha(1), \alpha(2), \ldots\ldots .\alpha(\text{n})$.

that is $$\hat{x}(n/\underline{\boldsymbol{y}}_n) = \sum_{k=1}^{n} b_k \, \alpha(\text{k})$$

Rewrite :

$$\hat{x}(n/\underline{\boldsymbol{y}}_n) = \underbrace{\sum_{k=0}^{n-1} b_k \, \alpha(\mathrm{k})}_{\hat{x}(n-1/\underline{\boldsymbol{y}}_{n-1})} + b_n \, \alpha(\mathrm{n})$$

then $$b_k = \frac{\mathrm{E}[x(n)\alpha^*(k)]}{\mathrm{E}[\alpha(k)\alpha^*(k)]} \qquad 1 \le k \le n$$

where we have used α's are orthogonal and

$x(n) - \hat{x}(n/\underline{\boldsymbol{y}})$ is minimized.

Recursive Estimation :

$$\hat{x}(n/\underline{\mathbf{y}}_n) = \hat{x}(n-1/\underline{\mathbf{y}}_{n-1}) + b_n \alpha(n)$$

$$b_n = \frac{\mathrm{E}[x(n)\alpha^*(n)]}{\mathrm{E}[\alpha(n)\alpha^*(n)]}$$

Notice the correction term $b_n \alpha(n)$ is directly proportional to the innovation.

This gives a general approach to recursive estimation from the concept of innovation. The current updated estimate $\hat{x}(n/\underline{\mathbf{y}}_n)$ is simply obtained from the previous estimate $\hat{x}(n-1/\underline{\mathbf{y}}_{n-1})$ plus the correction term based on the scalar multiple of the innovation.

Problem Statement

Given the observed data $\underline{y}(1), \underline{y}(2), \ldots\ldots \underline{y}(n)$

for $n \geq 1$, find the minimum mean square estimates

of the state $\underline{x}(i)$.

Filtering Problem	:	If $i = n$
Prediction Problem	:	If $i > n$
Smoothing Problem	:	If $1 \leq i \leq \text{n}$

The Innovation Process : Following the scalar variable case, we define ;

a. $\underline{\alpha}(n) = \underline{y}(n) - \underline{\hat{y}}(n / \underline{\mathbf{y}}_{n-1})$

M x 1 vector

b. $E[\underline{\alpha}(n)\underline{y}^H(k)] = 0 \qquad 1 \le k \le n-1$

c. $E[\underline{\alpha}(n)\underline{\alpha}^H(k)] = \underline{0} \qquad 1 \le k \le n-1$

d. $\{\underline{y}(1), \underline{y}(2), \ldots\ldots \underline{y}(n)\} \Leftrightarrow \{\underline{\alpha}(1), \underline{\alpha}(2), \ldots. \underline{\alpha}(n)\}$

Steps In Obtaining Kalman estimation

I. Compute the correlation matrix of the Innovation Process

Express the innovation process as

$$\underline{\alpha}(n) = \underline{y}(n) - \underset{\sim}{C}(n)\underline{\hat{x}}(n / \underline{\boldsymbol{y}}_{n-1})$$

Substitute $\underline{y}(n) = \underset{\sim}{C}(n)\underline{x}(n) + \underline{v}_2(n)$

$$\underline{\alpha}(n) = \underset{\sim}{C}(n)\underline{\varepsilon}(n, n-1) + \underline{v}_2(n)$$

where $\underset{\sim}{\varepsilon}(n,n-1) = \underbrace{\underline{x}(n) - \hat{\underline{x}}(n/\underline{\boldsymbol{y}}_{n-1})}_{\text{predicted state error vector}}$

Define $\quad \underset{\sim}{\Sigma}(n) = \mathrm{E}[\underline{\alpha}(n)\underline{\alpha}^{H}(n)]$

Correlation matrix $\quad \underset{\sim}{\Sigma}(n) = \underset{\sim}{C}(n)\,\underset{\sim}{K}(\mathrm{n},\mathrm{n}-1)\,\underset{\sim}{C}^{H}(n) + \underset{\sim}{Q}_{2}(n)$

where $\underset{\sim}{\mathrm{K}}(n,n-1) = \mathrm{E}[\underline{\varepsilon}(n,n-1),\underline{\varepsilon}^{H}(n,n-1)]$

state - error correlation

Step II : Estimation of the State using the innovation process

Start from : $\underline{\hat{x}}(i/\underline{\mathbf{y}}_n) = \sum_{k=1}^{n} B_i(k)\underline{\alpha}(k)$

where $\{B_i(k)\}$ is a set of matrices to be determined.

Using the orthogonality property :

$$\mathrm{E}[\underline{\varepsilon}(i,n)\underline{\alpha}^H] = \underline{0}$$

and using $\mathrm{E}[\underline{\alpha}(\mathrm{n})\underline{\alpha}^H(k)] = \underline{0}$

we can obtain ：

$$E[\underline{x}(i)\underline{\alpha}^{H}(m)] = B_i(m)\underset{\sim}{\Sigma}(m)$$

$$\therefore \quad B_i(m) = E[\underline{x}(i)\underline{\alpha}^{H}(m)]\underset{\sim}{\Sigma}^{-1}(m)$$

$$\sum_{k=1}^{n-1} E[\underline{x}(n+1)\underline{\alpha}^{H}(k)]\underset{\sim}{\Sigma}^{-1}(k)\underline{\alpha}(k)$$

$$= \Phi(n+1,n)\hat{x}(n/\underline{\boldsymbol{y}}_{n-1})$$

Step III : Kalman Gain

Define the M x N matrix

$$\underset{\sim}{G}(n) = E[\underline{x}(n+1)\underline{\alpha}^H(\mathrm{n})]\underset{\sim}{\Sigma}^{-1}(n)$$

$$\hat{x}(n+1/\underline{\boldsymbol{y}}_{\mathrm{n}}) = \underset{\sim}{\Phi}(n+1,n)\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_{\mathrm{n\text{-}1}}) + \underset{\sim}{G}(n)\underline{\alpha}(n)$$

Notice the similarity of this to the scalar recursive estimation case :

The Kalman Gain equation can be manipulated to :

$$\underset{\sim}{G}(n) = \underset{\sim}{\Phi}(n+1,n)\underset{\sim}{K}(n,n-1)\underset{\sim}{C}^H(n)\underset{\sim}{\Sigma}^{-1}(n)$$

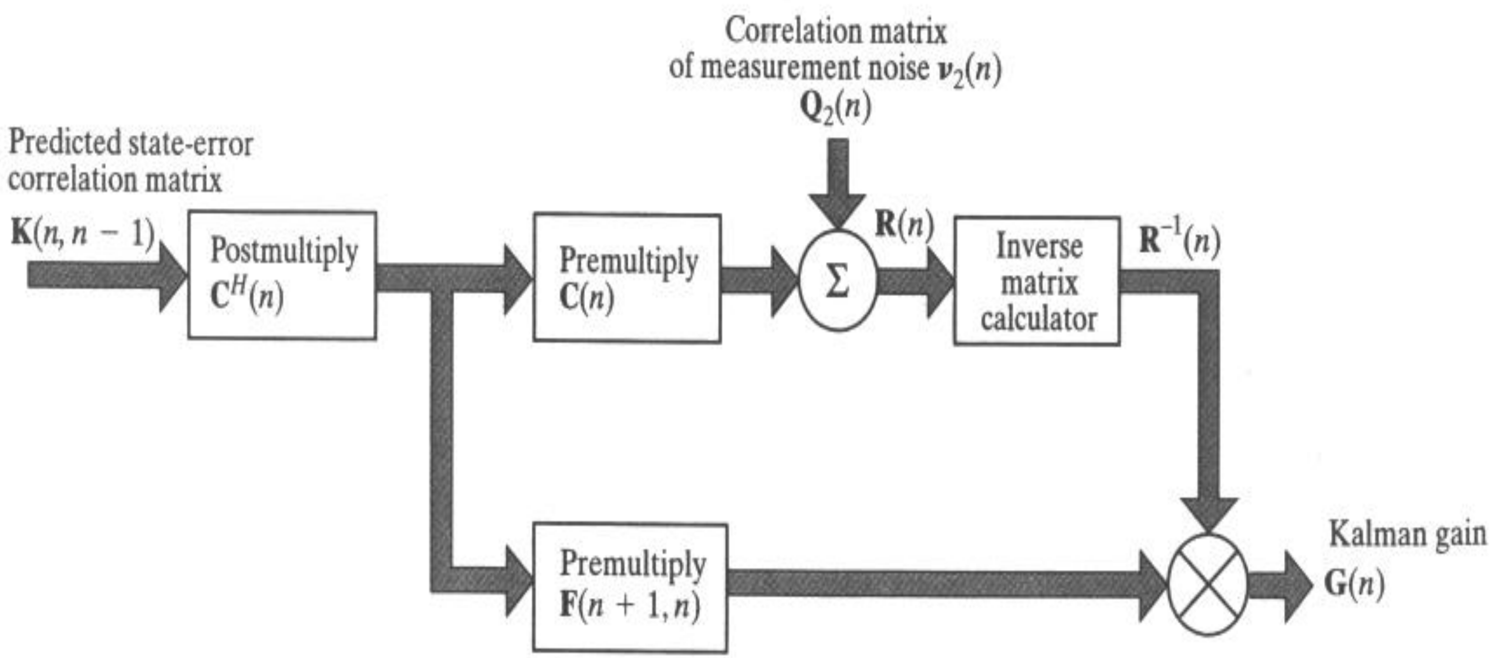


FIGURE 10.3 Kalman gain computer.

Step IV: Recursive equation for the predicted state-error correlation matrix

$$\underset{\sim}{K}(n) = \underset{\sim}{K}(n,n-1) - \underset{\sim}{\Phi}(n,n+1)\,\underset{\sim}{G}(n)\,\underset{\sim}{C}(n)\,\underset{\sim}{K}(n,n-1)$$

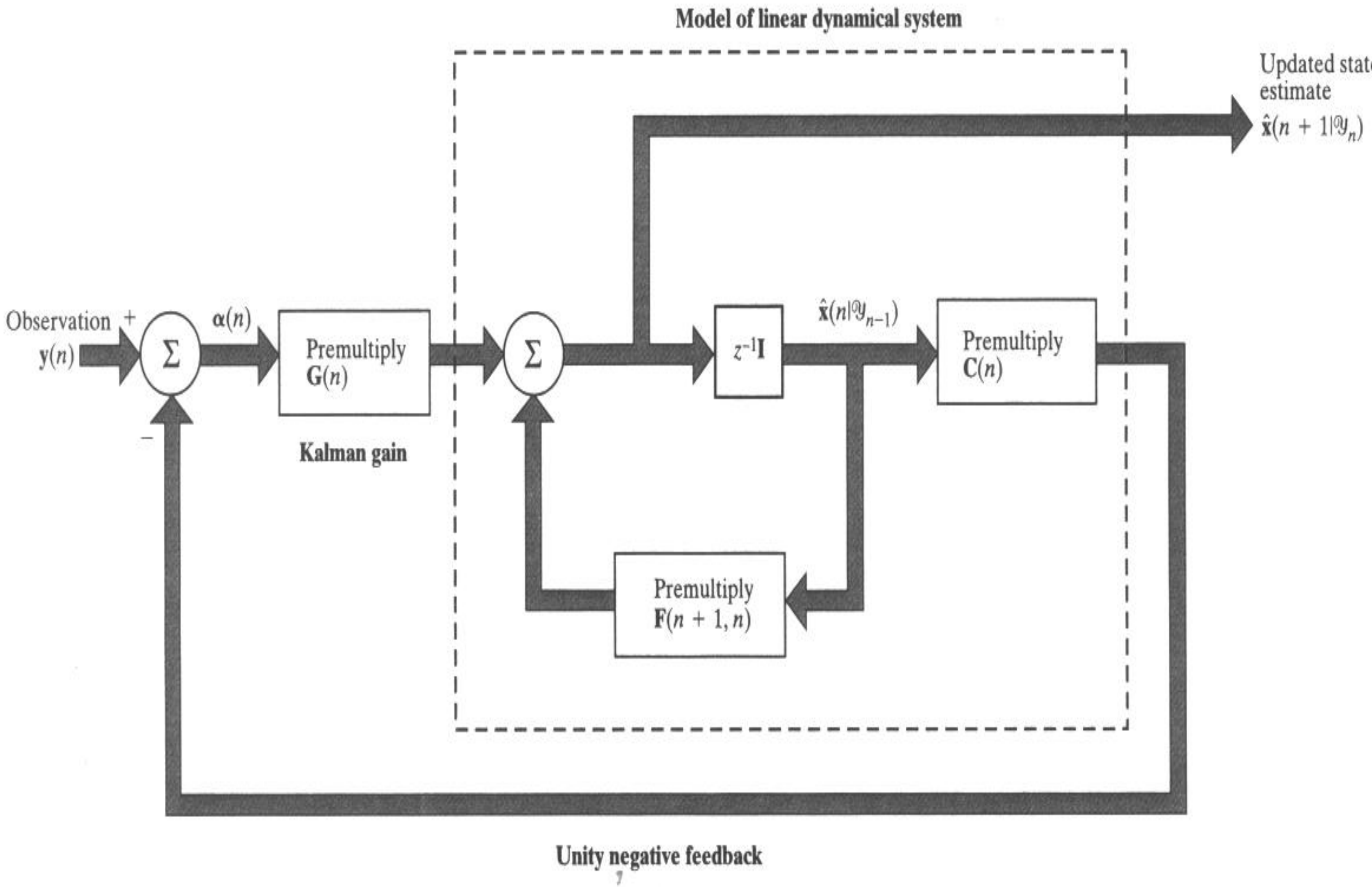


FIGURE 10.4 One-step state predictor; given the old estimate $\hat{\mathbf{x}}(n|\mathcal{Y}_{n-1})$ and the observation $\mathbf{y}(n)$, the predictor computes the new state $\hat{\mathbf{x}}(n+1|\mathcal{Y}_n)$.

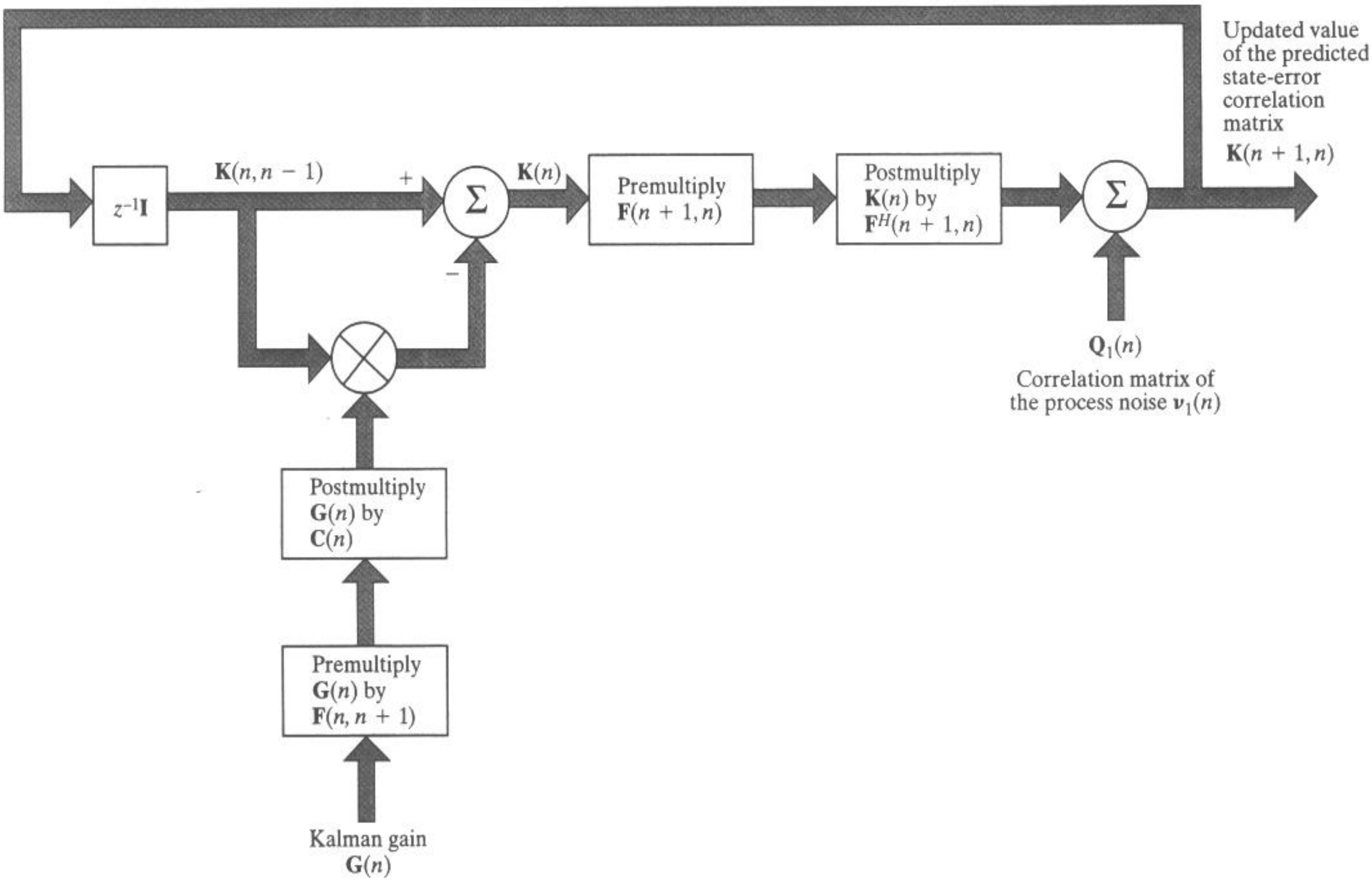


FIGURE 10.5 Riccati equation solver for propagating the predicted state-error correlation matrix.

Filtering from one-step prediction

Start from:

$$\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n) = \Phi(n+1,n)\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_n)$$

Filtered Estimates Follow:

$$\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_n) = \Phi^{-1}(n+1,n)\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n)$$

Since,

$$\Phi^{-1}(n+1,n) = \Phi(n,n+1)$$

$$\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_n) = \Phi(n,n+1)\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n)$$

Rewrite $\underline{\alpha}(n)$ at n - 1 for filtering

$$\underline{\alpha}(n) = \underline{y}(n) - \underset{\sim}{C}(n)\,\underset{\sim}{\Phi}(n, n-1)\underline{\hat{x}}(n-1/\underline{\boldsymbol{y}}_{n-1})$$

and the state vector is rewritten :

$$\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_{n}) = \underset{\sim}{\Phi}(n, n-1)\underline{\hat{x}}(n-1/\underline{\boldsymbol{y}}_{n-1}) + \underset{\sim}{\Phi}(n, n+1)\,\underset{\sim}{G}(n)\underline{\alpha}(n)$$

where

$$\underset{\sim}{G}(n) = \underset{\sim}{\Phi}(n+1, n)\,\underset{\sim}{K}(n, n-1)\,\underset{\sim}{C}^{H}(n)\underset{\sim}{\Sigma}^{-1}(n)$$

$$\underset{\sim}{\Sigma}(n) = \underset{\sim}{C}(n)\,\underset{\sim}{K}(n, n-1)\,\underset{\sim}{C}^{H}(n) + \underset{\sim}{Q}_{2}(n)$$

TABLE 10.1 Summary of the Kalman Variables and Parameters

Variable	Definition	Dimension
$\mathbf{x}(n)$	State at time n	M by 1
$\mathbf{y}(n)$	Observation at time n	N by 1
$\mathbf{F}(n+1, n)$	Transition matrix from time n to time $n+1$	M by M
$\mathbf{C}(n)$	Measurement matrix at time n	N by M
$\mathbf{Q}_1(n)$	Correlation matrix of process noise $\boldsymbol{\nu}_1(n)$	M by M
$\mathbf{Q}_2(n)$	Correlation matrix of measurement noise $\boldsymbol{\nu}_2(n)$	N by N
$\hat{\mathbf{x}}(n \mid \mathcal{Y}_{n-1})$	Predicted estimate of the state at time n given the observations $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n-1)$	M by 1
$\hat{\mathbf{x}}(n \mid \mathcal{Y}_n)$	Filtered estimate of the state at time n, given the observations $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n)$	M by 1
$\mathbf{G}(n)$	Kalman gain at time n	M by N
$\boldsymbol{\alpha}(n)$	Innovations vector at time n	N by 1
$\mathbf{R}(n)$	Correlation matrix of the innovations vector $\boldsymbol{\alpha}(n)$	N by N
$\mathbf{K}(n, n-1)$	Correlation matrix of the error in $\hat{\mathbf{x}}(n \mid \mathcal{Y}_{n-1})$	M by M
$\mathbf{K}(n)$	Correlation matrix of the error in $\hat{\mathbf{x}}(n \mid \mathcal{Y}_n)$	M by M

TABLE 10.2 Summary of the Kalman Filter Based on One-Step Prediction

Input vector process
Observations = $\{\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n)\}$
Known parameters
Transition matrix = $\mathbf{F}(n+1, n)$
Measurement matrix = $\mathbf{C}(n)$
Correlation matrix of process noise = $\mathbf{Q}_1(n)$
Correlation matrix of measurement noise = $\mathbf{Q}_2(n)$
Computation: $n = 1, 2, 3, \ldots$
$\mathbf{G}(n) = \mathbf{F}(n+1, n)\mathbf{K}(n, n-1)\mathbf{C}^H(n)\left[\mathbf{C}(n)\mathbf{K}(n, n-1)\mathbf{C}^H(n) + \mathbf{Q}_2(n)\right]^{-1}$
$\boldsymbol{\alpha}(n) = \mathbf{y}(n) - \mathbf{C}(n)\hat{\mathbf{x}}(n \mid \mathcal{Y}_{n-1})$
$\hat{\mathbf{x}}(n+1 \mid \mathcal{Y}_n) = \mathbf{F}(n+1, n)\hat{\mathbf{x}}(n \mid \mathcal{Y}_{n-1}) + \mathbf{G}(n)\boldsymbol{\alpha}(n)$
$\mathbf{K}(n) = \mathbf{K}(n, n-1) - \mathbf{F}(n, n+1)\mathbf{G}(n)\mathbf{C}(n)\mathbf{K}(n, n-1)$
$\mathbf{K}(n+1, n) = \mathbf{F}(n+1, n)\mathbf{K}(n)\mathbf{F}^H(n+1, n) + \mathbf{Q}_1(n)$
Initial conditions:
$\hat{\mathbf{x}}(1 \mid \mathcal{Y}_0) = E[\mathbf{x}(1)]$
$\mathbf{K}(1, 0) = E\left[(\mathbf{x}(1) - E[\mathbf{x}(1)])(\mathbf{x}(1) - E[\mathbf{x}(1)])^H\right] = \Pi_0$

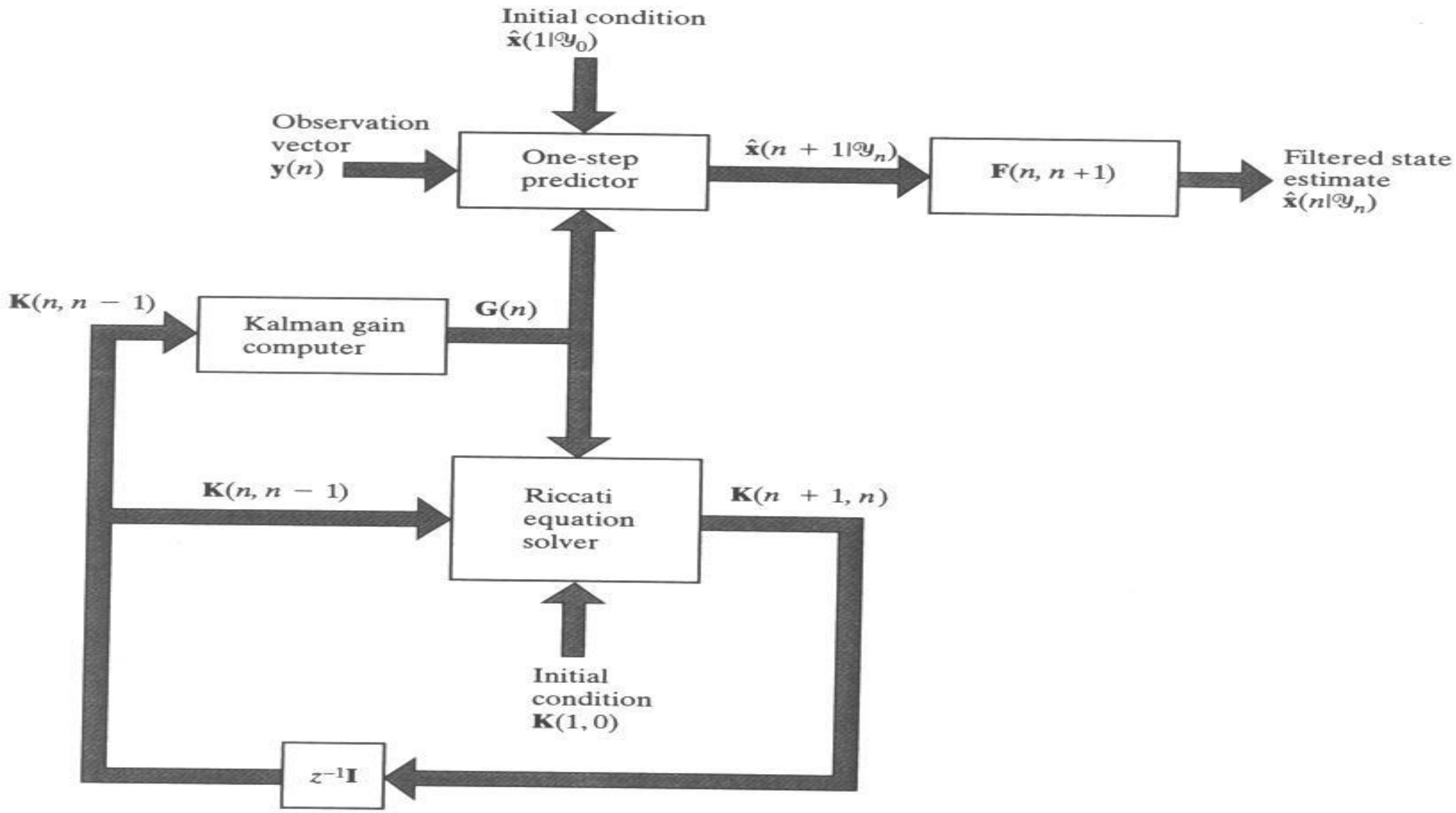


FIGURE 10.6 Block diagram of the Kalman filter based on one-step prediction.

A block diagram representation of the Kalman filter is given in Fig. 10.6, which is based on three functional blocks:

- The one-step predictor, described in Fig. 10.4
- The Kalman gain computer, described in Fig. 10.3
- The Riccati equation solver, described in Fig. 10.5

Summary : Two Important Equations

$$\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n) = \underset{\sim}{\Phi}(n+1,n)\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_{n-1}) + \underset{\sim}{G}(n)\underline{\alpha}(n)$$

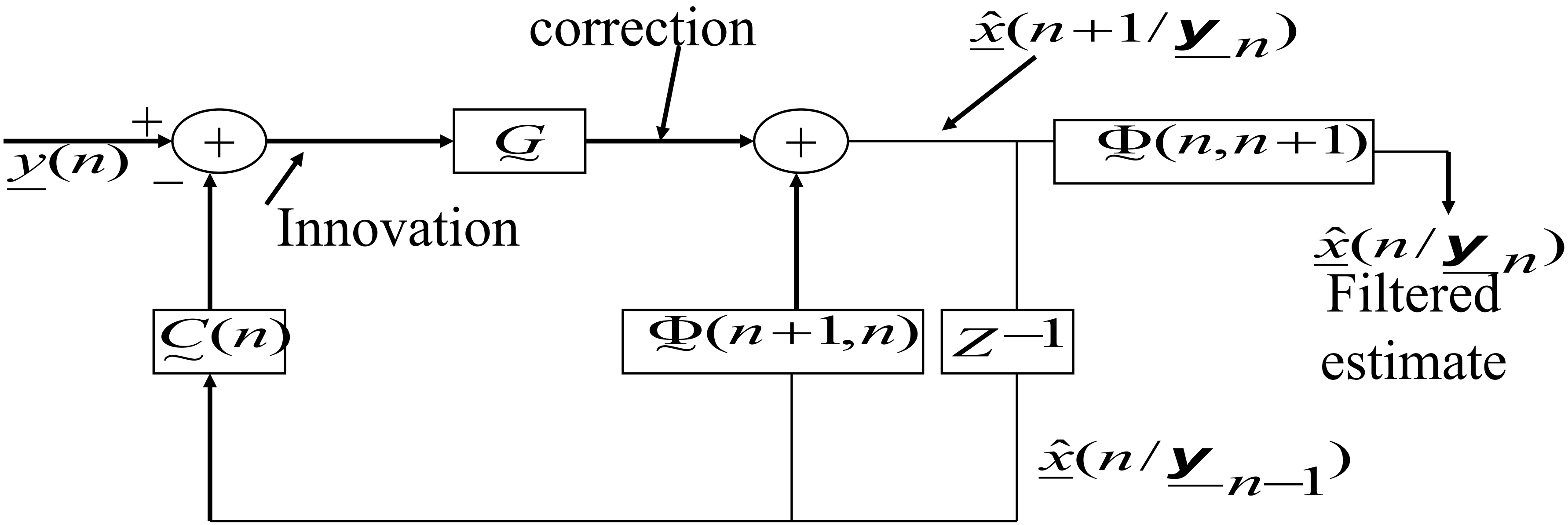
Prediction Estimates : (suubstitute for $\underline{\alpha}$)

$$\underbrace{\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n)}_{\text{updated estimate}} = \underbrace{\underset{\sim}{\Phi}(n+1/n)\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_{n-1})}_{\text{prediction estimate}} + \underbrace{\underset{\sim}{G}(n)[\underline{y}(n) - \underset{\sim}{C}(n)\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_{n-1})]}_{\text{correction term}}$$

Filtered Estimate

$$\underline{\hat{x}}(n/\underline{\boldsymbol{y}}_n) = \underset{\sim}{\Phi}(n,n+1)\underline{\hat{x}}(n+1/\underline{\boldsymbol{y}}_n)$$

Block Diagram



Initial Conditions :

$$\hat{\underline{x}}(0/\underline{\boldsymbol{y}}_0) = \mathrm{E}[\hat{\underline{x}}(0)]$$

$$\underset{\sim}{K}(0) = \mathrm{E}[\underline{x}(0)\underline{x}^H(0)] = \underset{\sim}{P}_0$$

$$\hat{\underline{x}}(0/\underline{\boldsymbol{y}}_0) = \underline{0} \quad \text{(you can choose to set)}$$

$$\hat{\underline{x}}(1/\underline{\boldsymbol{y}}_0) = \underset{\sim}{\Phi}(1,0)\hat{\underline{x}}(0/\underline{\boldsymbol{y}}_0)$$

$$= \underset{\sim}{\Phi}(1,0)\mathrm{E}[\hat{\underline{x}}(0)]$$

$$\underset{\sim}{K}(1,0) = \underset{\sim}{\Phi}(1,0)\,\underset{\sim}{K}(0)\,\underset{\sim}{\Phi}^H(1,0) + \underset{\sim}{Q}_1(0)$$

$$= \underset{\sim}{\Phi}(1,0)\,\underset{\sim}{P}_0\,\underset{\sim}{\Phi}^H(1,0) + \underset{\sim}{Q}_1(0)$$