# Finite Impulse Response (FIR)

# Digital Filters (II)

# <u>Ideal Impulse Response Design Examples</u>

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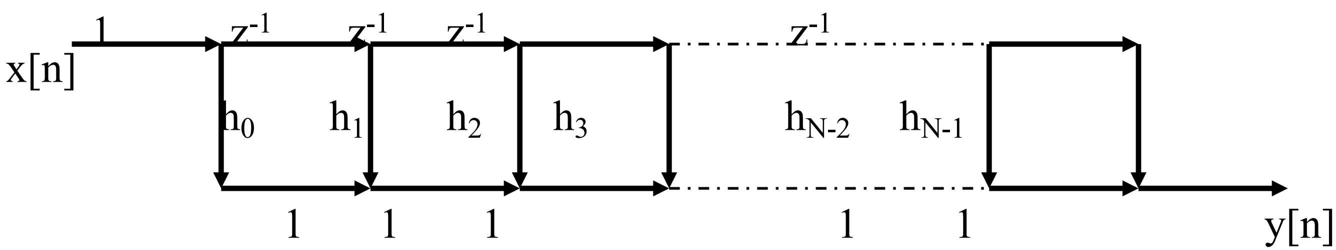
- FIR Filter Design Problem

Given H(z) or $H(e^{jw})$, find filter coefficients $\{b_0, b_1, b_2, \ldots.. b_{N-1}\}$ which are equal to $\{h_0, h_1, h_2, \ldots.h_{N-1}\}$ in the case of FIR filters.




Consider a general (infinite impulse response) definition:

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

From complex variable theory, the inverse transform is:

$$h[n] = \frac{1}{2\pi j} \oint_C H(z) z^{n-1} dz$$

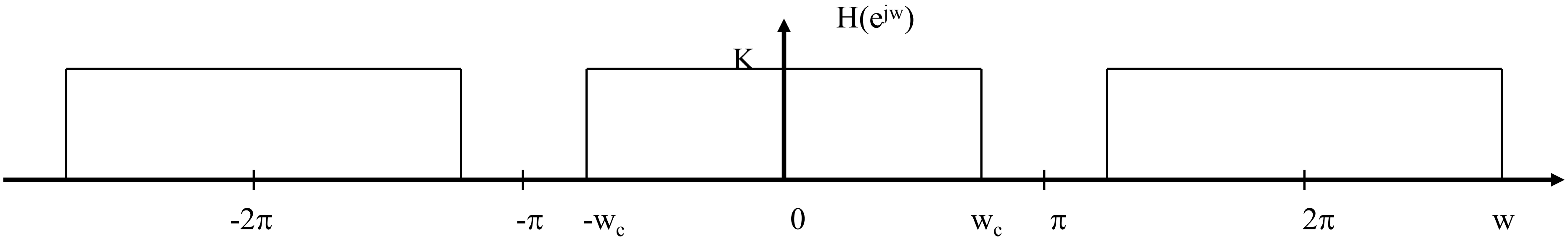
Where C is a counterclockwise closed contour in the region of convergence of H(z) and encircling the origin of the z-plane

- Evaluating H(z) on the unit circle ( $z = e^{jw}$ ) :

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n] e^{-jnw}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw}) e^{jnw} dw$$ where $dz = je^{jw}\, dw$

- Design of an ideal low pass FIR digital filter




Find ideal low pass impulse response {h[n]}

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{jw})e^{jnw}dw$$

$$= \frac{1}{2\pi}\int_{-w_c}^{w_c} Ke^{jnw}dw$$

Hence

$$h_{LP}[n] = \frac{K}{n\pi}\sin(nw_c)$$

n = 0, ±1, ±2, …. ±∞

Let K = 1, $w_c = \pi/4$, n = 0, ±1, …, ±10

The impulse response coefficients are

| | | | |
|---|---|---|---|
| n = 0, | h[n] = 0.25 | n = ±4, | h[n] = 0 |
| = ±1, | = 0.225 | = ±5, | = -0.043 |
| = ±2, | = 0.159 | = ±6, | = -0.053 |
| = ±3, | = 0.075 | = ±7, | = -0.032 |

| | |
|---|---|
| n = ±8, | h[n] = 0 |
| = ±9, | = 0.025 |
| = ±10, | = 0.032 |

Non Causal
FIR Impulse
Response

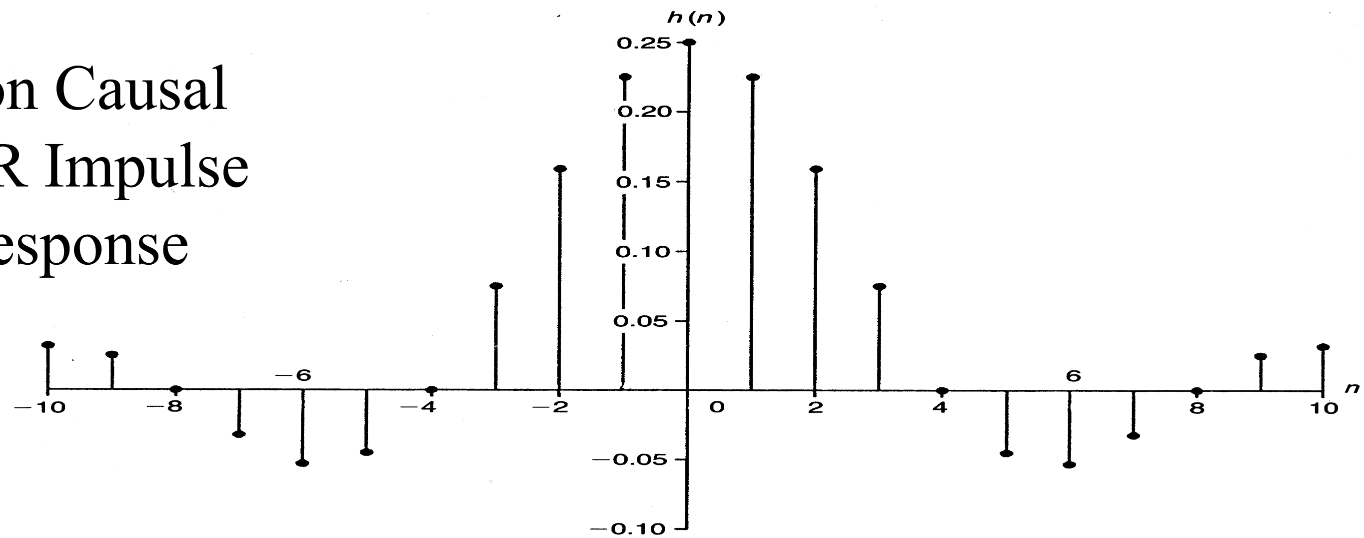
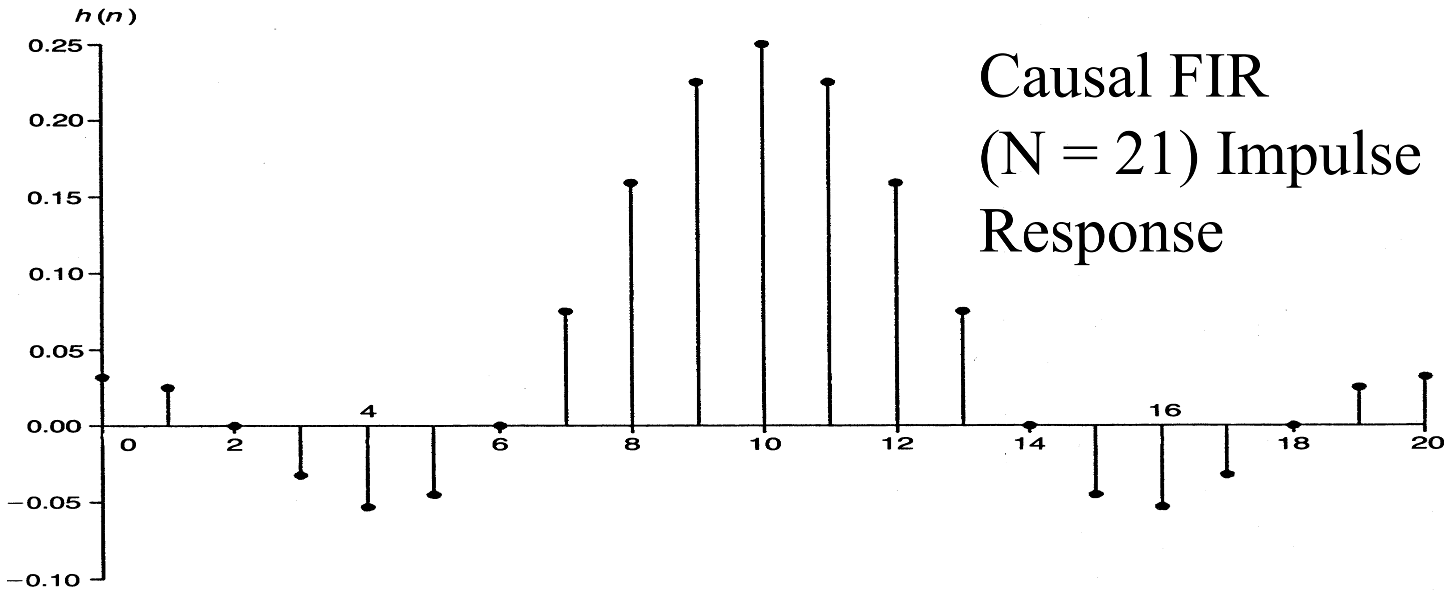



**FIGURE 9.6(a)** Coefficients for noncausal, nonrecursive filter

We can make it causal if we shift $h_{LP}[n]$ by 10 units to the right:

$$h_{LP}[n] = \frac{K}{(n-10)\pi}\sin((n-10)w_c) \qquad n = 0, 1, 2, \ldots.\ 20$$




Causal FIR (N = 21) Impulse Response

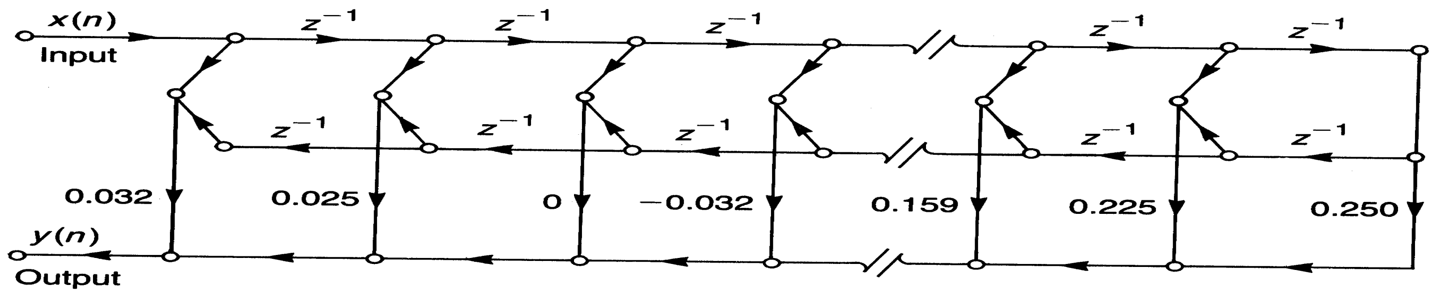
Notice the symmetry: h[n] = h[N-1-n] which satisfies the linear phase condition.

- Filter Transfer Function

$$H_{LP}(e^{jw}) = \sum_{n=0}^{20} h_{LP}[n]e^{-jnw} \qquad \text{since } z = e^{jw}$$

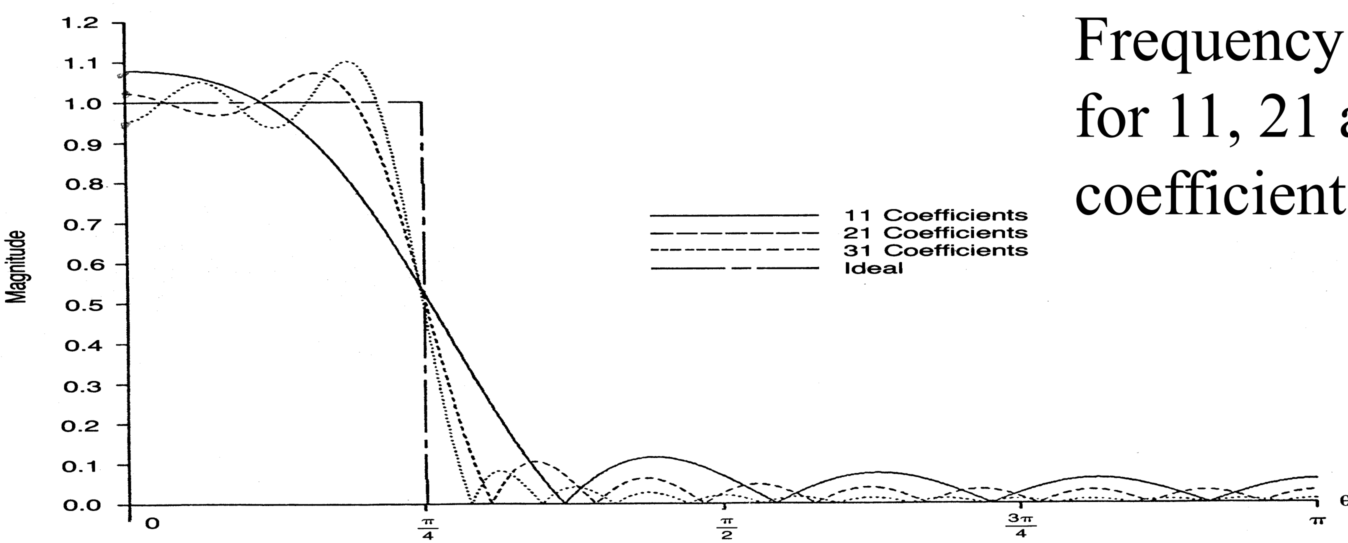
Implies

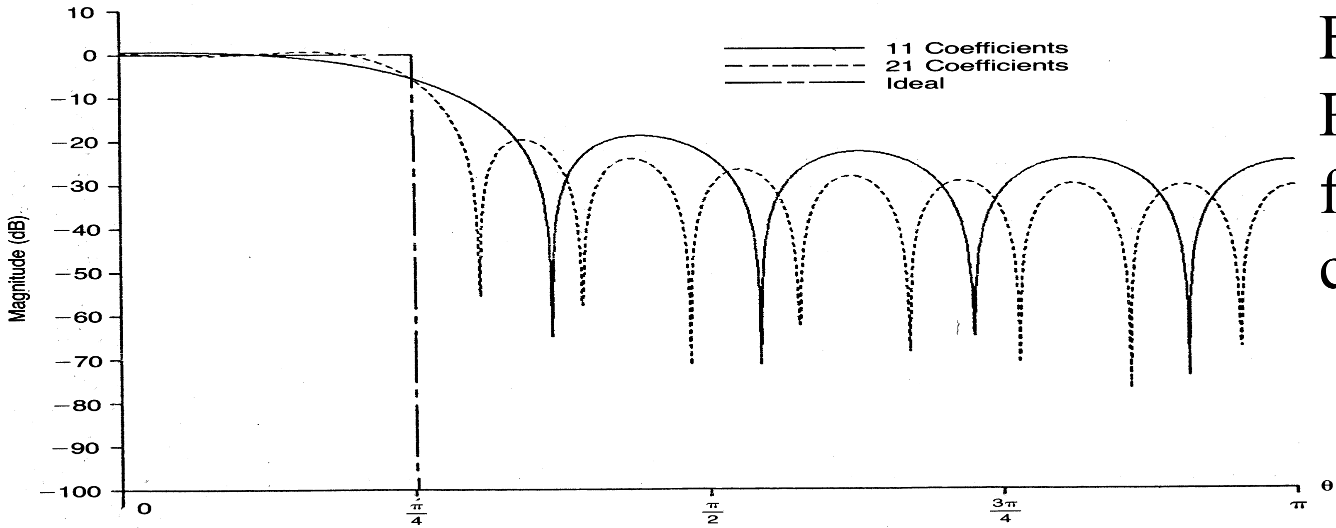
$$H_{LP}(z) = \sum_{n=0}^{20} h_{LP}[n] z^{-n}$$



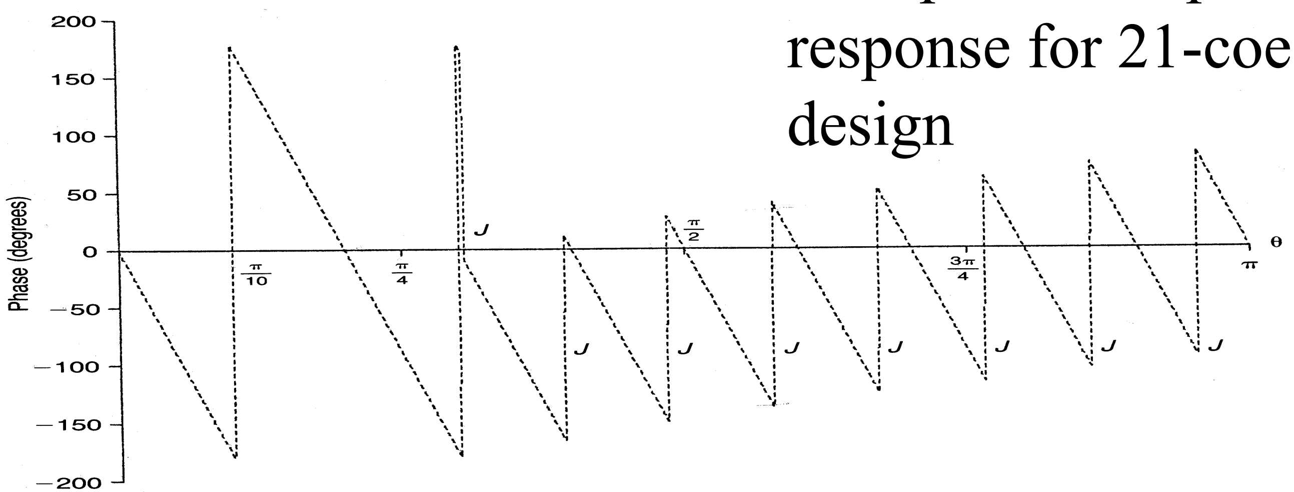

Filter Implementation for 21 coefficients

Magnitude of filter Frequency Response for 11, 21 and 31 coefficient lengths






Low Pass filter Frequency Response in dB for 11 and 21 coefficients




Low pass filter phase response for 21-coefficient design

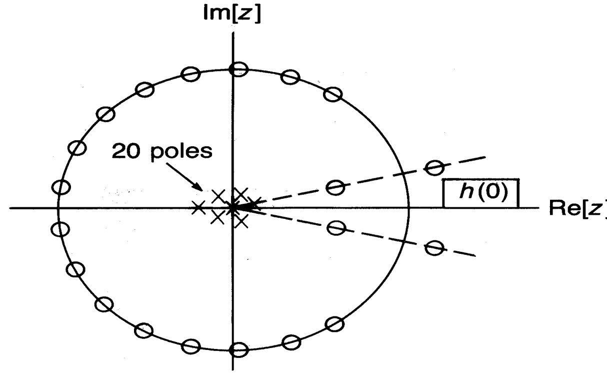



**FIGURE 9.8(d)** Pole-zero plot for $H(z)$

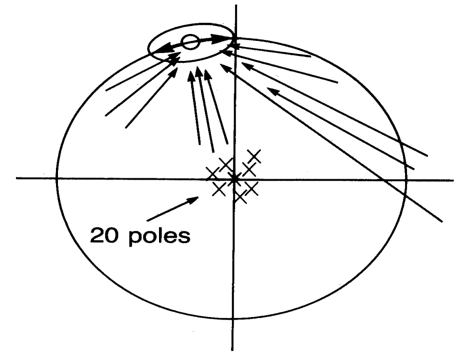
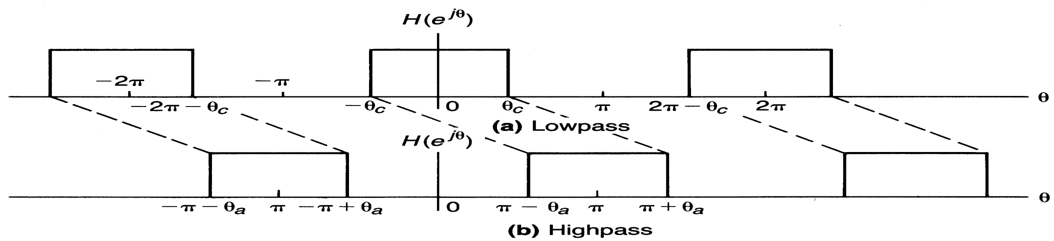



**FIGURE 9.8(e)** Vectors for frequency response near $\theta = 1.813$

**FIGURE 9.9** Magnitude characteristics of ideal low-pass and highpass filters




The above figure implies

$$H_{HP}(e^{jw}) = H_{LP}(e^{j(w-\pi)})$$

$$h_{HP}[n] = h_{LP}[n](-1)^n$$

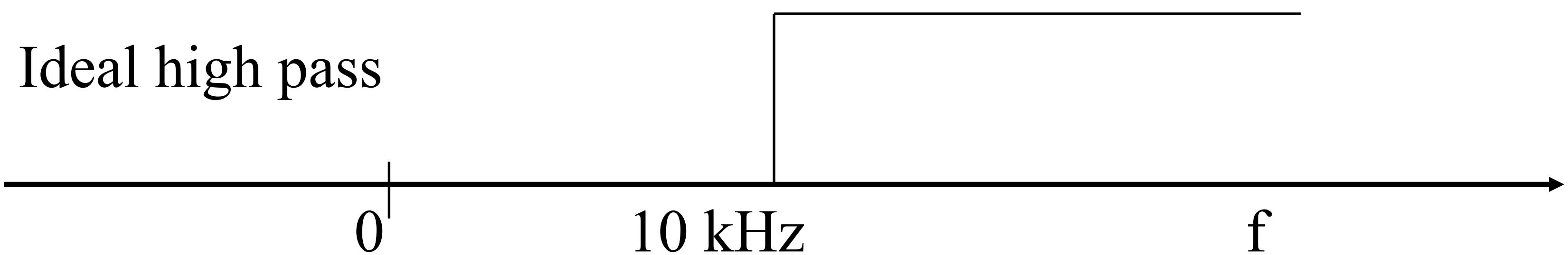
Now:

$$H_{HP}(e^{jw}) = \sum_{n=-\infty}^{\infty} h_{LP}[n] e^{-jn(w-\pi)} \qquad = \sum_{n=-\infty}^{\infty} h_{LP}[n] e^{-jnw} (-1)^n$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} h_{LP}[n](-1)^n}_{h_{HP}[n]} e^{-jnw}$$

Example: Analog high pass filter specification

Ideal high pass




Given: $f_s$ = 50kHz, it follows: $w = \Omega T$

$= 2\pi f/f_s = 0.4\pi$ rad

Digital specification: $0.4\pi = \pi - w_a$

$\therefore$ $w_c$ (LPF) = $\pi$ - $w_a$ = $\pi$ - 0.4$\pi$ = 0.6$\pi$

First design ideal LPF with $w_c$= 0.6$\pi$

Answer :

$$h_{LP}[n] = \frac{1}{n\pi}\sin(0.6\pi n)$$

n = 0, ±1, ±2, …. ±∞

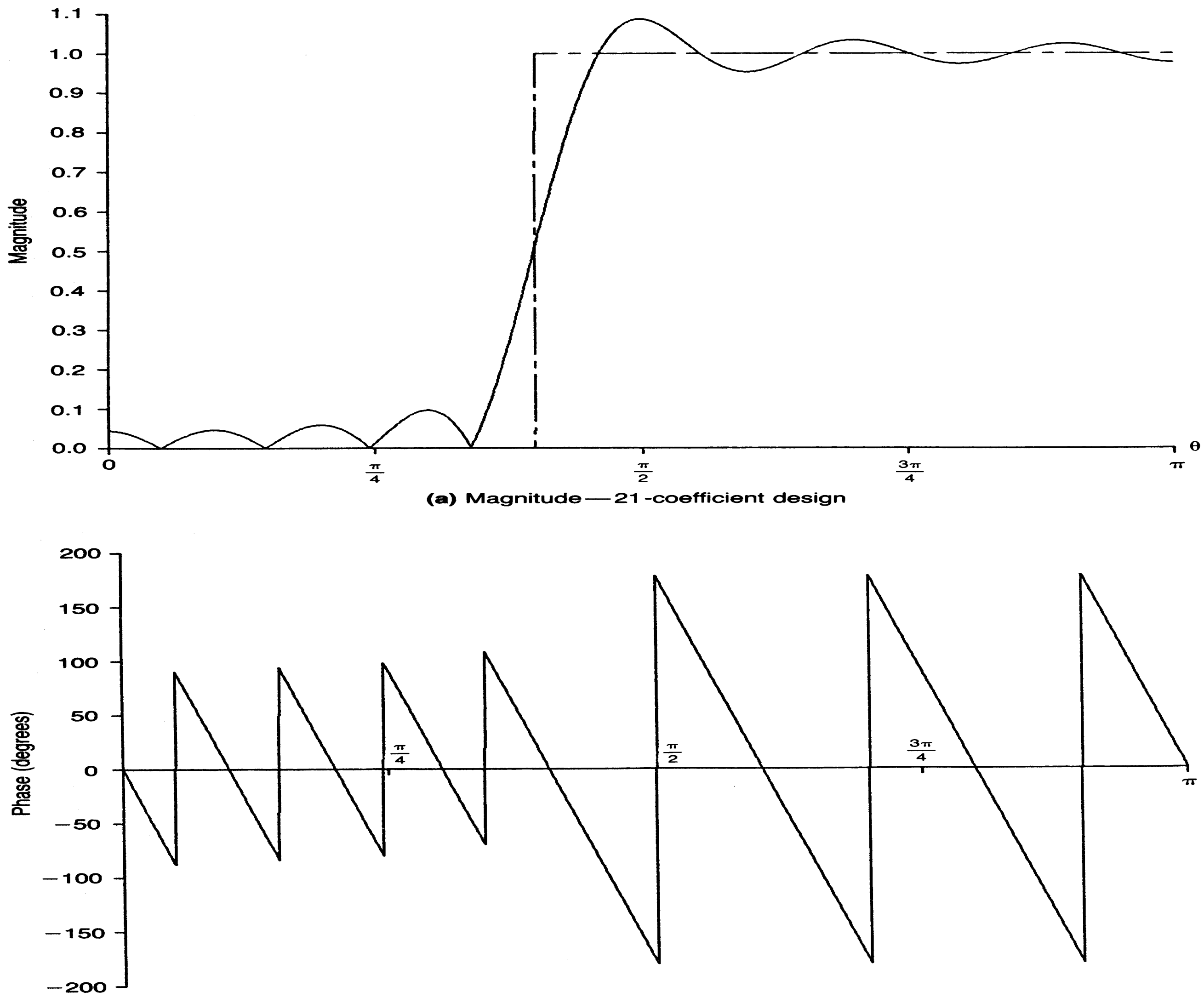
Follows now:

$$h_{HP}[n] = h_{LP}[n](-1)^n$$

$$h_{HP}[n] = \frac{(-1)^{n-I}}{\pi(n-I)}\sin(0.6\pi(n-I))$$

The I indicates the length of the filter

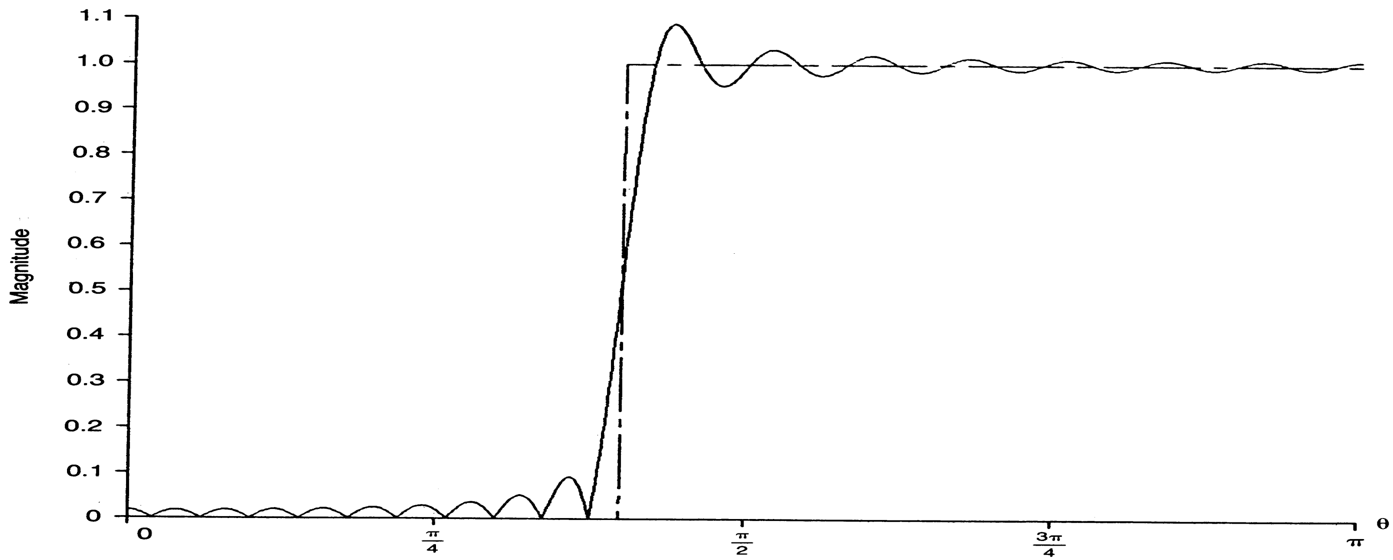
Hence the above filter is a causal FIR high pass digital filter



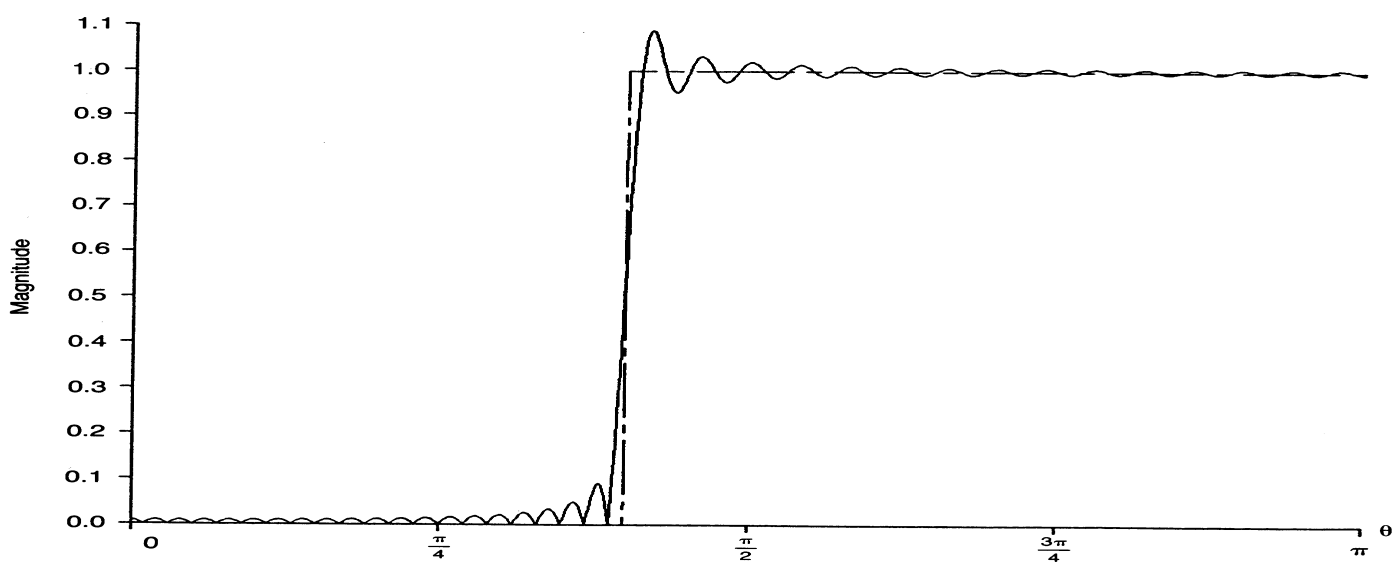

(a) Magnitude—21-coefficient design

(b) Phase—21-coefficient design

**FIGURE 9.10** Highpass filter frequency response

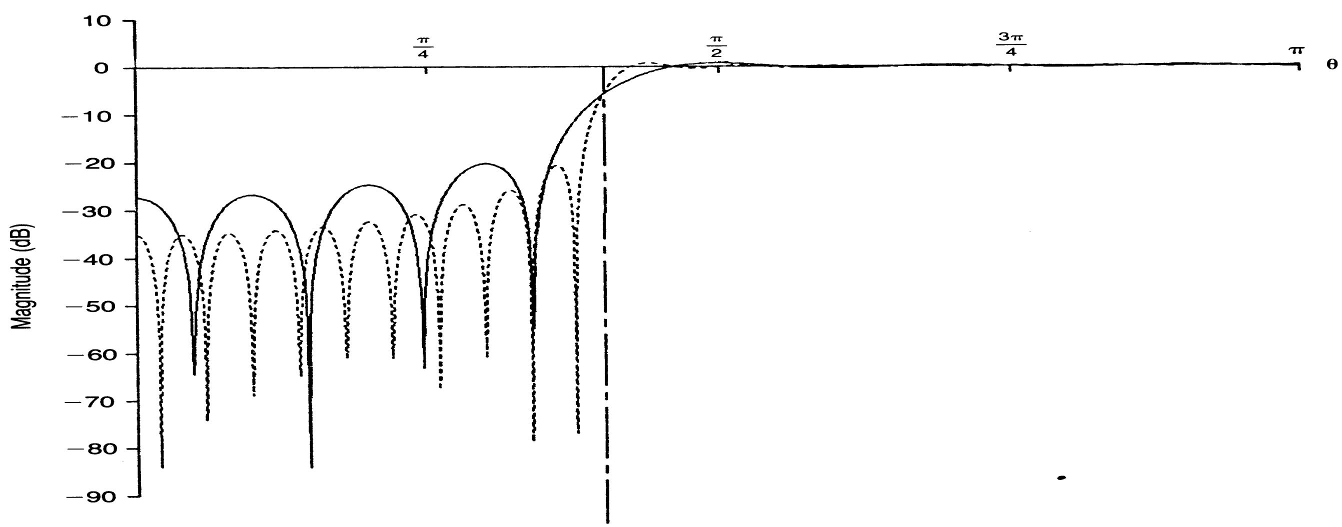



(c) Magnitude—51-coefficient design




(d) Magnitude—101-coefficient design

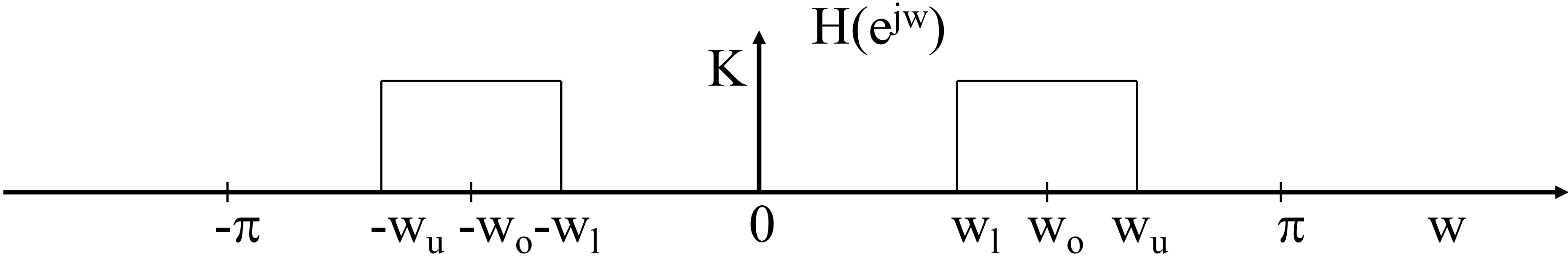
**FIGURE 9.10** Highpass filter frequency response




**(e) Magnitude(dB)—21 and 51-coefficient designs**

**FIGURE 9.10** Highpass filter frequency response

# Bandpass FIR filter design:




Shift to right: $H(e^{jw})\, e^{jwo}$

Shift to left: $H(e^{jw})\, e^{-jwo}$

Follows:

$$h_{BP}[n] = [2\cos n w_o] h_{LP}[n]$$

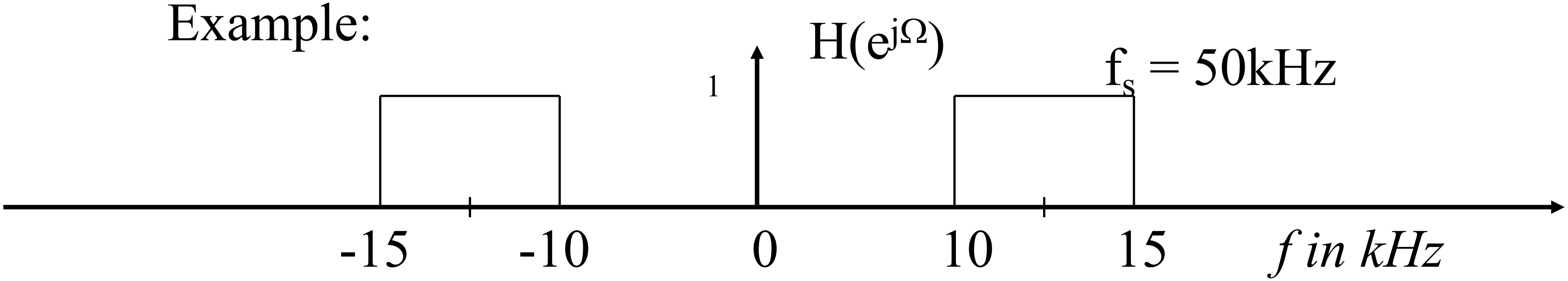
$n = 0, \pm 1, \pm 2, \ldots \pm I$

where

$$w_u - w_l = 2 w_c (lowpass)$$

$$w_o = \frac{w_u + w_l}{2}$$

Example:




Digital specifications:

$$w_l = \frac{2\pi(10^4)}{(5)(10^4)} = 0.4\pi$$

$$w_u = \frac{2\pi(10^4)(1.5)}{(5)(10^4)} = 0.6\pi$$

$$w_o = 0.5\pi$$

$$w_c = \frac{w_u - w_l}{2} = 0.1\pi$$

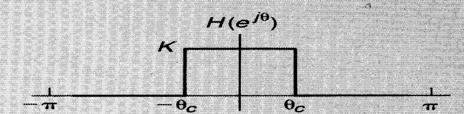
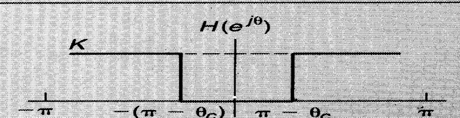
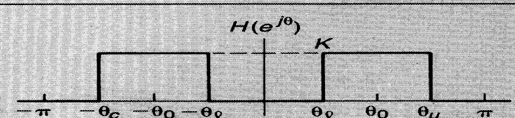
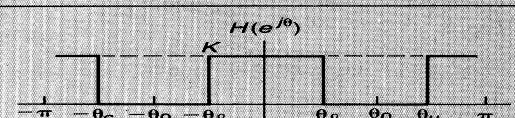
# First design LPF:

$$h_{LP}[n] = \frac{1}{n\pi}\sin(0.1\pi n) \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Follows:

$$h_{BP}[n] = [2\cos(0.5\pi n) h_{LP}[n] \qquad n = 0, \pm 1, \pm 2, \ldots.$$

**TABLE 9.1 FREQUENCY TRANSFORMATIONS**

| | | |
|---|---|---|
| **Lowpass** | | |
| $h_{LP}(n) = \frac{K}{\pi n}\sin(n\theta_c)$, $n = 0, \pm 1, \pm 2, \ldots, \pm I$ |   | (9.60) |
| **Highpass** | | |
| $h_{HP}(n) = (-1)^n h_{LP}(n)$, $n = 0, \pm 1, \pm 2, \ldots, \pm I$ |   | (9.61) |
| **Bandpass** | | |
| $h_{BP}(n) = [2\cos(n\theta_0)]h_{LP}(n)$, $n = 0, \pm 1, \pm 2, \ldots, \pm I$ <br> $\theta_u - \theta_\ell = 2\theta_c$, $\theta_0 = \frac{\theta_u + \theta_\ell}{2}$ |   | (9.62) |
| **Bandstop** | | |
| $h_{BS}(0) = K - h_{BP}(0)$ <br> $h_{BS}(n) = -h_{BP}(n)$, $n = \pm 1, \pm 2, \ldots, \pm I$ <br> $\theta_u - \theta_\ell = 2\theta_c$ <br> $\theta_0 = \frac{\theta_u + \theta_\ell}{2}$ |   | (9.63) |

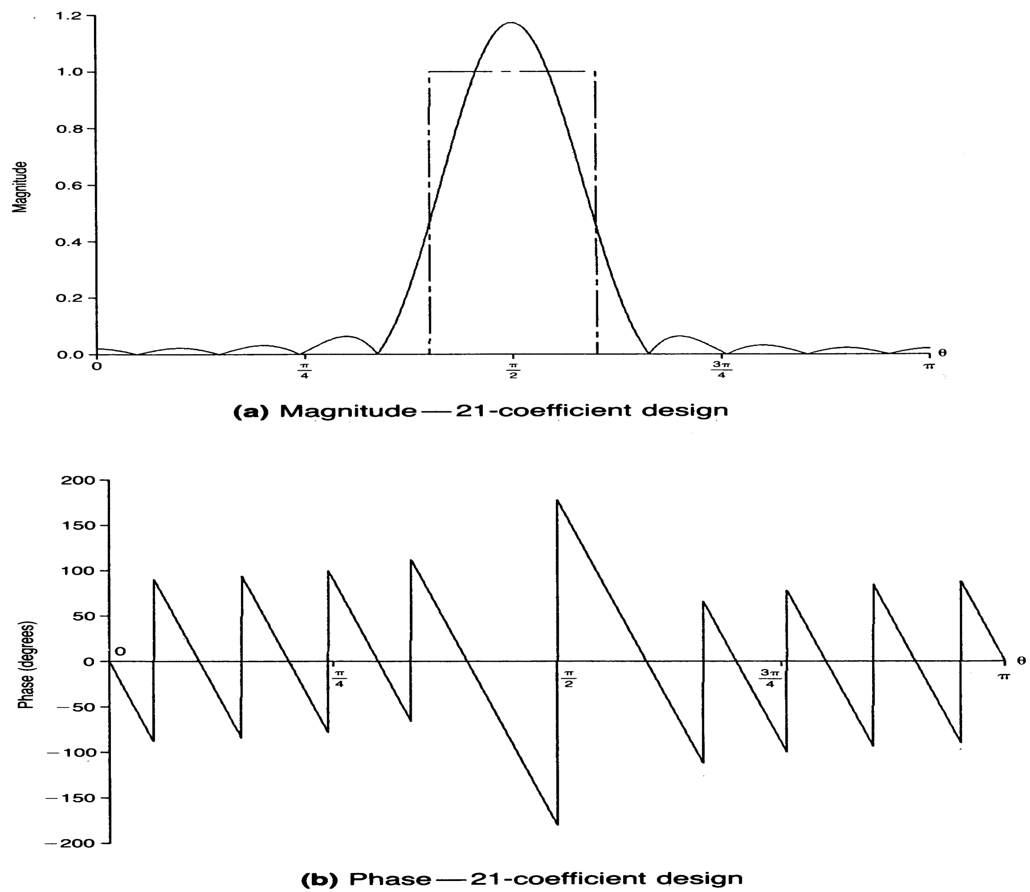
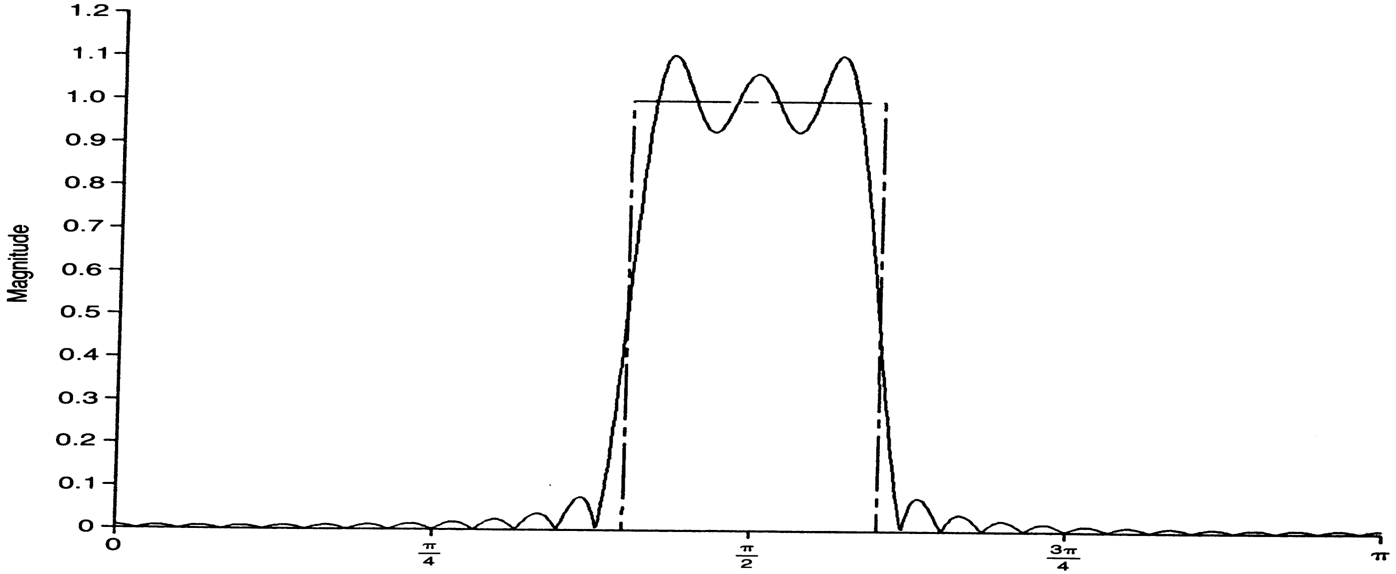
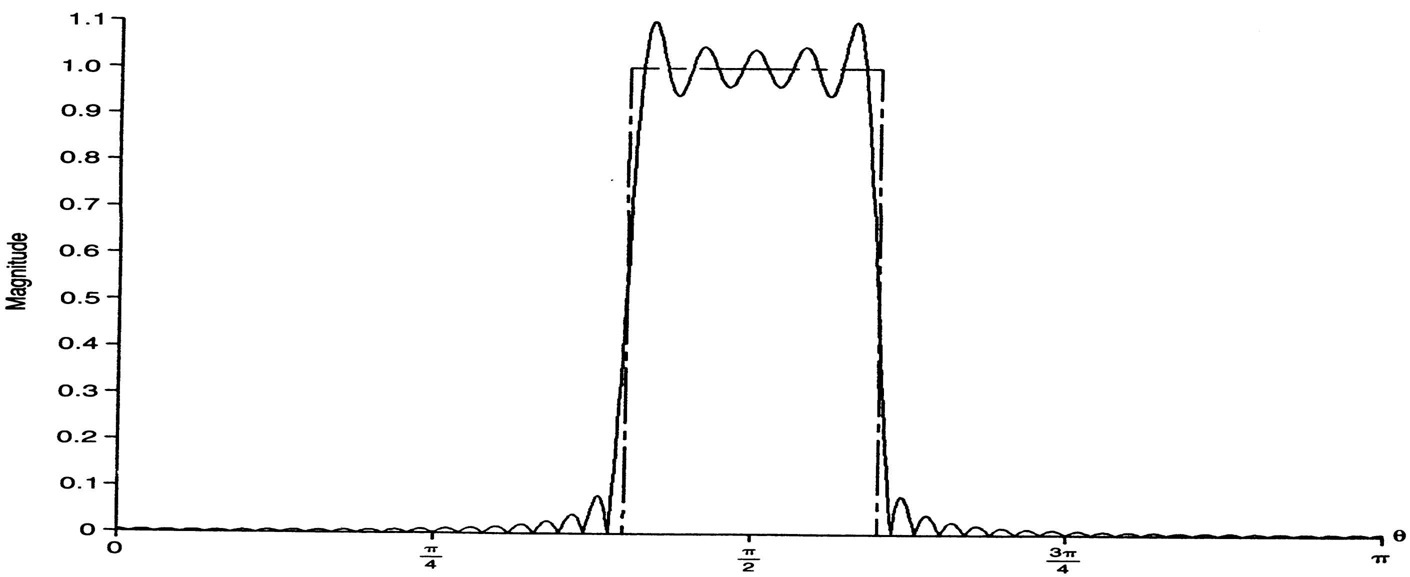



**FIGURE 9.12** Bandpass filter frequency response

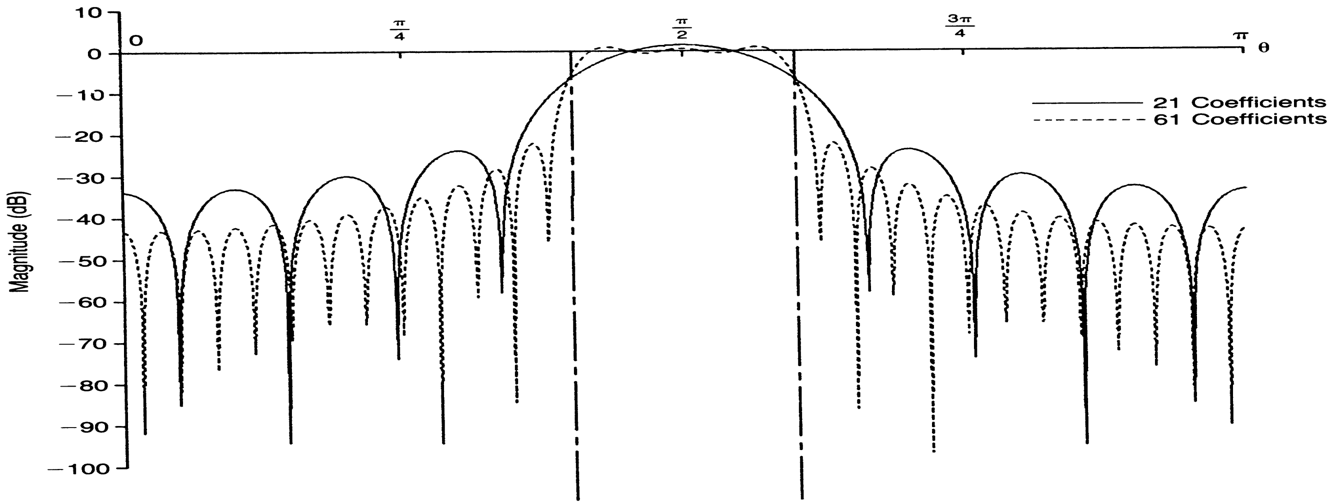



(c) Magnitude — 61-coefficient design




(d) Magnitude — 101-coefficient design

**FIGURE 9.12** Bandpass filter frequency response




(e) Magnitude (dB)—21- and 61-coefficient designs

Bandpass filter frequency response

# Homework # 6

1. Design a length-5 FIR bandpass filter with an antisymmetric impulse response h[n], i.e. h[n]= -h[4-n], $0 \le n \le 4$, satisfying the following magnitude response values : $|H(e^{\frac{j\pi}{4}})| = 0.5$ and $|H(e^{\frac{j\pi}{2}})| = 1.$ Determine the exact expression for the frequency response of the filter designed.

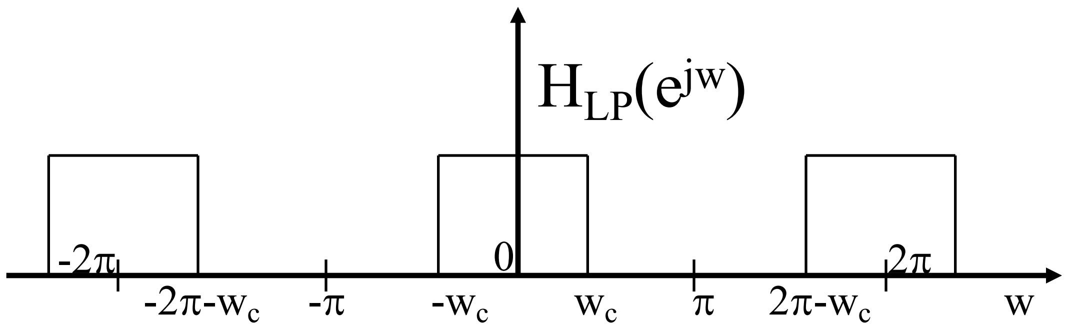
2. An FIR filter of length 5 is defined by a symmetric impulse response i.e. h[n]= h[4-n], $0 \le n \le 4$,. Let the input to this filter be a sum of 3 cosine sequences of angular frequencies: 0.2 rad/samples, 0.5 rad/samples, and 0.8 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the midfrequency component of the input.

3. The frequency response $H(e^{j\omega})$ of a length-4 FIR filter with a real and antisymmetric impulse response has the following specific values: $H(e^{j\pi})=8$, and $H(e^{j\pi/2}) = -2+j2$. Determine H(z).

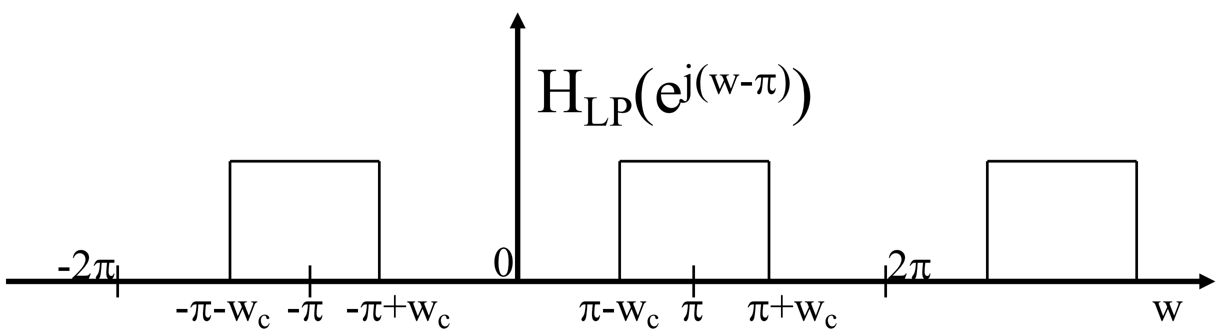
# Frequency transformations of ideal filters.

## High pass Filter

The ideal low filter frequency response is shown below. $h_{LP}[n]$ represents the non-causal impulse response of $H_{LP}(e^{jw})$




Shifting the low pass filter frequency by π, we get



The above figure is the frequency response of an ideal high pass filter with a cutoff frequency of $\pi$-$w_c$ ($H_{HP}(e^{jw})$ ).Hence,

$$H_{HP}(e^{jw}) = H_{LP}(e^{j(w-\pi)})$$

From the shifting property of the Fourier transform, we get

$$h_{HP}[n] = h_{LP}[n]e^{j\pi n}$$

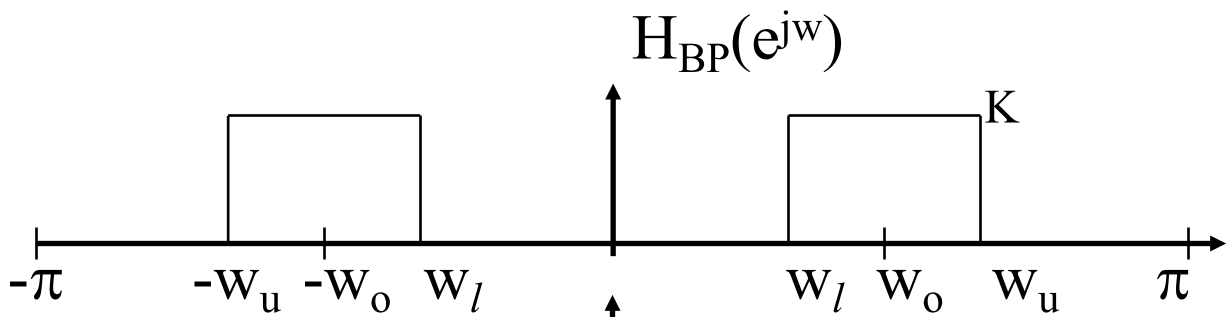
$$h_{HP}[n] = (-1)^n h_{LP}[n]$$

where $$h_{LP}[n] = \frac{1}{\pi n}\sin(nw_c)$$

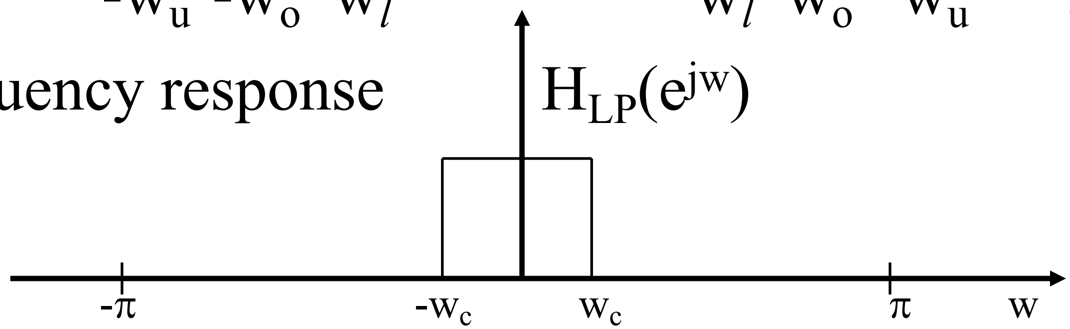
Hence, to design an ideal high pass filter with cutoff frequency of $w_a$, first design a ideal low pass filter with a cutoff frequency of ($\pi$-$w_a$). Then use the above transformation to obtain the impulse response of the ideal high pass filter.

## Band pass Filter

The frequency response for an ideal band pass filter is shown below. The center frequency is $w_o$.

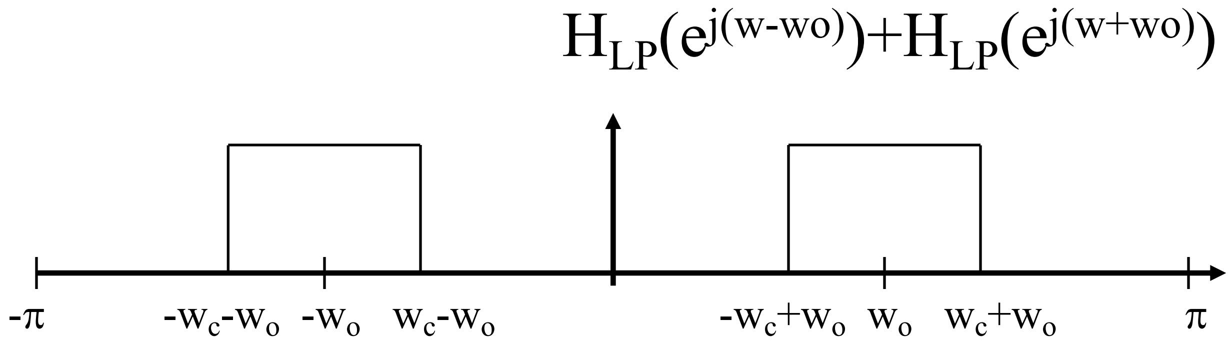



Low pass frequency response




Looking at the frequency responses of a band pass filter and a low pass filter, we can observe that a band pass filter is obtained by shifting the low pass filter to the left and to the right by $w_o$ and adding the two shifted responses.

Shifting the low pass filter by wo to the left and to the right and adding , we get




Comparing the ideal band pass filter and the above figure, we can observe that

$$w_l = -w_c + w_o$$

$$w_u = w_c + w_o$$

$$w_c = \frac{w_u - w_l}{2}$$

$$w_o = \frac{w_u + w_l}{2}$$

The frequency response of the band pass filter $H_{BP}(e^{jw})$ is

$$H_{BP}(e^{jw}) = H_{LP}(e^{j(w-w_o)}) + H_{LP}(e^{j(w+w_o)})$$

From the shifting property of the Fourier transform , we have

$$h_{BP}[n] = h_{LP}[n]e^{jw_o n} + h_{LP}[n]e^{-jw_o n}$$

$$h_{BP}[n] = h_{LP}[n]\{e^{jw_o n} + e^{-jw_o n}\}$$

$$h_{BP}[n] = h_{LP}[n]\{2\cos(nw_o)\}$$

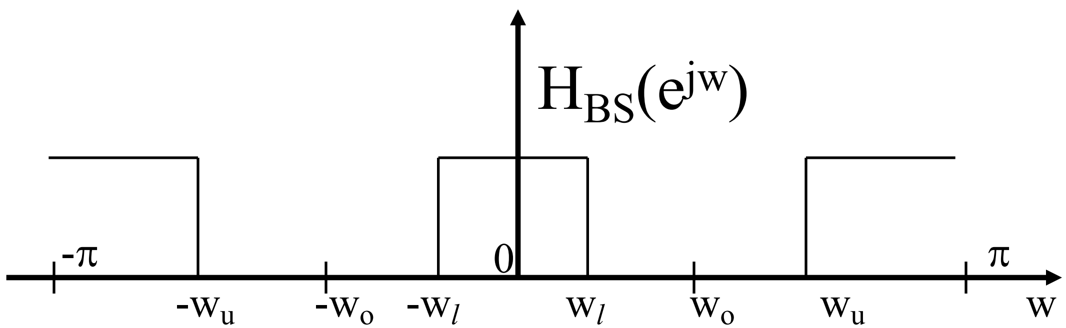
To design a band pass filter, first design a low pass filter with cutoff frequency $w_c$ given by

$$w_c = \frac{w_u - w_l}{2}$$

And use the above transformation to get the impulse response of the band pass filter .

## Band stop Filter

The frequency response of an ideal band stop filter ($H_{BS}(e^{jw})$) is




Looking at the frequency responses of band stop and band pass filters, we observe that the band stop filter is obtained by subtracting the band pass from 1.

$$H_{BS}(e^{jw}) = 1 - H_{BP}(e^{jw})$$

$$\sum_{n=-\infty}^{\infty} h_{BS}[n]e^{-jnw} = 1 - \sum_{n=-\infty}^{\infty} h_{BP}[n]e^{-jnw}$$

If we work out the first few terms, we get

$$\begin{aligned} &\ldots\ldots + h_{BS}[-2]e^{j2w} + h_{BS}[-1]e^{jw} + h_{BS}[0] + h_{BS}[1]e^{-jw} + h_{BS}[2]e^{-j2w} \\ &= \ldots\ldots + h_{BP}[-2]e^{j2w} + h_{BP}[-1]e^{jw} + 1 - h_{BP}[0] + h_{BP}[1]e^{-jw} + h_{BP}[2]e^{-j2w} \end{aligned}$$

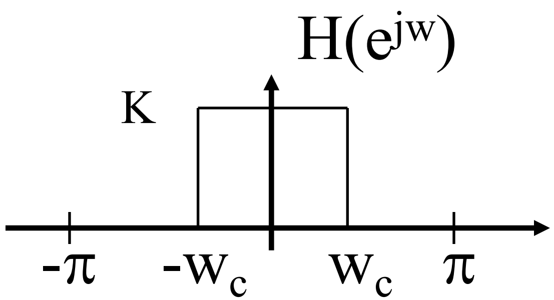
Hence, the impulse response coefficients for a band stop filter are obtained as

$$h_{BS}[0] = 1 - h_{BP}[0] = 1 - 2h_{LP}[0]$$

$$h_{BS}[n] = -h_{BP}[n] = -h_{LP}[n]\{2\cos(nw_o\} \qquad n \neq 0$$

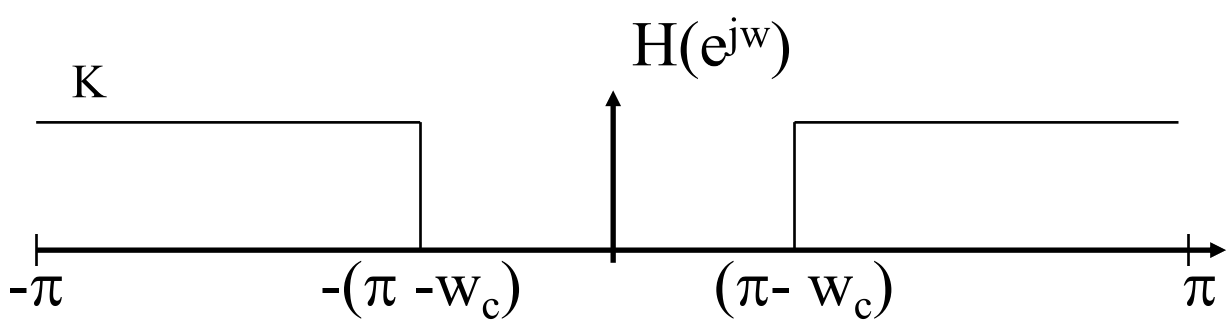
## Low pass

$$h_{LP}[n] = \frac{K}{n\pi}\sin(nw_c) \qquad n = 0, \pm 1, \pm 2, \ldots, \pm I$$




## High pass

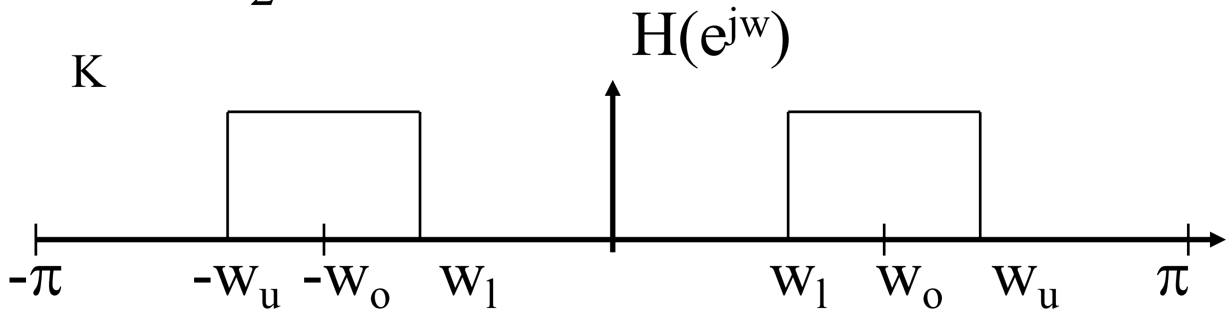
$$h_{HP}[n] = (-1)^n h_{LP}[n], \qquad n = 0, \pm 1, \pm 2, \ldots, \pm I$$



# Band pass

$$h_{BP}[n] = [2\cos(nw_o)]h_{LP}[n], \qquad n = 0, \pm1, \pm2, \ldots, \pm I$$

$$w_u - w_l = 2w_c, w_0 = \frac{w_u + w_l}{2}$$

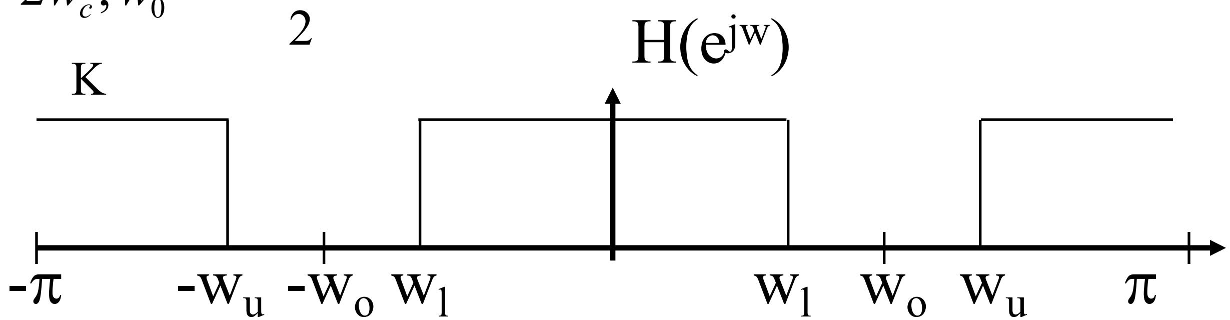



# Band stop

$$h_{BS}[0] = K - h_{BP}[0],$$

$$h_{BS}[n] = -h_{BP}[n], \qquad n = \pm1, \pm2, \ldots, \pm I$$

$$w_u - w_l = 2w_c, w_0 = \frac{w_u + w_l}{2}$$



# Examples

**1.** An FIR filter is defined by a symmetric impulse response, i.e. h[0] = h[2]. Input to the filter is a sum of two cosine sequences of angular frequencies 0.2 rad/s and 0.5 rad/s

Determine the impulse response coefficients so that it passes only the high frequency component of the input

**Solution:**

**Since h[0] = h[2]** ⇨

$$H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega}$$

$$= h[0]\,e^{-j\omega}(e^{j\omega} + e^{-j\omega}) + h[1]\,e^{-j\omega}$$

$$= e^{-j\omega}(\,2h[0]\cos(\omega_0) + h[1]\,)$$

Since only high frequency component can pass through the filter,

*when*,

$$\omega_0 = 0.2 \quad \Rightarrow \quad H(e^{j\omega_0}) = 0 \quad \text{and}$$

$$\omega_0 = 0.5 \quad \Rightarrow \quad H(e^{j\omega_0}) = 1$$

i.e.

$$H(e^{j0.2}) = 2h[0]\cos(0.2) + h[1] = 0$$

$$H(e^{j0.5}) = 2h[0]\cos(0.5) + h[1] = 1$$

Solving these two equations,

$$h[0] = h[2] = -4.8788 \quad \text{and} \quad h[1] = 9.5631$$

**2.** Design a length – 4 FIR bandpass filter with an anti symmetric impulse response i.e. h[n] = h[-n-4], for 0�n�4 satisfying the following magnitude response Values: $|H(e^{j\pi/4})| = 1$ and $|H(e^{j\pi/2})| = 0.5$

$$
\begin{aligned}
H(e^{j\omega}) &= h_o + h_1 e^{-jw} + h_2 e^{-j2w} + h_3 e^{-j3w} \\
&= h_o + h_1 e^{-jw} - h_1 e^{-j2w} - h_0 e^{-j3w} \\
&= h_0(1 - e^{-j3w}) + h_1 e^{-jw}(1 - e^{-jw}) \\
&= h_0 e^{-j3w/2}(e^{j3w/2} - e^{-j3w/2}) + h_1 e^{-jw}(e^{-jw/2}(e^{jw/2} - e^{-jw/2})) \\
&= je^{-j3w/2}(2h_0 \sin(3w/2) + 2h_1 \sin(w/2))
\end{aligned}
$$

Since we have $|H(e^{j\pi/4})| = 1$ and $|H(e^{j\pi/2})| = 0.5$

$$H(e^{j\pi/4}) = 2h_0 \sin(3\pi/8) + 2h_1 \sin(\pi/8) = 1$$

*and*

$$H(e^{j\pi/2}) = 2h_0 \sin(3\pi/4) + 2h_1 \sin(\pi/4) = 0.5$$

Solving equations, we get

$h_0 = -0.0381$ and $h_1 = 0.7452$

**3.** The frequency response of a length-4 FIR filter has values:

$$|H(e^{j0})| = 2 \quad |H(e^{j\pi/2})| = 7 - j3 \quad \text{and} \, |H(e^{j\pi})| = 0$$

Determine H(z)

**Solution**

Using the symmetry property of DTFT of a real sequence, we observe that

$$|H(e^{j3\pi/2})| = |H^*(e^{j\pi/2})| = 7 + j3$$

Thus the 4 point DFT of the sequence is given by:

$$H(k) = [2, 7 - j3, 0, 7 + j3]$$

The inverse DFT of the H(k) gives h(n) as:

h(n) = (4 , 2 , -3 , -1) (using **ifft** in MATLAB)

Therefore H(z) is:

$$H(z) = 4 + 2z^{-1} - 3z^{-2} - z^{-3}$$