DIGITAL (FIR)
Weiner Filters

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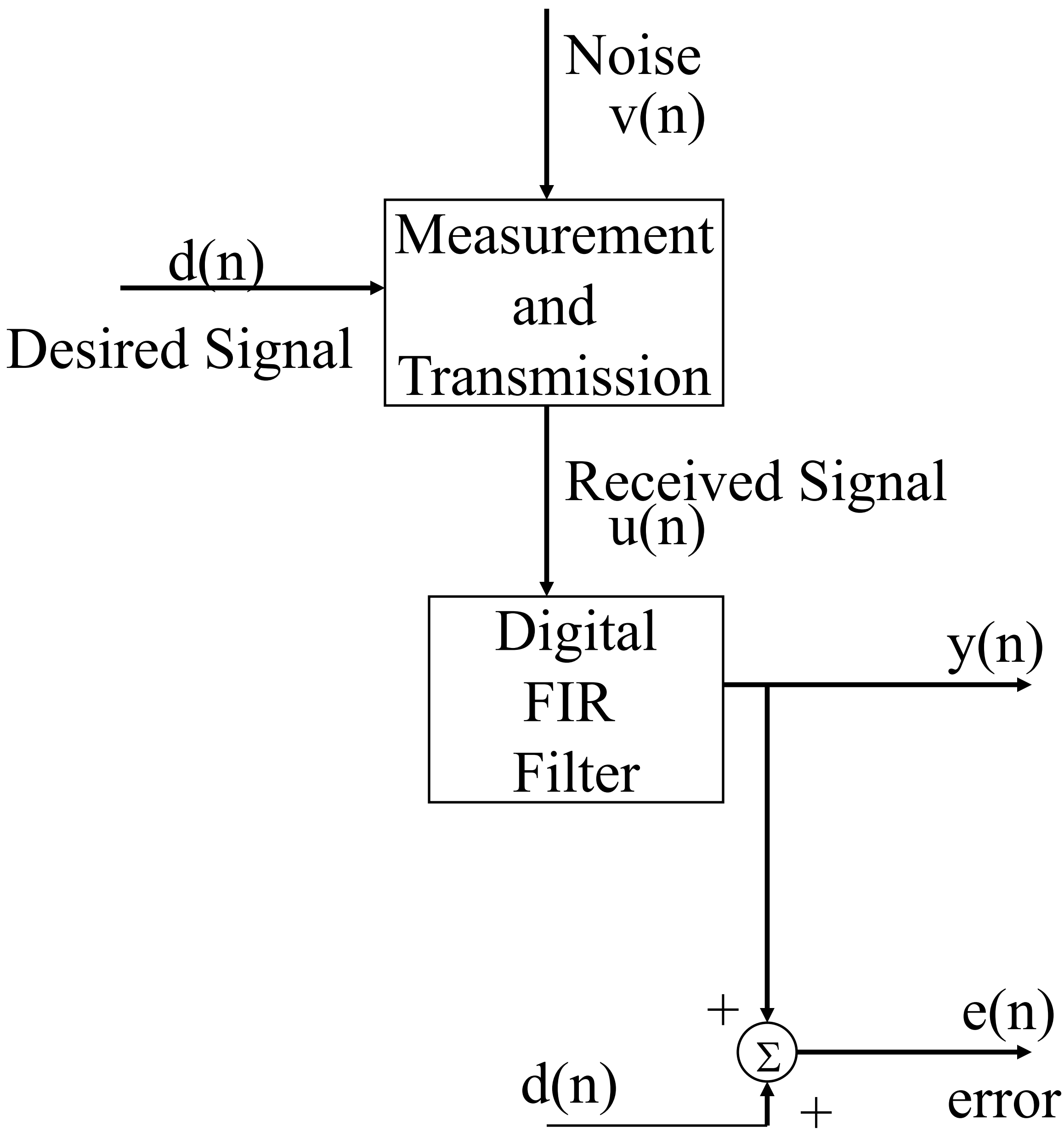
- Digital FIR (Transversal) Weiner Filter : Problem Formulation



d(n) : Stationary process (desired)

v(n) : Stationary noise process

Objective : Given the power spectral densities of d(n) and v(n), design a digital FIR filter, such that the output y(n) ($\hat{d}$) will be as close as possible to d(n) .

- **Minimum Mean-squared Error**

Given {d(n)} and {u(n)}; the desired and observed signals

Find { w(n); n = 0,. . . . . M - 1} : the impulse response values (filter tap weights) of the FIR filter :

$$y(n) = \sum_{k=0}^{M-1} w(k)u(n-k) \qquad n = 0,1,2$$

such that, MSE :

$$J = \mathrm{E}[e^2(n)]$$
$$= \mathrm{E}[(d(n) - y(n))^2] \quad \text{is minimized.}$$

Rewrite J :

$$J = \mathrm{E}[d^2(n)] - \sum_{k=0}^{M-1} w(k)\mathrm{E}[u(n-k)d(k)]$$

$$- \sum_{k=0}^{M-1} w(k)\mathrm{E}[u(n-k)d(n)]$$

$$+ \sum_{k=0}^{M-1} \sum_{i=0}^{M-1} w_k w_i \mathrm{E}[u(n-k)u(n-i)]$$

First Term :

$$\sigma_d^2 = r_d(0) = \mathrm{E}[d^2(n)]$$

= the variance of the desired response d(n) (zero mean)

Second term :

$$r_{ud} = \mathrm{E}[u(n-k)d(n)] = p(-k)$$

= the cross-correlation between u and d

For complex valued case :

$$r_{\mathrm{ud}} = \mathrm{E}[u(n-k)d^{*}(n)] = p(-k)$$

Third term : same as the second

Fourth term :

$$r(\mathrm{i-k}) = \mathrm{E}[u(n-k)u(n-i)]$$

$$= \mathrm{E}[u(n-k)u^{*}(n-i)] \quad \text{complex case}$$

= the autocorrelation of the filter input for lag value of i - k

$$J = \sigma_d^2 - 2\sum_{k=0}^{M-1} w(k)\, p(-k) + \sum_{k=0}^{M-1}\sum_{i=0}^{M-1} w_k w_i r(i-k)$$

$J =$ Second order function of the filter weights or coefficients

A bowl-shaped (M + 1) - dimensional surface with M degrees of freedom.

$J_{\min}$: At the bottom or minimum point of the error-performance, $J_{\min}$ is when J attains its minimum

$$\nabla_{\mathrm{k}}(J) = 0 \qquad k = 0,1,\ldots M-1$$

when the minimum is attained.

$$\frac{\partial J}{\partial w(k)} = 2\sum_{i=0}^{M-1} w_i r(i-k) - 2p(-k)$$
$$= 0$$

Optimum Filter Weights:

$$\sum_{i=0}^{M\text{-}1} w_{oi} r(i-k) = p(-k)$$
$$k = 0, 1, \ldots \ldots M-1$$
Weiner-Hopf equations

These equations are also known as the Normal Equations.

- Matrix Representation

Define : $\underset{\sim}{R} = \mathrm{E}[u(n)u^{H}(n)]$

where

$$\underline{u}(n) = [\,u(n), u(n-1), \ldots \ldots u(n-M+1)]^{T}$$

$$\underset{\sim}{R} = \begin{bmatrix} r(0) & r(1) & . & . & r(M-1) \\ r^{*}(1) & r(0) & . & . & r(M-2) \\ . & . & & & . \\ . & . & & & . \\ r^{*}(M-1) & r^{*}(M-2) & . & . & r(0) \end{bmatrix}$$

$$\underline{P} = \mathrm{E}[\,u(n)d^{*}(n)]$$

$$= [\,p(0), p(-1), \ldots \ldots\; p(1-M)]^{T}$$

$\underset{\sim}{R}$: $\underset{\sim}{R}$ is symmetric for real valued data

$\underset{\sim}{R}^H = \underset{\sim}{R}$ Hermitian for complex - valued data

$$r(-k) = r^*(k)$$

$\underset{\sim}{R}$ is Toeplitz (true for a stationary discrete time stochastic process)

<u>Rewrite the Optimum solution</u>

$\underset{\sim}{R}\,\underline{w}_o = \underline{p}$ Weiner - Hopf equations

where $\underline{w}_o = [w_{o0}, w_{o1}, \cdots\cdots w_{oM-1}]^T$

$\underset{\sim}{R}$: $M \times M$ correlation matrix

$\underline{w}_o$: Filter weight vector

$\underline{p}$: cross - correlation vector

Filter weight vector solution :

$$\underline{w}_o = \underset{\sim}{R}^{-1}\,\underline{p}$$

Minimum Mean-squared Error : (MMSE)

Scalar case :

Let $\hat{d}(n) = y_o(n)$ the optimal estimate of $d(n)$

then :

$$e_o(n) = d(n) - y_o(n)$$
$$= d(n) - \hat{d}(n)$$

or

$$d(n) = \hat{d} + e_o(n)$$

Also let

$$J_{\min} = \mathrm{E}[e_o^2(n)]$$

we can then write :

$$\sigma_d^2 = \sigma_{\hat{d}}^2 + J_{\min} \qquad \text{(zero mean is assumed)}$$

or

$$\boxed{J_{\min} = \sigma_d^2 - \sigma_{\hat{d}}^2}$$

<u>Vector case</u> :

$$\hat{d} = \underline{w}_o^H \, \underline{u}(n)$$

$\underline{w}_o$: vector

$\underline{u}$: vector

$$\sigma_{\hat{d}}^2 = \underline{w}_o^H \, \underset{\sim}{R} \, \underline{w}_o$$

$$= \underline{p}^H \, \underline{w}_o \qquad \text{(since } \underset{\sim}{R} \, \underline{w}_o = \underline{p} \text{)}$$

Now :

$$J_{\min} = \sigma_d^2 - \underline{p}^H \underline{w}_o$$
$$= \sigma_d^2 - \underline{p}^H \underset{\sim}{R}^{-1} p$$

- Another representation of MSE :

From page 4, we can write J as follows :

$$J = \sigma_d^2 - \underline{w}^H \underline{p} - p^H \underline{w} + \underline{w}^H \underset{\sim}{R} \underline{w} \qquad (1)$$

we know : $\underset{\sim}{R} \underline{w}_o = \underline{p}$ (2)

$$J_{\min} = \sigma_d^2 - \underline{p}^H \underline{w}_o \quad (3)$$

Eliminate σ_{d}^2 in (1) and (2) that is :

$$J(\underline{w}) = J_{\min} + \underline{p}^H \underline{w}_o - \underline{p}^H \underline{w} - \underline{w}^H \underline{p} + \underline{w}^H \underset{\sim}{R} \underline{w} \qquad (4)$$

use (2) and eliminate $\underline{p}$

$$J(\underline{w}) = J_{\min} + \underline{w}_o^H \underset{\sim}{R} \underline{w}_o - \underline{w}_o^H \underset{\sim}{R} \underline{w} - \underline{w}^H \underset{\sim}{R} \underline{w}_o + \underline{w}^H \underset{\sim}{R} \underline{w}$$

where $\underset{\sim}{R}^H = \underset{\sim}{R}$ is used

We can write $J(\underline{w})$ in canonical form :

$$\boxed{J(\underline{w}) = J_{\min} + (\underline{w} - \underline{w}_o)^H \underset{\sim}{R} (\underline{w} - \underline{w}_o)}$$

This quadratic form shows explicitly the unique optimality of minimizing the filter weight vector $\underline{w}_o$.

- J as a function of principal eigen values : Canonical form

Unitary Similarity Transformation :
Let $\underline{q}_1, \underline{q}_2, \ldots . \underline{q}_M$ be the eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots . \lambda_M$ of the $\underset{\sim}{M}$ by $\underset{\sim}{M}$ correlation matrix $\underset{\sim}{R}$, define :

$$\underset{\sim}{Q} = [\, \underline{q}_1, \underline{q}_2, \ldots . \underline{q}_M \,]$$

$$\text{where } q_i^H q_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Also define :

M by M diagonal matrix

$$\underset{\sim}{\Lambda} = diag(\lambda_1, \lambda_2, \ldots\ldots \lambda_M)$$

then :

$$\underset{\sim}{Q}^H \underset{\sim}{R} \underset{\sim}{Q} = \underset{\sim}{\Lambda}$$

$$\underset{\sim}{R} = \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^H$$

Follows :

$$J = J_{\min} + (\underline{w} - \underline{w}_o)^H \underset{\sim}{Q} \underset{\sim}{\Lambda} \underset{\sim}{Q}^H (\underline{w} - \underline{w}_o)$$

define :

$$\underline{v} = Q^H (\underline{w} - \underline{w}_o)$$

then:

$$J = J_{\min} + \underline{v}^H \underset{\sim}{\Lambda} \underline{v}$$

$$= J_{\min} + \sum_{k=1}^{M} \lambda_k v_k v_k^*$$

$$= J_{\min} + \sum_{k=1}^{M} \lambda_k \left| v_k \right|^2$$

the vector v_k constitute the principal axes of the error-performance surface, useful representation since there are no cross terms when designing adaptive FIR filters.

NUMERICAL EXAMPLE

To illustrate the filtering theory developed above, we consider the example depicted in Fig. 5.5. The desired response $d(n)$ is modeled as an AR process of order 1; that is, it may be produced by applying a white-noise process $v_1(n)$ of zero mean and variance $\sigma_1^2 = 0.27$ to the input of an all-pole filter of order 1, whose transfer function equals [see Fig. 5.5(a)]

$$H_1(z) = \frac{1}{1 + 0.8458z^{-1}}$$

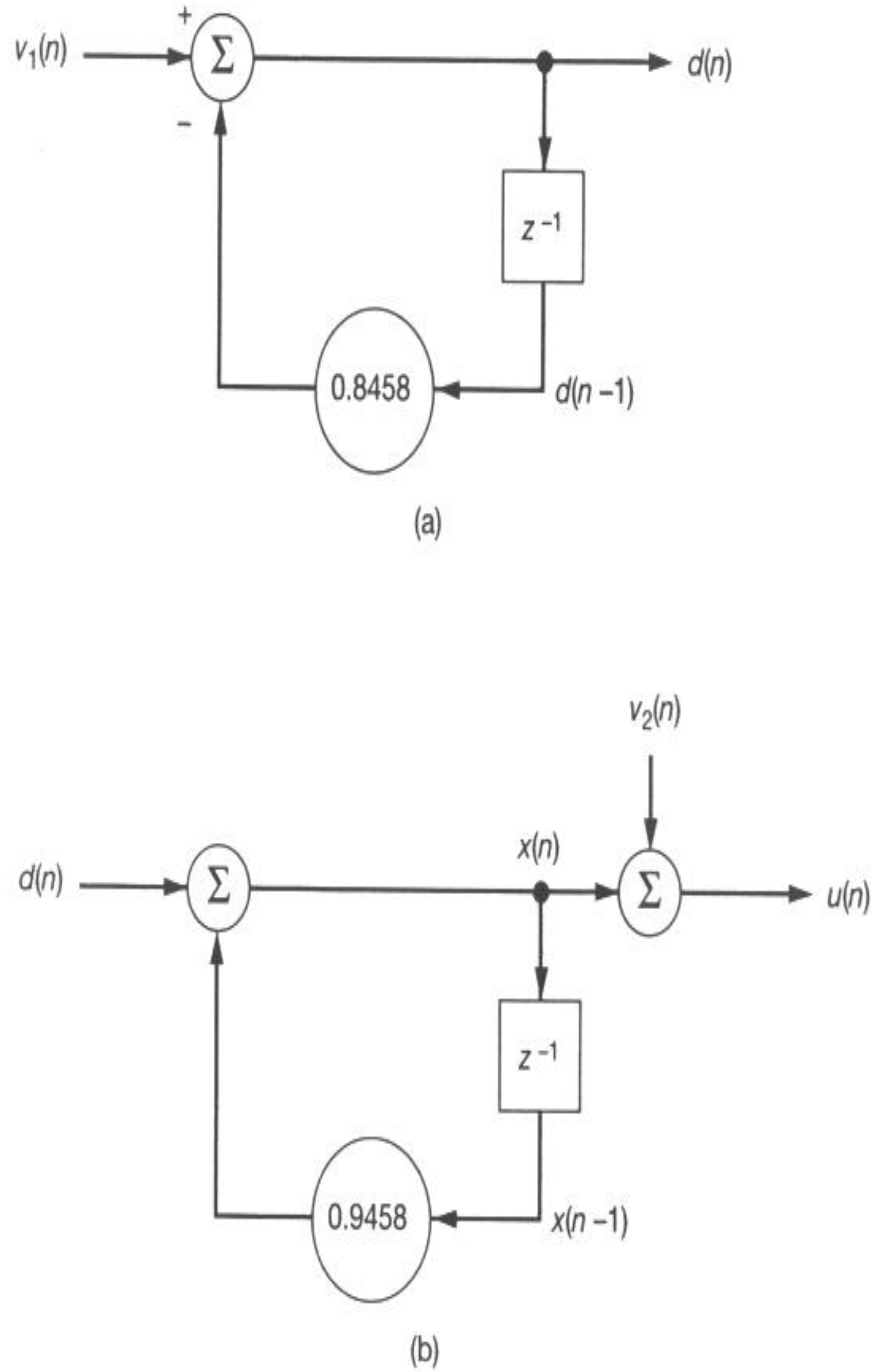
The process $d(n)$ is applied to a communication channel modeled by the all-pole transfer function

$$H_2(z) = \frac{1}{1 - 0.9458z^{-1}}$$

The channel output $x(n)$ is corrupted by an additive white-noise process $v_2(n)$ of zero mean and variance $\sigma_2^2 = 0.1$, so a sample of the received signal $u(n)$ equals [see Fig. 5.5(b)]

$$u(n) = x(n) + v_2(n)$$

The white-noise processes $v_1(n)$ and $v_2(n)$ are uncorrelated. It is also assumed that $d(n)$ and $u(n)$, and therefore $v_1(n)$ and $v_2(n)$, are all real valued.



(a) Autoregressive model of desired response $d(n)$; (b) model of noisy communication channel.

The requirement is to specify a Wiener filter consisting of a transversal filter with two taps, which operates on the received signal $u(n)$ so as to produce an estimate of the desired response that is optimum in the mean-square sense.

Statistical Characterization of the Desired Response $d(n)$ and the Received Signal $u(n)$

We begin the analysis by considering the difference equations that characterize the various processes described by the models of Fig. 5.5. First, the generation of the desired response $d(n)$ is governed by the first-order difference equation

$$d(n) + a_1 d(n-1) = v_1(n) \tag{5.58}$$

where $a = 0.8458$. The variance of the process $d(n)$ equals (see Problem 4 of Chapter 2)

$$\sigma_d^2 = \frac{\sigma_1^2}{1 - a_1^2}$$

$$= \frac{0.27}{1 - (0.8458)^2} \tag{5.59}$$

$$= 0.9486$$

The process $d(n)$ acts as input to the channel. Hence, from Fig. 5.5(b), we find that the channel output $x(n)$ is related to the channel input $d(n)$ by the first-order difference equation

$$x(n) + b_1 x(n-1) = d(n) \tag{5.60}$$

where $b_1 = -0.9458$. We also observe from the two parts of Fig. 5.5 that the channel output $x(n)$ may be generated by applying the white-noise process $v_1(n)$ to a second-order all-pole filter whose transfer function equals

$$H(z) = H_1(z)H_2(z) \tag{5.61}$$

$$= \frac{1}{(1 + 0.8458z^{-1})(1 - 0.9458z^{-1})}$$

Accordingly, $x(n)$ is a second-order AR process described by the difference equation

$$x(n) + a_1 x(n-1) + a_2 x(n-2) = v(n) \tag{5.62}$$

where $a_1 = -0.1$ and $a_2 = -0.8$. Note that both AR processes $d(n)$ and $x(n)$ are wide-sense stationary.

To characterize the Wiener filter, we need to solve the Wiener–Hopf equations (5.34). This set of equations requires knowledge of two quantities: (1) the correlation matrix $\mathbf{R}$ pertaining to the received signal $u(n)$, and (2) the cross-correlation vector $\mathbf{p}$ between $u(n)$ and the desired response $d(n)$. In our example, $\mathbf{R}$ is a 2-by-2 matrix and $\mathbf{p}$ is a 2-by-1 vector, since the transversal filter used to implement the Wiener filter is assumed to have two taps.

The received signal $u(n)$ consists of the channel output $x(n)$ plus the additive white noise $v_2(n)$. Since the process $x(n)$ and $v_2(n)$ are uncorrelated, it follows that the correlation matrix $\mathbf{R}$ equals the correlation matrix of $x(n)$ plus the correlation matrix of $v_2(n)$. That is,

$$\mathbf{R} = \mathbf{R}_x + \mathbf{R}_2 \tag{5.63}$$

For the correlation matrix $\mathbf{R}_x$, we write [since the process $x(n)$ is real valued]

$$\mathbf{R}_x = \begin{bmatrix} r_x(0) & r_x(1) \\ r_x(1) & r_x(0) \end{bmatrix}$$

where $r_x(0)$ and $r_x(1)$ are the autocorrelation functions of the received signal $x(n)$ for lags of 0 and 1, respectively. From Section 2.9 we have

$$r_x(0) = \sigma_x^2$$

$$= \left(\frac{1 + a_2}{1 - a_2}\right) \frac{\sigma_1^2}{[(1 + a_2)^2 - a_1^2]}$$

$$= \left(\frac{1 - 0.8}{1 + 0.8}\right) \frac{0.27}{[(1 - 0.8)^2 - (0.1)^2]}$$

$$= 1$$

$$r_x(1) = \frac{-a_1}{1 + a_2}$$

$$= \frac{0.1}{1 - 0.8}$$

$$= 0.5$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \qquad (5.64)$$

Next we observe that since $v_2(n)$ is a white-noise process of zero mean and variance $\sigma_2^2 = 0.1$, the 2-by-2 correlation matrix $\mathbf{R}_2$ of this process equals

$$\mathbf{R}_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad (5.65)$$

Thus, substituting Eqs. (5.64) and (5.65) in Eq. (5.63), we find that the 2-by-2 correlation matrix of the received signal $x(n)$ equals

$$\mathbf{R} = \begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix} \qquad (5.66)$$

For the 2-by-1 cross-correlation vector $\mathbf{p}$, we write

$$\mathbf{p} = \begin{bmatrix} p(0) \\ p(-1 \end{bmatrix}$$

where $p(0)$ and $p(-1)$ are the cross-correlation functions between $d(n)$ and $u(n)$ for lags of 0 and -1, respectively. Since these two processes are real valued, we have

$$p(k) = p(-k) = E[u(n-k)d(n)], \qquad k = 0, 1 \qquad (5.67)$$

Substituting Eqs. (5.57) and (5.60) into Eq. (5.67), and recognizing that the channel output $x(n)$ is uncorrelated with the white-noise process $v_2(n)$, we get

$$p(k) = r_x(k) + b_1 r_x(k-1), \qquad k = 0, 1$$

Putting $b_1 = -0.9458$ and using the element values for the correlation matrix $\mathbf{R}_x$ given in Eq. (5.64), we obtain

$$\begin{aligned} p(0) &= r_x(0) + b_1 r_x(-1) \\ &= 1 - 0.9458 \times 0.5 \\ &= 0.5272 \end{aligned}$$

$$\begin{aligned} p(1) &= r_x(1) + b_1 r_x(0) \\ &= 0.5 - 0.9458 \times 1 \\ &= -0.4458 \end{aligned}$$

Hence,

$$\mathbf{p} = \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix} \qquad (5.68)$$

Error-Performance Surface

The dependence of the mean-squared error on the 2-by-1 tap-weight vector **w** is defined by Eq. (5.50). Hence, substituting Eqs. (5.59), (5.66), and (5.68) into (5.50), we get

$$J(w_0, w_1) = 0.9486 - 2[0.5272, -0.4458]\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + [w_0, w_1]\begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

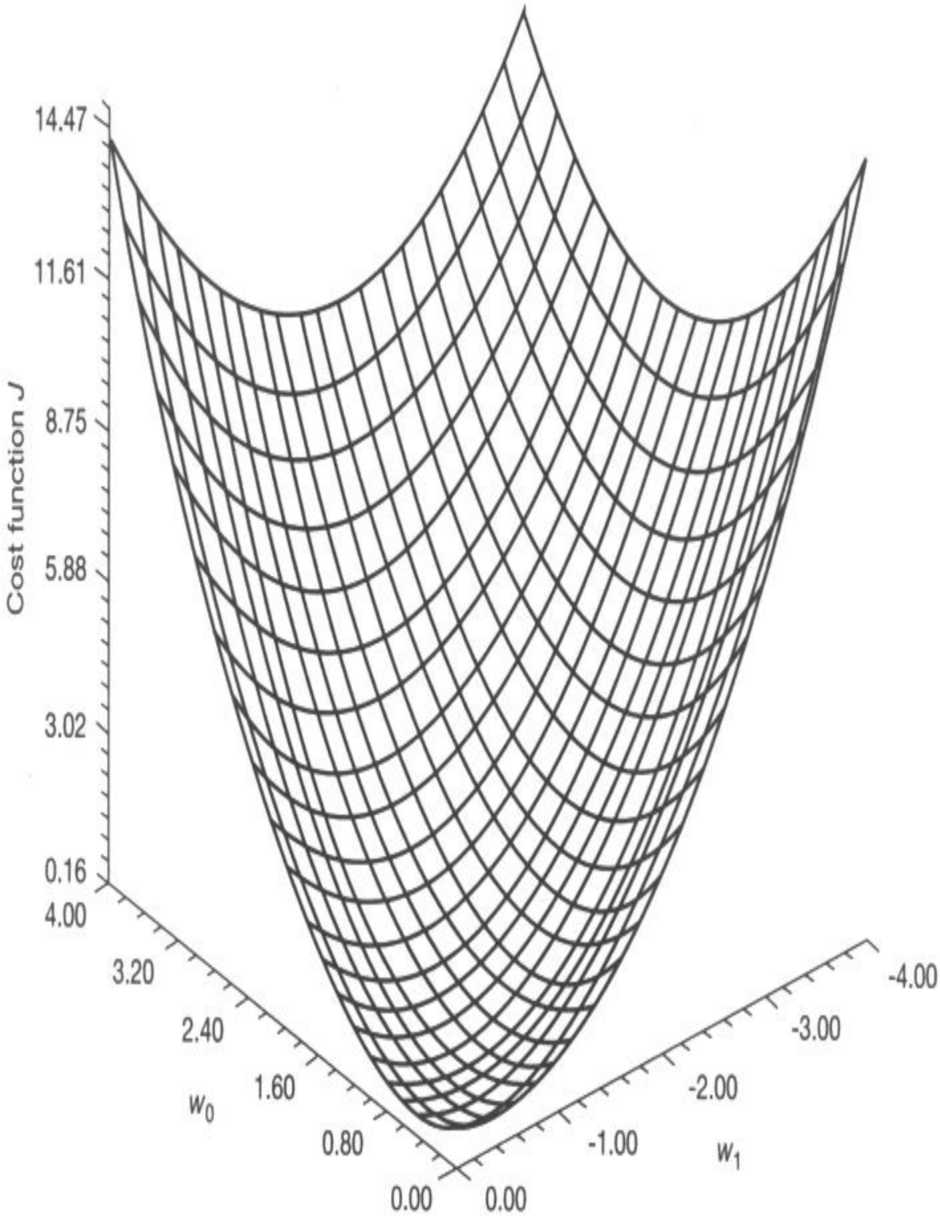
$$= 0.9486 - 1.0544w_0 + 0.8916w_1 + w_0w_1 + 1.1(w_0^2 + w_1^2)$$

Using a three-dimensional computer plot, the mean-squared error $J(w_0, w_1)$ is plotted versus the tap weights w_0 and w_1. The result is shown in Fig. 5.6.

Figure 5.7 shows contour plots of the tap weight w_1 versus w_0 for varying values of the mean-squared error J. We see that the locus of w_1 versus w_0 for a fixed J is in the form of an ellipse. The elliptical locus shrinks in size as the mean-squared error J approaches the minimum value $J_{\min}$. For $J = J_{\min}$, the locus reduces to a point with coordinates w_{o0} and w_{o1}.

Wiener Filter

The 2-by-1 optimum tap-weight vector $\mathbf{w}_o$ of the Wiener filter is defined by Eq. (5.36). In particular, it consists of the inverse matrix $\mathbf{R}^{-1}$ multiplied by the cross-correlation vector **p**. Inverting the correlation matrix **R** of Eq. (5.66), we get



Error-performance surface of the two-tap transversal filter described in the numerical example.

$$\mathbf{R}^{-1} = \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix}^{-1}$$

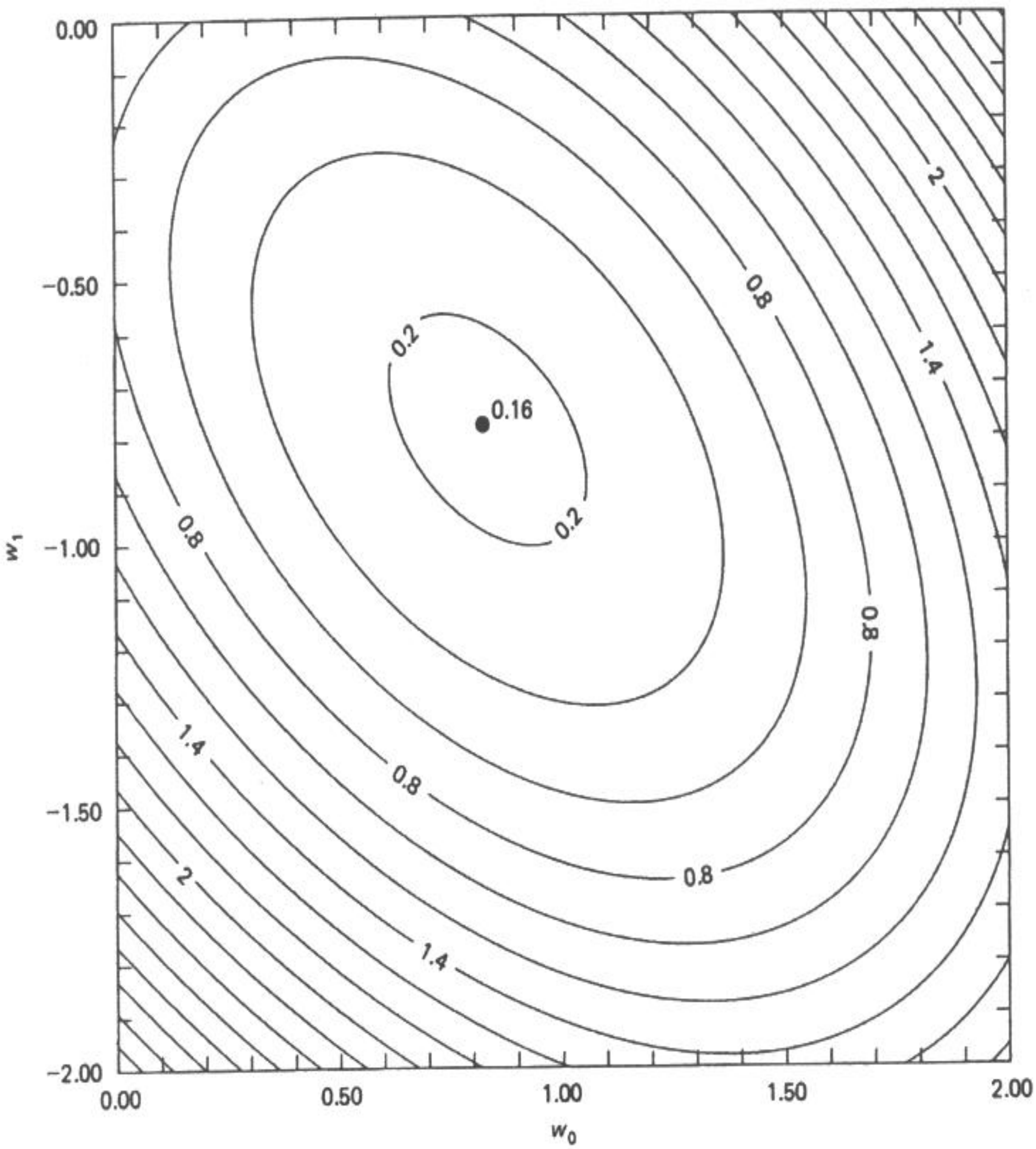
$$= \frac{1}{r^2(0) - r^2(1)} \begin{bmatrix} r(0) & -r(1) \\ -r(1) & r(0) \end{bmatrix} \qquad (5.69)$$

$$= \begin{bmatrix} 1.1456 & -0.5208 \\ -0.5208 & 1.1456 \end{bmatrix}$$

Hence, substituting Eqs. (5.68) and (5.69) into Eq. (5.36), we get the desired result:

$$\mathbf{w}_o = \begin{bmatrix} 1.1456 & -0.5208 \\ -0.5208 & 1.1456 \end{bmatrix} \begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8360 \\ -0.7853 \end{bmatrix} \qquad (5.70)$$



Contour plots of the error-performance surface depicted in Fig. 5.6.

Minimum Mean-Squared Error

To evaluate the minimum value of the mean-squared error, J_{min}, which results from the use of the optimum tap-weight vector $\mathbf{w}_o$, we use Eq. (5.49). Hence, substituting Eqs. (5.59), (5.68), and (5.70) into Eq. (5.49), we get

$$\begin{aligned} J_{min} &= 0.9486 - [0.5272, -0.4458]\begin{bmatrix} 0.8360 \\ -0.7853 \end{bmatrix} \\ &= 0.1579 \end{aligned} \quad (5.71)$$

The point represented jointly by the optimum tap-weight vector $\mathbf{w}_o$ of Eq. (5.70) and the minimum mean-squared error of Eq. (5.71) defines the bottom of the error-performance surface in Fig. 5.6, or the center of the contour plots in Fig. 5.7.

Canonical Error-Performance Surface

The characteristic equation of the 2-by-2 correlation matrix **R** of Eq. (5.66) is

$$(1.1 - \lambda)^2 - (0.5)^2 = 0$$

The two eigenvalues of the correlation matrix **R** are therefore

$$\lambda_1 = 1.6$$

$$\lambda_2 = 0.6$$

The canonical error-performance surface is therefore defined by [see Eq. (5.57)]

$$J(v_1, v_2) = J_{\min} + 1.6v_1^2 + 0.6v_2^2 \qquad (5.72)$$

The locus of v_2 versus v_1, as defined in Eq. (5.72), traces an *ellipse* for a fixed value of $J - J_{\min}$. In particular, the ellipse has a minor axis of $[(J - J_{\min})/\lambda_1]^{1/2}$ along the v_1-coordinate and a major axis of $[(J - J_{\min}/\lambda_2)]^{1/2}$ along the v_2-coordinate; this assumes that $\lambda_1 > \lambda_2$, which is how they are related.

EXAMPLE

To illustrate the optimum filtering theory developed in the preceding sections, consider a regression model of order $m = 3$ with its parameter vector denoted by

$$\mathbf{a} = \left[a_0, a_1, a_2\right]^T.$$

The statistical characterization of the model, assumed to be real valued, is as follows:

(a) The correlation matrix of the input vector $\mathbf{u}(n)$ is

$$\mathbf{R}_4 = \begin{bmatrix} 1.1 & 0.5 & 0.1 & -0.05 \\ 0.5 & 1.1 & 0.5 & 0.1 \\ 0.1 & 0.5 & 1.1 & 0.5 \\ -0.05 & 0.1 & 1.5 & 1.1 \end{bmatrix},$$

where the dashed lines are included to identify the submatrices that correspond to varying filter lengths.

(b) The cross-correlation vector between the input vector $\mathbf{u}(n)$ and observable data $d(n)$ is

$$\mathbf{p} = [0.5272, -0.4458, -0.1003, -0.0126]^T,$$

where the value of the fourth entry ensures that the model parameter a_3 is zero (i.e., the model order m is 3, as prescribed; see Problem 9).

(c) The variance of the observable data is

$$\sigma_d^2 = 0.9486.$$

(d) The variance of the additive white noise is

$$\sigma_\nu^2 = 0.1066.$$

The requirement is to do three things:

- Investigate the variation of the minimum mean-square error $J_{\min}$ produced by a Wiener filter of varying length $M = 1, 2, 3, 4$.
- Display the error-performance surface of a Wiener filter with length $M = 2$.
- Compute the canonical form of the error-performance surface.

Variation of J_{min} with filter length M With model order $M = 3$, the real-valued regression model is described by

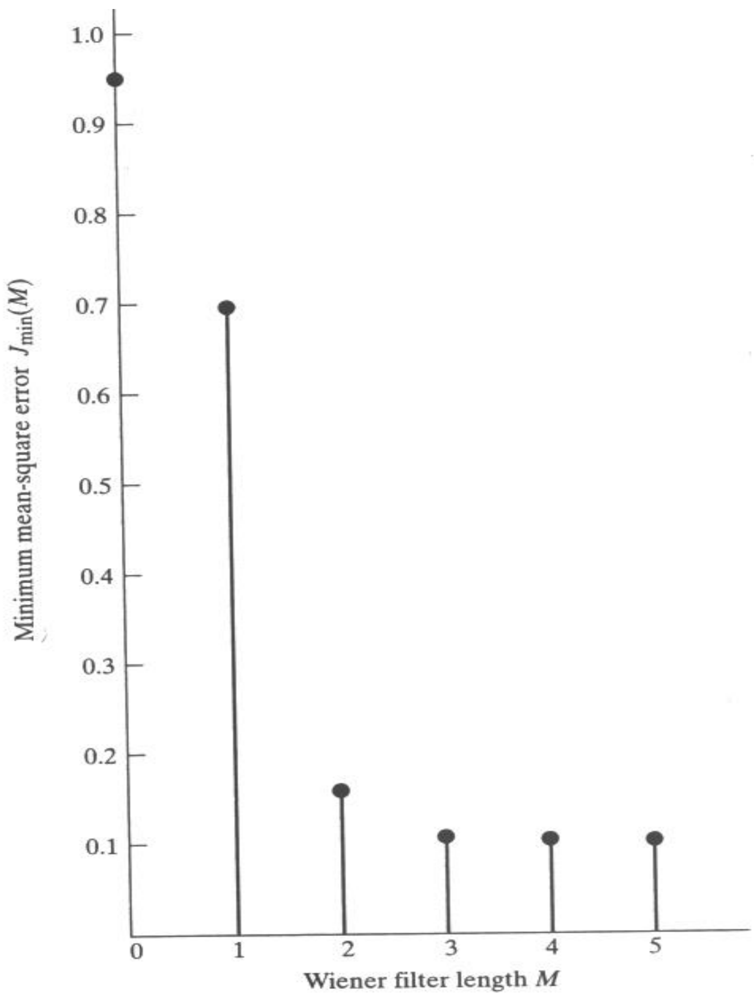
$$d(n) = a_o u(n) + a_1 u(n-1) + a_2 u(n-2) + \nu(n), \tag{2.66}$$

where $a_k = 0$ for all $k \geq 3$. Table 2.1 summarizes the computations of the M-by-1 optimum tap-weight vector and minimum mean-square error $J_{min}(M)$ produced by the Wiener filter for $M = 1, 2, 3, 4$. The table also includes the pertinent values of the correlation matrix $\mathbf{R}$ and cross-correlation vector $\mathbf{p}$ that are used in Eqs. (2.36) and (2.49) to perform the computations.

Figure 2.6 displays the variation of the minimum mean-square error $J_{min}(M)$ with the Wiener filter length M. The figure also includes the point corresponding to the worst

Summary of Wiener filter computations for varying filter length M.

Filter length M	Correlation matrix $\mathbf{R}$	Cross-correlation vector $\mathbf{p}$	Optimum tap-weight vector $\mathbf{w}_o$	Minimum mean-square error $J_{min}(M)$
1	$[1.1]$	$[0.5272]$	$[0.4793]$	0.6959
2	$\begin{bmatrix} 1.1 & 0.5 \\ 0.5 & 1.1 \end{bmatrix}$	$\begin{bmatrix} 0.5272 \\ -0.4458 \end{bmatrix}$	$\begin{bmatrix} 0.8360 \\ -0.7853 \end{bmatrix}$	0.1576
3	$\begin{bmatrix} 1.1 & 0.5 & 0.1 \\ 0.5 & 1.1 & 0.5 \\ 0.1 & 0.5 & 1.1 \end{bmatrix}$	$\begin{bmatrix} 0.5272 \\ -0.4458 \\ -0.1003 \end{bmatrix}$	$\begin{bmatrix} 0.8719 \\ -0.9127 \\ 0.2444 \end{bmatrix}$	0.1066
4	$\begin{bmatrix} 1.1 & 0.5 & 0.1 & -0.05 \\ 0.5 & 1.1 & 0.5 & 0.1 \\ 0.1 & 0.5 & 1.1 & 0.5 \\ -0.05 & 0.1 & 0.5 & 1.1 \end{bmatrix}$	$\begin{bmatrix} 0.5272 \\ -0.4458 \\ -0.1003 \\ -0.0126 \end{bmatrix}$	$\begin{bmatrix} 0.8719 \\ -0.9129 \\ 0.2444 \\ 0 \end{bmatrix}$	0.1066



Variation of $J_{min}(M)$ with Wiener filter length M