LINEAR PREDICTION
Levinson-Durbin Algorithm
Lattice Filter Structures

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- Forward Linear Prediction

Given a set of data samples :
$\{u(n-1),\ u(n-2), ... u(n-M)\}$,

the past M samples, find an estimate of the current value (a linear one step prediction of $u(n)$) of a random process $u(n)$ from M previous values :

$$\hat{u}(n) = -a_1 u(n-1) - a_2 u(n-2), ...\ a_M u(n-M)$$

Define the forward prediction error :

$$f_M(n) = u(n) - \hat{u}(n)$$

Rewrite :

$$f_M(n) = u(n) + a_1 u(n-1) + a_2 u(n-2) + \dots .$$
$$\dots + a_M u(n-M)$$

Define matrix form :

$$\underline{u}(n) = \begin{bmatrix} u(n-M) \\ u(n-M+1) \\ . \\ . \\ u(n) \end{bmatrix}$$

and $\underline{a} = \begin{bmatrix} 1 \\ a_1 \\ . \\ . \\ a_M \end{bmatrix}$ prediction coefficients

$P_M = \mathrm{E}[f_M(n)^2]$ Minimum Mean-Square prediction error

Matrix Representation :

$$f_M(n) = a^T \underline{\tilde{u}}(n)$$

where

$$\underline{\tilde{u}} = \begin{bmatrix} u(n) \\ u(n-1) \\ . \\ . \\ u(n-M) \end{bmatrix}$$

Find the optimal filter coefficients by using the orthogonality theorem :

$$\mathrm{E}[u(n-i) f_M^*(n)] = 0 \qquad i = 1,2,.\ .\ .\ .\ .M$$

$$P_M = \mathrm{E}[u(n) f_M^*(n)]$$

More compactly :

$$\mathrm{E}[\underline{\tilde{u}}(n) f_M^*(n)] = \begin{bmatrix} P_M \\ 0 \\ . \\ . \\ 0 \end{bmatrix} = P_M \underline{l}$$

where $$\underline{l} = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

For real valued case :

$$\mathrm{E}[\tilde{u}(n)(\underline{a}^T \underline{\tilde{u}})] = \mathrm{E}[\underline{\tilde{u}}(\underline{\tilde{u}}^T \underline{a})] = \mathrm{E}[\underline{\tilde{u}}\,\underline{\tilde{u}}^T]\underline{a}$$
$$= P_M \underline{l}$$

since $$\mathrm{E}[\tilde{u}\tilde{u}^T] = \underset{\sim}{\tilde{R}}_u$$

$$\underset{\sim}{\tilde{R}}\,\underline{a} = P_M \underline{l}$$

Finding optimal forward linear prediction coefficients lead to the Normal Equations :

$$\begin{bmatrix} r_u(0) & r_u(1) & . \; . & r_u(M) \\ r_u(-1) & r_u(0) & . \; . & r_u(M-1) \\ . & . & . \; . & . \\ . & . & . \; . & . \\ r_u(-M) & r_u(-M+1) & . \; . & r_u(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ . \\ . \\ a_M \end{bmatrix} = \begin{bmatrix} P_M \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

- Backward linear prediction

  predict or estimate the point $u(n-M)$ from the points :

$$\underbrace{u(n-M+1),. \; . \; . \; . \; . \; .u(n)}_{\text{future points}}$$

Then :

$$\hat{u}(n-M) = -g_1 u(n-M+1) - g_2 u(n-M+2) \\ -.\,.\,.\,.\,.g_M u(n)$$

Backward prediction Error :

$$b_M(n) = u(n-M) - \hat{u}(n-M)$$

$$or$$

$$b_M(n-M) = u(n-M) + g_1 u(n-M+1) \\ +.\,.\,.\,.\,.g_M u(n)$$

Define :

$$\underline{g} = \begin{bmatrix} 1 \\ g_1 \\ . \\ . \\ g_M \end{bmatrix}$$

then : $b_M(n) = \underline{g}^T \, \underline{u}(n)$

the orthogonality theorem implies

$$\mathrm{E}[\underline{u}(n)b_M(n)]=\begin{bmatrix} P_M^{(b)} \\ 0 \\ . \\ . \\ 0 \end{bmatrix}=P_M^{(b)}\underline{l}$$

<u>Follows :</u>

$$\mathrm{E}[\underline{u}(\underline{g}^T\underline{u})]=\mathrm{E}[\underline{u}\underline{u}^T]\underline{g}=P_M^{(b)}\underline{l}$$

$$\underset{\sim}{R}_u\,\underline{g}=P_M^{(b)}\underline{l}$$

$$\begin{bmatrix} r_u(0) & r_u(-1) & . \; . \; . & r_u(-M) \\ r_u(1) & r_u(0) & . \; . \; . & r_u(-M+1) \\ . & . & . \; . \; . & . \\ . & . & . \; . \; . & . \\ r_u(M) & r_u(M-1). \; . & & r_u(0) \end{bmatrix}\begin{bmatrix} 1 \\ g_1 \\ . \\ . \\ g_M \end{bmatrix}=\begin{bmatrix} P_M^{(b)} \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

The normal equations for backward prediction differ only in that the correlation matrix is in unreversed form.

Since $\tilde{R}_u = R_u^*$ (complex conjugate)

Implies : $\boxed{\underline{g} = \underline{a}^*}$

$$\boxed{P_M = P_M^{(b)}}$$

For real random process, forward prediction filter is identical to the backward prediction error filter. For Gaussian processes, this equivalence between forward and backward prediction models is unique.

- **LEVINSON ALGORITHM**

Normal equations arise in solving digital Weiner filter equations and in solving linear prediction problems and also in modelling AR processes by linear filters (Yule - Walker). For stationary processes, the correlation matrix has a special structure called Toeplitz. Levinson algorithm allows the solution of these linear equations in O(p^2) operations instead of O(p^3) operations when solved by algorithms based on Gaussian elimination procedures.

- Forward *Prediction* Normal Equations :

$$\underset{\sim}{\tilde{R}}_u^{(M)} \underline{a}_M = \begin{bmatrix} P_M \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \qquad (1)$$

where

$$\underset{\sim}{\tilde{R}}_u^{(M)} = \begin{bmatrix} R_u(0) & R_u(1) & . & . & R_u(M) \\ R_u(-1) & R_u(0) & . & . & R_u(M-1) \\ . & . & . & . & . \\ . & . & . & . & . \\ R_u(-M) & R_u(-M+1) & . & . & R_u(0) \end{bmatrix}$$

$$\underline{a}_M = \begin{bmatrix} 1 \\ a_1^{(M)} \\ . \\ . \\ a_M^{(M)} \end{bmatrix}$$

- Backward Prediction Normal Equations

$$\underset{\sim}{R}_u^{(M)} \, \underline{g}_M = \begin{bmatrix} P'_M \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \qquad (2)$$

where

$$\underset{\sim}{R}_u^{(M)} = \begin{bmatrix} R_u(0) & R_u(-1) & . & . & R_u(-M) \\ R_u(1) & R_u(0) & . & . & R_u(1-M) \\ . & . & . & . & . \\ . & . & . & . & . \\ R_u(M) & R_u(M-1) & . & . & R_u(0) \end{bmatrix}$$

$$\underline{g}_M = \begin{bmatrix} 1 \\ g_1^{(M)} \\ . \\ . \\ g_M^{(M)} \end{bmatrix}$$

Define :

$$\underline{r}_M = \begin{bmatrix} R_u(1) \\ R_u(2) \\ . \\ . \\ R_u(M+1) \end{bmatrix}$$

Then :

$$\underset{\sim}{\widetilde{R}}{}_u^{(M)} = \left[\begin{array}{ccc|c} \underset{\sim}{\widetilde{R}}{}_u^{(M-1)} & . & . & \widetilde{\underline{r}}_{M-1} \\ . & . & . & \\ . & . & . & \\ - & - & - & - \\ \widetilde{\underline{r}}{}^{*T}_{M-1} & . & . & R_u(0) \end{array}\right]$$

and

$$\underset{\sim}{R}_{u}^{(M)} = \begin{bmatrix} \underset{\sim}{R}_{u}^{(M-1)} & . & . & | & \underline{\tilde{r}}^{*}_{M-1} \\ . & . & . & | & \\ . & . & . & | & \\ - & - & - & | & - \\ \underline{\tilde{r}}^{T}_{M-1} & . & . & | & R_{u}(0) \end{bmatrix}$$

The Development of the Levinson Algorithm :

Assume that the linear prediction parameters of order M-1 are known. They satisfy (1) with M replaced by M-1 .

Consider the forward problem as follows :

$$\underset{\sim}{\tilde{R}}_u^{(M)} \begin{bmatrix} \underline{a}_{M-1} \\ . \\ . \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\tilde{R}}_u^{(M-1)} & . & . & | & \tilde{r}_{M-1} \\ . & . & . & | & \\ . & . & . & | & \\ - & - & - & | & - \\ \tilde{r}^{*T}_{M-1} & . & . & | & R_u(0) \end{bmatrix} \begin{bmatrix} \underline{a}_{M-1} \\ . \\ . \\ - \\ 0 \end{bmatrix} = \begin{bmatrix} P_{M-1} \\ 0 \\ . \\ . \\ \Delta_M \end{bmatrix} \qquad (3)$$

where

$$\Delta_M = \underline{\tilde{r}}^{*T}_{M-1} \underline{a}_{M-1} = \underline{r}^{*T}_{M-1} \underline{\tilde{a}}_{M-1}$$

Consider the Backward problem :

$$\underset{\sim}{R}_u^{(M)}\begin{bmatrix} \underline{g}_{M-1} \\ . \\ . \\ - \\ 0 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R}_u^{(M-1)} & . & . & | & \tilde{r}^*_{M-1} \\ . & . & . & | & \\ . & . & . & | & \\ - & - & - & | & - \\ \tilde{\underline{r}}^T_{M-1} & . & . & | & R_u(0) \end{bmatrix} \begin{bmatrix} \underline{g}_{M-1} \\ . \\ . \\ - \\ 0 \end{bmatrix} = \begin{bmatrix} P'_{M-1} \\ 0 \\ . \\ . \\ \Delta'_M \end{bmatrix}$$

(4)

where

$$\Delta'_M = \tilde{\underline{r}}^T_{M-1}\underline{g}_{M-1} = \underline{r}^T_{M-1}\tilde{\underline{g}}_{M-1}$$

Suppose we reverse (4)

$$\tilde{R}_u^{(M)}\begin{bmatrix} 0 \\ --- \\ . \\ . \\ \tilde{\underline{g}}_{M-1} \end{bmatrix} = \begin{bmatrix} \Delta'_M \\ 0 \\ . \\ . \\ P'_{M-1} \end{bmatrix} \qquad (5)$$

Suppose we multiply (5) by C^{I} and add to (3), the result is :

$$\tilde{R}_u^{(M)}\left[\begin{bmatrix}\underline{a}_{M-1}\\ .\\ .\\ 0\end{bmatrix}+c_1\begin{bmatrix}0\\ .\\ .\\ \underline{\tilde{g}}_{M-1}\end{bmatrix}\right]=\begin{bmatrix}P_{M-1}\\ 0\\ .\\ \Delta_M\end{bmatrix}+c_1\begin{bmatrix}P'_M\\ 0\\ .\\ P'_{M-1}\end{bmatrix} \qquad (6)$$

Compare this (6) to (1). Since the solution to (1) is unique (normal equation of order M), it follows :

$$\begin{bmatrix}P_{M-1}\\ 0\\ .\\ \Delta_M\end{bmatrix}+c_1\begin{bmatrix}\Delta'_M\\ 0\\ .\\ P'_{M-1}\end{bmatrix}=\begin{bmatrix}P_M\\ 0\\ .\\ 0\end{bmatrix} \qquad (7)$$

and
$$\begin{bmatrix}\underline{a}_{M-1}\\ .\\ ---\\ 0\end{bmatrix}+c_1\begin{bmatrix}0\\ ---\\ .\\ \underline{\tilde{g}}_{M-1}\end{bmatrix}=\underline{a}_M \qquad (8)$$

(7) requires that :

$$P_{M-1} + c_1 \Delta'_M = P_M \qquad (9a)$$

$$\Delta_M + c_1 P'_{M-1} = 0 \qquad (9b)$$

Repeat this procedure for the backward problem :

$$R_u^{(M)} \begin{bmatrix} 0 \\ --- \\ . \\ . \\ \tilde{a}_{M-1} \end{bmatrix} = \begin{bmatrix} \Delta_p \\ 0 \\ . \\ . \\ P_{M-1} \end{bmatrix} \qquad (10)$$

Multiply (10) by C_2 and add to (4)

$$R_u^{(M)}\left[\begin{bmatrix}\underline{g}_{M-1}\\ \cdot\\ ---\\ 0\end{bmatrix}+c_2\begin{bmatrix}0\\ ---\\ \cdot\\ \tilde{a}_{M-1}\end{bmatrix}\right]=\begin{bmatrix}P'_{M-1}\\ 0\\ \cdot\\ \Delta'_M\end{bmatrix}+c_2\begin{bmatrix}\Delta_M\\ 0\\ \cdot\\ P_{M-1}\end{bmatrix} \qquad (11)$$

Now compare this to (2) : Follows :

$$P'_{M-1}+c_2\Delta_M=P'_M \qquad (12a)$$

$$\Delta'_{M-1}+c_2P_{M-1}=0 \qquad (12b)$$

and

$$\begin{bmatrix}\underline{g}_{M-1}\\ \cdot\\ ---\\ 0\end{bmatrix}+c_2\begin{bmatrix}0\\ ---\\ \cdot\\ \underline{\tilde{a}}_{M-1}\end{bmatrix}=\underline{g}_M \qquad (13)$$

Find c_1 and c_2 from (9b) and (12b)

From (9b) :

$$c_1 = \frac{-\Delta_M}{P'_{M-1}} \qquad (14a)$$

From (12b) :

$$c_2 = \frac{-\Delta'_M}{P_{M-1}} \qquad (14b)$$

Summary : Let $\Gamma_{\mathrm{M}} = -c_1$

$\Gamma'_{\mathrm{M}} = -c_2$

The notation is for future reference when they are known as reflection coefficients.

Initialization :	$\underline{r}_o = R_u(1), \underline{a}_o = \underline{g}_o = 1, p_o = p'_o = R_u(0)$	
I	$\Delta_M = \underline{r}^{*T}_{M-1}\tilde{a}_{M-1}$	$\Delta'_M = \underline{r}^{T}_{M-1}\underline{\tilde{g}}_{M-1}$
II	$\Gamma_M = \dfrac{\Delta_M}{P'_{M-1}}$	$\Gamma'_M = \dfrac{\Delta'_M}{P_{M-1}}$
III	$\underline{a}_M = \begin{bmatrix} \underline{a}_{M-1} \\ . \\ \cdots\cdots \\ 0 \end{bmatrix} - \Gamma_M \begin{bmatrix} 0 \\ \cdots\cdots \\ . \\ \underline{\tilde{g}}_{M-1} \end{bmatrix}$	$\underline{g}_M = \begin{bmatrix} \underline{g}_{M-1} \\ . \\ \cdots\cdots \\ 0 \end{bmatrix} - \Gamma'_M \begin{bmatrix} 0 \\ \cdots\cdots \\ . \\ \underline{\tilde{a}}_{M-1} \end{bmatrix}$
IV	$P_M = P_{M-1} - \Gamma_M \Delta'_M$ $P_M = (1 - \Gamma_M \Gamma'_M) P_{M-1}$	$P'_M = P'_{M-1} - \Gamma'_M \Delta_M$ $P'_M = (1 - \Gamma'_M \Gamma_M) P'_{M-1}$

Special Case : By using the relationship between the forward and the backward prediction filters :

In step I :

we know (page 9) $\begin{cases} \underline{g}_{M-1} = \underline{a}^*_{M-1} \\ P'_{M-1} = P_{M-1} \end{cases}$

then : Step I will change to :

$$\Delta'_M = r^T_{M-1}\underline{\tilde{g}}_{M-1} = r^T_{M-1}\tilde{a}^*_{M-1} = \Delta^*_M$$

Also Step II will change to :

$$\Gamma'_M = \frac{\Delta'_M}{P_{M-1}} = \frac{\Delta^*_M}{P'_{M-1}} = \Gamma^*_M$$

Revised Levinson Algorithm summary :

$$\Gamma_M = \frac{\underline{r}^{*T}_{M-1}\,\underline{\tilde{a}}_{M-1}}{P_{M-1}} \qquad (1)$$

$$\underline{a}_M = \begin{bmatrix} \underline{a}_{M-1} \\ . \\ 0 \end{bmatrix} - \Gamma_M \begin{bmatrix} 0 \\ . \\ \tilde{a}^{*}_{M-1} \end{bmatrix} \qquad (2)$$

$$P_M = (1 - |\Gamma_M|^2) P_{M-1} \qquad (3)$$

Initialize : $a_o = 1;\quad \underline{r}_o = R_u(1);\quad P_o = R_u(0)$ and compute for m = 1,2,.........., M

Additional Observations :

(a) last element of $\tilde{a}^{*}_{M-1}$ is equal to 1, follows:

$$a^{(M)}_{M} = -\Gamma_M$$

(b) note (3) in the boxed section :

$$P^2_M \geq 0 \text{ and } P^2_{M-1} \geq 0$$

implies that $|\Gamma_M| \leq 1$ and $P_M \leq P_{M-1}$

Also $\boxed{0 \leq |\Gamma_M| < 1}$

Levinson - Durbin Algorithm :
(levinson (1947) Durbin (1960))
Restated (textbook notation)

I. Forward Prediction Filter is order updated :

$$\underline{a}_M = \begin{bmatrix} \underline{a}_{M-1} \\ 0 \end{bmatrix} + \Gamma_M \begin{bmatrix} o \\ a^{B*}_{M-1} \end{bmatrix} \qquad (A)$$

scalar case k = 0,1, ...M

$$a_{M,K} = a_{M-1,K} + \Gamma_M a^{*}_{M-1,M-K}$$

$$a_{M-1,0} = 1 \qquad a_{M-1,M} = 0$$

$\underline{a}_M$: the tap - weight vector of a forward prediction - error filter of order M

$\underline{a}^{B*}_M$: reversed $\underline{a}_M$ and complex conjugate of the resulting vector.

$$\underline{a}^{B*}_{M} = \begin{bmatrix} 0 \\ \underline{a}^{B*}_{M-1} \end{bmatrix} + \Gamma^{*}_{M} \begin{bmatrix} \underline{a}_{M-1} \\ 0 \end{bmatrix} \qquad (B)$$

scalar case

$$a^{*}{}_{M,M-K} = a^{*}{}_{M-1,M-1} + \Gamma^{*}_{M} a_{M-1,K}$$

$$\mathrm{K} = 0,1,.\ .\ .\ .\ .,\mathrm{M}$$

$$\begin{bmatrix} P_M \\ . \\ \underline{0}_M \end{bmatrix} = \begin{bmatrix} P_{M-1} \\ \underline{0}_{M-1} \\ \Delta_{M-1} \end{bmatrix} + \Gamma_M \begin{bmatrix} \Delta^{*}_{M-1} \\ \underline{0}_{M-1} \\ P_{M-1} \end{bmatrix}$$

Follows :

$$P_M = P_{M-1} + \Gamma_M \Delta^{*}_{M-1}$$

$$0 = \Delta_{M-1} + \Gamma_M P_{M-1}$$

$$\Gamma_M = -\frac{\Delta_{M-1}}{P_{M-1}}$$

$$P_M = P_{M-1}(1 - \left|\Gamma_M\right|^2)$$

$$\Gamma_M = a_{M,M}$$

Example Suppose we are given the normal equations as follows :

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} P_2 \\ 0 \\ 0 \end{bmatrix}$$

Compute : $\Gamma_1 = -\frac{r_o a_o}{P_o} = -\frac{2.1}{3} = -2/3$

$$(\underline{r}_o = R_u(1) = 2, \quad P_o R_u(0) = 3)$$

$$\Gamma_{M=} - \Gamma_M \text{ (text book)}$$

$M = 1$

$$\underline{a}_1 = \begin{bmatrix} \underline{a}_0 \\ . \\ 0 \end{bmatrix} + \Gamma_1 \begin{bmatrix} 0 \\ . \\ a_0^{B*} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2/3 \end{bmatrix}$$

$$P_1 = (1 - |\Gamma_1|^2) P_0 = (1 - (2/3)^2).3 = 5/3$$

$$M = 2$$

$$\Gamma_2 = -\frac{r_1^{*T} a_1^B}{P_1} = -\frac{1}{5/3}\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}-2/3 \\ 1\end{bmatrix} = \frac{1}{5} = +0.2$$

$$\underline{a}_2 = \begin{bmatrix}\underline{a}_1 \\ . \\ 0\end{bmatrix} + \Gamma_2\begin{bmatrix}0 \\ . \\ a_1^{B*}\end{bmatrix} = \begin{bmatrix}1 \\ -2/3 \\ 0\end{bmatrix} + \frac{1}{5}\begin{bmatrix}0 \\ -2/3 \\ 1\end{bmatrix} = \begin{bmatrix}1 \\ -0.8 \\ 0.2\end{bmatrix}$$

$$P_2 = 1.6$$

Example

Suppose we are given reflection coefficients directly from the data; $\Gamma_1, \Gamma_2, \Gamma_3$ and average power P_o :

Start from scalar form :

$$a_{M,K} = a_{M-K,K} + \Gamma_{\mathrm{M}} a^*_{M-1,M-K}$$

$$K = 0,1,. \; . \; . \; . \; .M$$

$$P_M = P_{M-1}(1 - |\Gamma_{\mathrm{M}}|^2)$$

$$\Gamma_{\mathrm{M}} = a_{M,M}$$

$M = 1$: $\quad a_{1,0} = 1 \quad a_{1,1} = \Gamma_1 \quad P_1 = P_0(1 - |\Gamma_1|^2)$

$M = 2$ $\quad a_{2,0} = 1 \quad a_{2,1} = \Gamma_1 + \Gamma_2 \Gamma_1^*$

$$a_{2,2} = \Gamma_2 \quad P_2 = P_1(1 - |\Gamma_2|^2)$$

$$M = 3$$

$$a_{3,0} = 1$$

$$a_{3,3} = a_{2,1} + \Gamma_3 \Gamma_2^*$$

$$a_{3,2} = \Gamma_2 + \Gamma_3 a_{2,1}^*$$

$$P_3 = P_2 (1 - \left| \Gamma_3 \right|^2)$$

$$a_{3,3} = \Gamma_3$$

- Lattice Structures to Implement Linear Prediction Error Filters

- Repeating (A) and (B)

$$\underline{a}_M = \begin{bmatrix} \underline{a}_{M-1} \\ 0 \end{bmatrix} + \Gamma_M \begin{bmatrix} 0 \\ \underline{a}^{B*}_{M-1} \end{bmatrix} \qquad (A)$$

$$\underline{a}^{B*}_M = \begin{bmatrix} 0 \\ \underline{a}^{B*}_{M-1} \end{bmatrix} + \Gamma^*_M \begin{bmatrix} \underline{a}_{M-1} \\ 0 \end{bmatrix} \qquad (B)$$

Define : $\{u(n), u(n-1), \; . \; . \; . \; .u(n-m)\}$ the input to the forward filter

$$\underline{u}_{M+1}(n) = \begin{bmatrix} \underline{u}_M(n) \\ . \\ u(n-M) \end{bmatrix}$$

or $$\underline{u}_{M+1}(n) = \begin{bmatrix} u(n) \\ . \\ \underline{u}_M(n-1) \end{bmatrix}$$

Now using (A), we may write as follows :

1.

$$f_M(n) = a_M^H \underline{u}_{M+1}(n)$$

forward prediction filter

2. $$[a_{M-1}^H|0]\underline{u}_{M+1}(n) = [\underline{a}_{M-1}^H|0] \begin{bmatrix} \underline{u}_M(n) \\ . \\ u(n-M) \end{bmatrix}$$

$$= \underline{a}_{M-1}^H \underline{u}_M(n) = f_{M-1}(n)$$

3. $$[0|\underline{a}_{M-1}^{BT}]\underline{u}_{M+1}(n) = [0|\underline{a}_{M-1}^{BT}] \begin{bmatrix} u(n) \\ . \\ \underline{u}_M(n-1) \end{bmatrix}$$

$$= \underline{a}_{M-1}^{BT} u_M(n-1)$$

$$= b_{M-1}(n-1)$$

backward prediction filter

Now combining 1, 2, and 3 :

$$f_M(n) = f_{M-n} + \Gamma_M^* b_{M-1}(n-1) \qquad (C)$$

Similarly , we can manipulate :

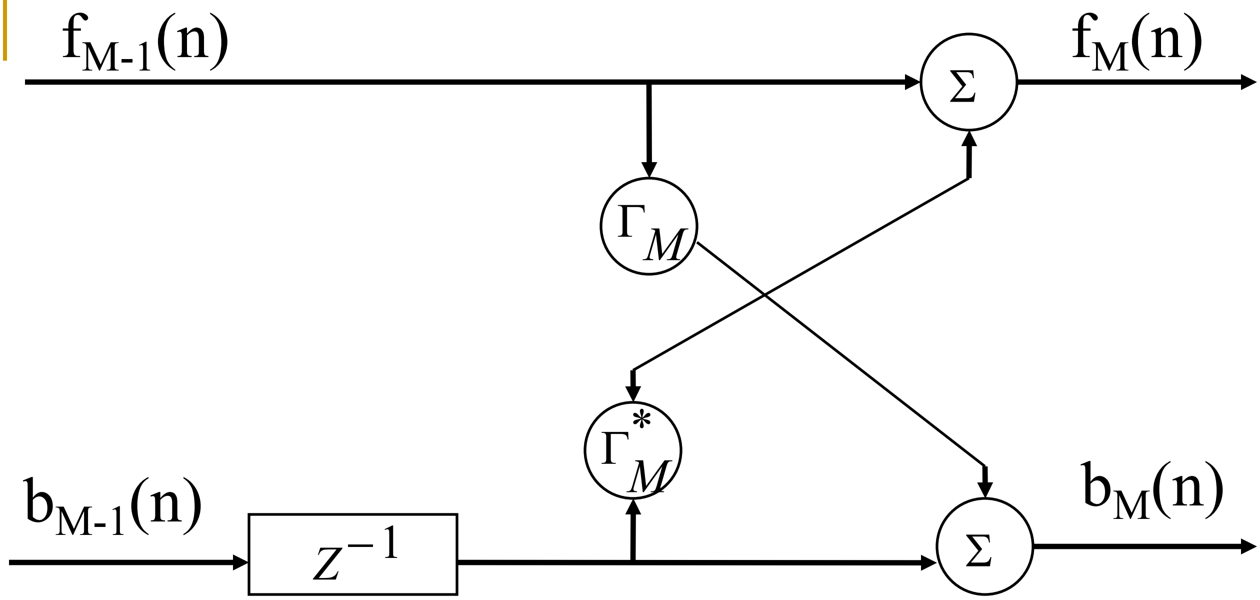
$$b_M(n) = b_{M-1}(n-1) + \Gamma_M f_{M-1}(n) \qquad (D)$$

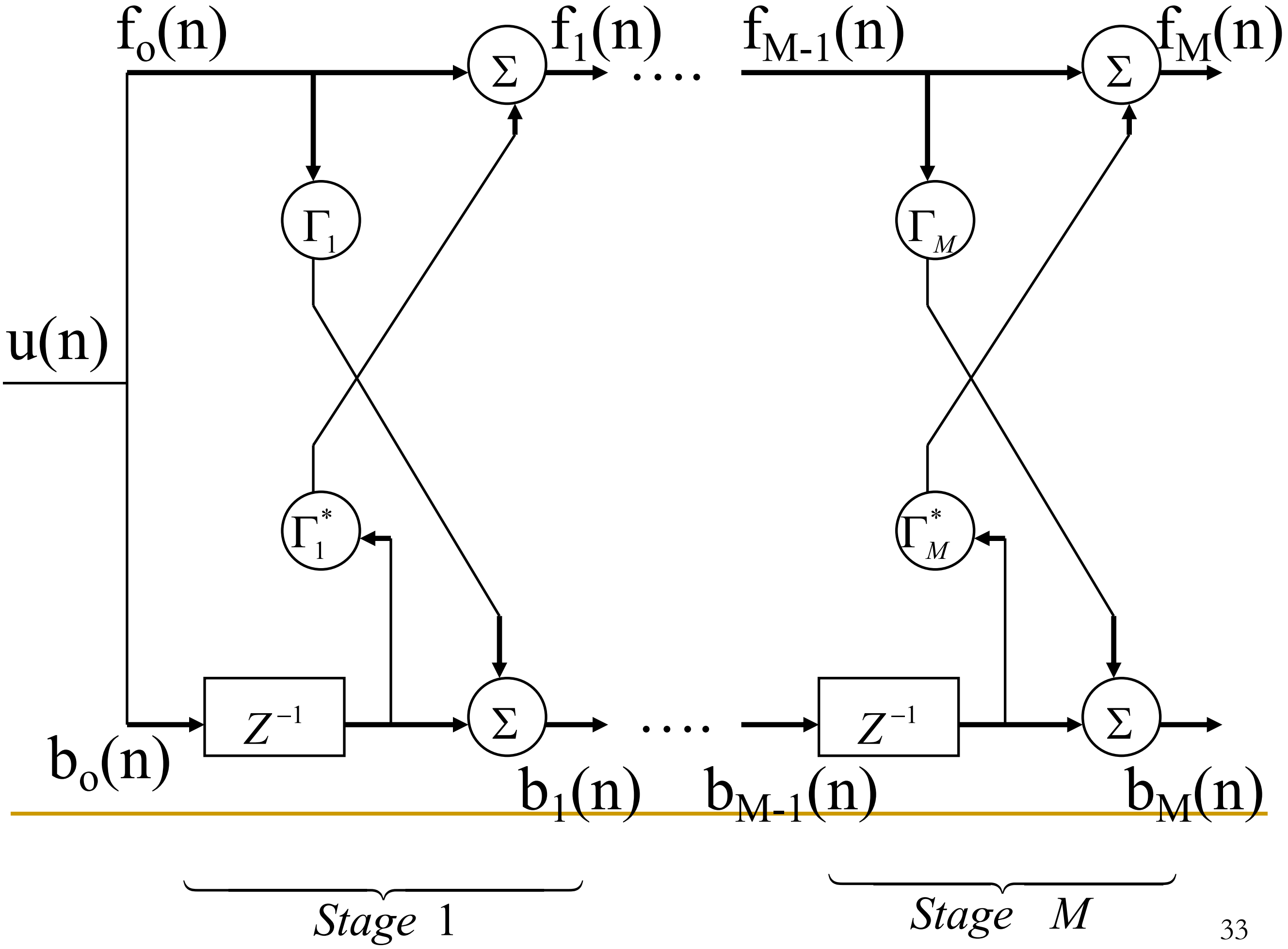
In Matrix Form :

$$\begin{bmatrix} f_M(n) \\ b_M(n) \end{bmatrix} = \begin{bmatrix} 1 & \Gamma_M^* \\ \Gamma_M & 1 \end{bmatrix} \begin{bmatrix} f_{M-1}(n) \\ b_{M-1}(n-1) \end{bmatrix} \; m = 1,2,\ldots .M$$

These equations when represented by a signal flow graph have lattice filter structure .

Mth stage


$f_{M-1}(n)$
$f_M(n)$
Σ
Γ_M
Γ_M^*
$b_{M-1}(n)$
z^{-1}
Σ
$b_M(n)$


$f_o(n)$
Σ
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....
$f_{M-1}(n)$
Σ
$f_M(n)$
Γ_1
Γ_M
$u(n)$
Γ_1^*
Γ_M^*
z^{-1}
Σ
....
z^{-1}
Σ
$b_o(n)$
$b_1(n)$
$b_{M-1}(n)$
$b_M(n)$
$Stage\ 1$
$Stage\ M$

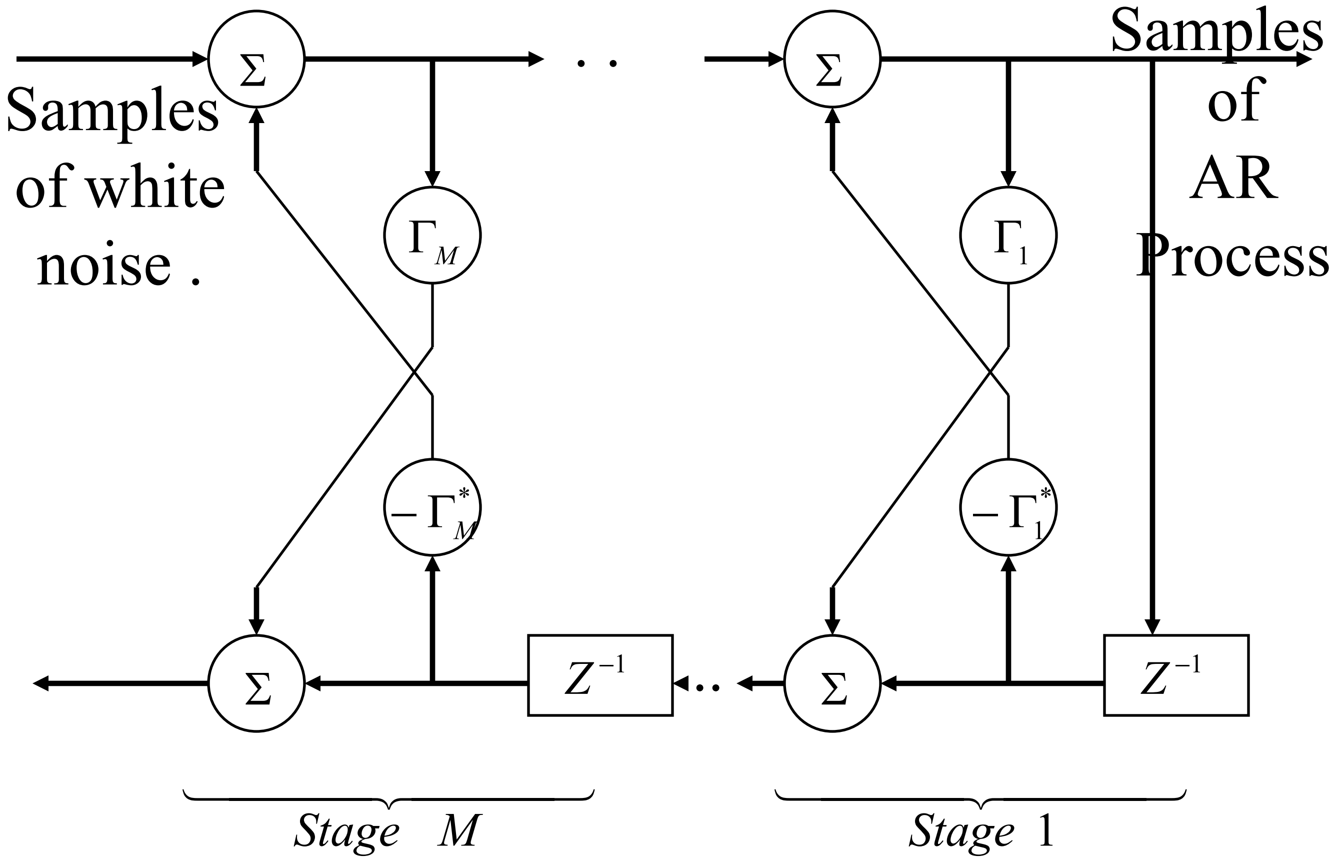
Notes on Lattice Filters for Linear Prediction :

1. Efficient Structure to generate forward and backward prediction errors .
2. Knowledge of reflection coefficients $\Gamma_1, \Gamma_2, ... \Gamma_M$ is all that is necessary once we initiate the filter $f_o(n) = b_0(n) = u(n)$.
3. Modular in structure. We can add required stages as they are needed .
4. Suitable for VLSI technology .
5. Note : $\Gamma_M = -\dfrac{E[b_{M-1}(n-1) f^*_{M-1}(n)]}{E[\left|f_{M-1}(n)\right|^2]}$

$$\Gamma_M = -\frac{1}{P_{M-1}} \sum_{k=0}^{M-1} a_{M-1,K} r(k-m)$$

- Lattice Inverse Filters

Lattice Filters are also a mechanism to represent (AR) process u(n) by way of reflection coefficients, $\{\Gamma_M\}$. We may generate samples of AR process by in-putting samples of white noise v(n) to the inverse lattice structure :



Samples of white noise .
Samples of AR Process
Σ
Γ_M
$-\Gamma_M^*$
Z^{-1}
Γ_1
$-\Gamma_1^*$
Stage M
Stage 1