

# **Finite Impulse Response (FIR)**

## **Digital Filters (III)**

### **Impulse Response Coefficients calculation with the Window method**

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# Ideal Impulse Response Design

- Ideal filter design equations:

$$H_D(\omega) = \sum_{n=-\infty}^{\infty} h_D[n] e^{-jn\omega}$$

$$h_D[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_D(\omega) e^{jn\omega} d\omega$$

where  $H_D(\omega)$  and  $h_D(n)$  are the ideal frequency response and the ideal impulse response respectively.

- Exact approximation of the desired  $H_D(\omega)$  requires an *infinite number of coefficients* for  $h_D(n)$ , which are also known as the Fourier coefficients.

- Ideal Low Pass impulse response design

We have, 
$$h_D[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_D(\omega) e^{jn\omega} d\omega \quad (1)$$

For the ideal low pass frequency response in Fig 1(a), the ideal impulse response shown in Fig 1b, is obtained as follows:

$$\begin{aligned} h_D(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \times e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{2f_c \sin(n\omega_c)}{n\omega_c}, \quad n \neq 0, -\infty \leq n \leq \infty \\ &= 2f_c, \quad n = 0 \text{ (using L'Hopital's rule)} \end{aligned} \quad (2)$$

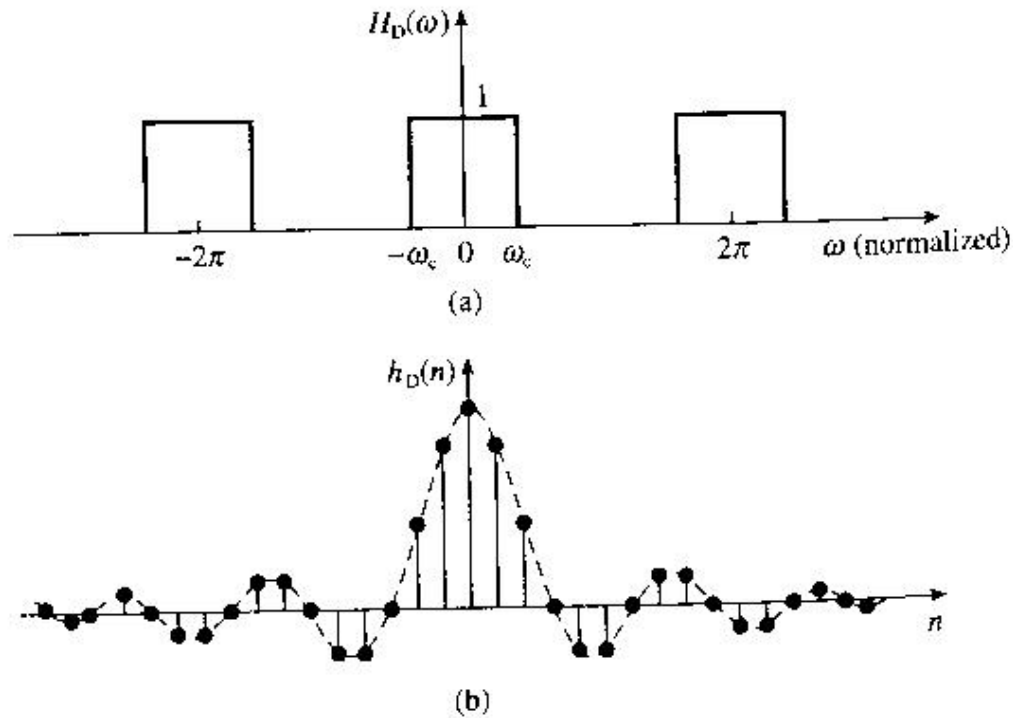


Fig 1<sup>1</sup>. (a) Ideal frequency response of a low pass filter (b) Impulse response of the ideal low pass filter

- The impulse responses of the ideal high pass, band pass and band stop filters are obtained from the low pass case of eqn (2) and are summarized in Table 1.

<i>Filter type</i>	<i>Ideal impulse response, <math>h_D(n)</math></i>	
	$h_D(n), n \neq 0$	$h_D(0)$
Lowpass	$2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$2f_c$
Highpass	$-2f_c \frac{\sin(n\omega_c)}{n\omega_c}$	$1 - 2f_c$
Bandpass	$2f_2 \frac{\sin(n\omega_2)}{n\omega_2} - 2f_1 \frac{\sin(n\omega_1)}{n\omega_1}$	$2(f_2 - f_1)$
Bandstop	$2f_1 \frac{\sin(n\omega_1)}{n\omega_1} - 2f_2 \frac{\sin(n\omega_2)}{n\omega_2}$	$1 - 2(f_2 - f_1)$

$f_c, f_1$  and  $f_2$  are the normalized passband or stopband edge frequencies;  $N$  is the length of filter.

Table 1<sup>1</sup>. Summary of ideal impulse responses for standard frequency selective filters

- FIR filter design requires the truncation of  $\{h[n]\}$  for some  $n = \pm M$

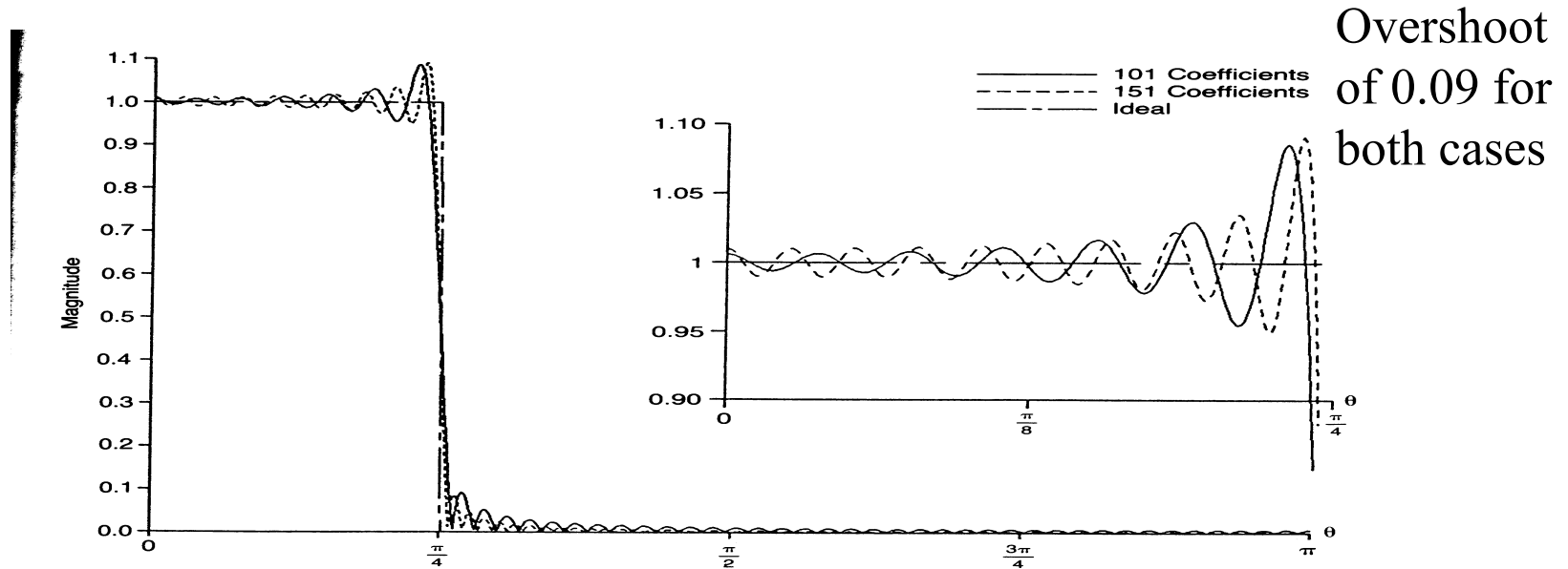


Fig 2. Low pass filter frequency response for 101 and 151 coefficient designs

- The direct truncation method exhibits Gibb's phenomenon implying there exists an overshoot of 8.95% at the point of discontinuity for a large  $M$ .

## Window functions

- A window function is used to obtain a finite impulse response from an ideal infinite impulse response.
- If  $h_D(n)$  is the impulse response for an ideal filter and  $w[n]$  is some window function which is zero outside the range  $0 \leq n \leq N - 1$  then a finite impulse response filter is obtained as

$$h[n] = \{h_D[n]\} \{w[n]\}$$

where  $h[n]$  is the finite impulse response.

### Windowing concept:

Given the infinite sequence from the design equation

$$h_D[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_D(\omega) e^{jn\omega} d\omega$$

$\{\dots h(-25), h(-24) \dots h(-1), h(0), h(1) \dots, h(24), h(25) \dots\}$

Multiply by  $w[n]$

$$w[n] = \{ \dots 0, 0, 0, 1, 1, 1 \dots 1, 1, 1, 0, 0, 0 \dots \}$$

$\uparrow \qquad \qquad \uparrow$   
 $n=-20 \qquad \qquad n=20$

Define the “windowed” filter coefficients:

$$h[n] = \{h_D[n]\} \{w[n]\}$$

$\uparrow \qquad \qquad \uparrow$   
Infinite sequence    Finite length window

$$H(\omega) = \sum_{n=-M}^M h[n] e^{-jn\omega}, \text{ where } M = \frac{N-1}{2}$$
$$= \sum_{n=-M}^M h[n] w[n] e^{-jn\omega}$$

Truncation is equivalent to a **rectangular window**:  $w[n] = \begin{cases} 1, & |n| \leq M \\ 0, & \text{otherwise} \end{cases}$



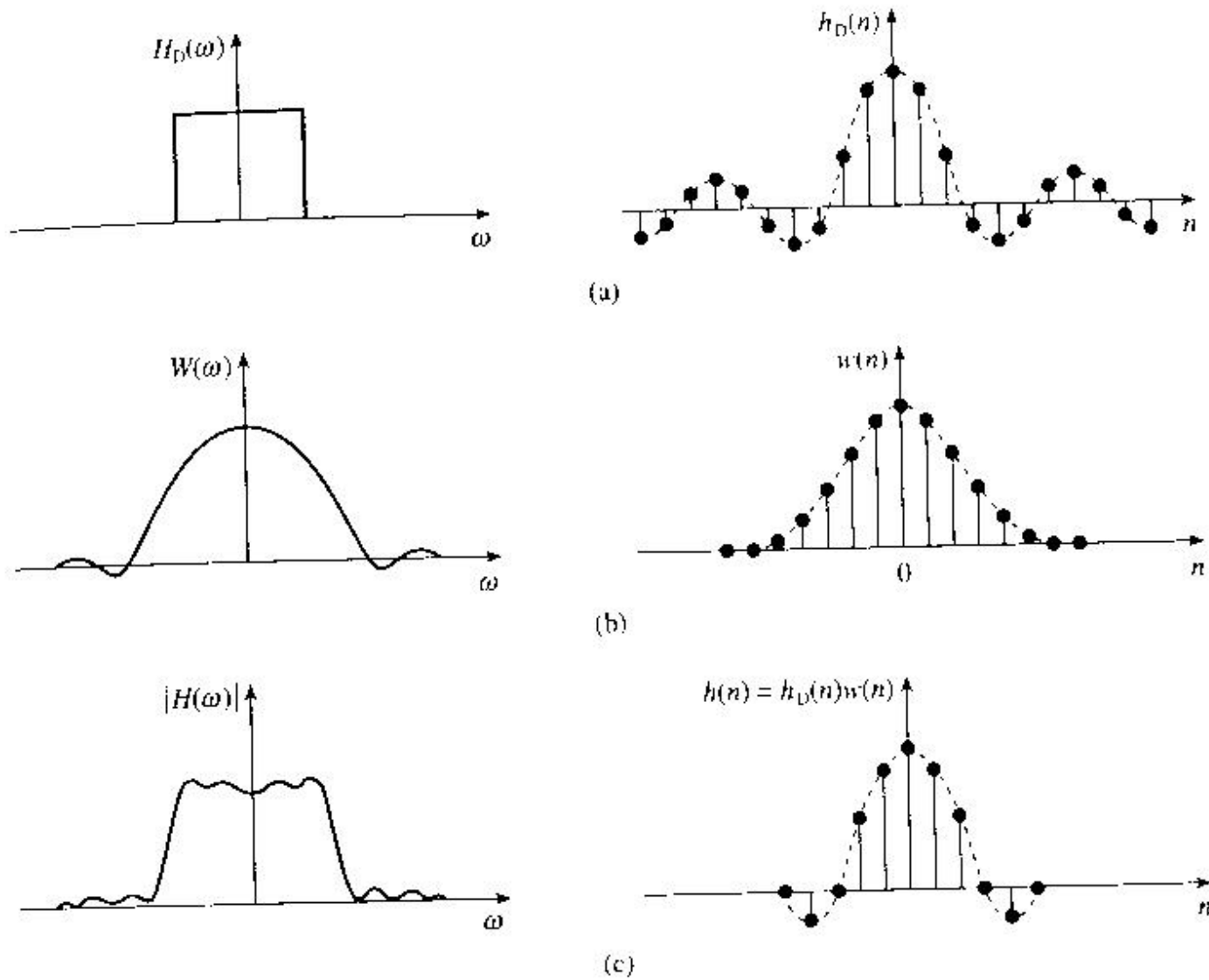


Fig 3<sup>1</sup>. An illustration of how the filter coefficients,  $h(n)$ , are determined by the window method

## Windowing: Convolution interpretation

Consider rectangular window design:

$$h[n] = h_D[n - M]w[n]$$
$$= \begin{cases} h_D[n - M] & 0 \leq n \leq 2M \\ 0 & \textit{otherwise} \end{cases}$$

$$h_D[n - M] \leftrightarrow H_D(e^{j\omega})e^{-j\omega M}$$

$$w[n] \leftrightarrow W(e^{j\omega})$$

$$W(e^{j\omega}) = DFT\{w[n]\} = \frac{\sin\left(\frac{\omega(2M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$H(e^{j\omega}) = H_D(e^{j\omega})e^{-j\omega M} * W(e^{j\omega})$$

## Desirable window properties

1. Small width of main lobe of the frequency response of the window which results in a filter with a lower transition width.
  2. Side lobes of the frequency response should decrease in energy as  $\omega \rightarrow \pi$ . This will lead to a filter with higher stop band attenuation.
- Various window functions have been proposed:-
    - Rectangular window
    - Hanning window
    - Hamming window
    - Blackmann window
    - Kaiser window

<i>Name of window function</i>	<i>Transition width (Hz) (normalized)</i>	<i>Passband ripple (dB)</i>	<i>Main lobe relative to side lobe (dB)</i>	<i>Stopband attenuation (dB) (maximum)</i>	<i>Window function <math>w(n),  n  \leq (N-1)/2</math></i>
Rectangular	$0.9/N$	0.7416	13	21	1
Hanning	$3.1/N$	0.0546	31	44	$0.5 + 0.5 \cos\left(\frac{2\pi n}{N}\right)$
Hamming	$3.3/N$	0.0194	41	53	$0.54 + 0.46 \cos\left(\frac{2\pi n}{N}\right)$
Blackman	$5.5/N$	0.0017	57	75	$0.42 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right)$
	$2.93/N (\beta = 4.54)$	0.0274		50	$\frac{I_0(\beta\{1 - [2n/(N-1)]^2\}^{1/2})}{I_0(\beta)}$
Kaiser	$4.32/N (\beta = 6.76)$	0.002 75		70	
	$5.71/N (\beta = 8.96)$	0.000 275		90	

Table 2<sup>1</sup>. Summary of important features of common window functions

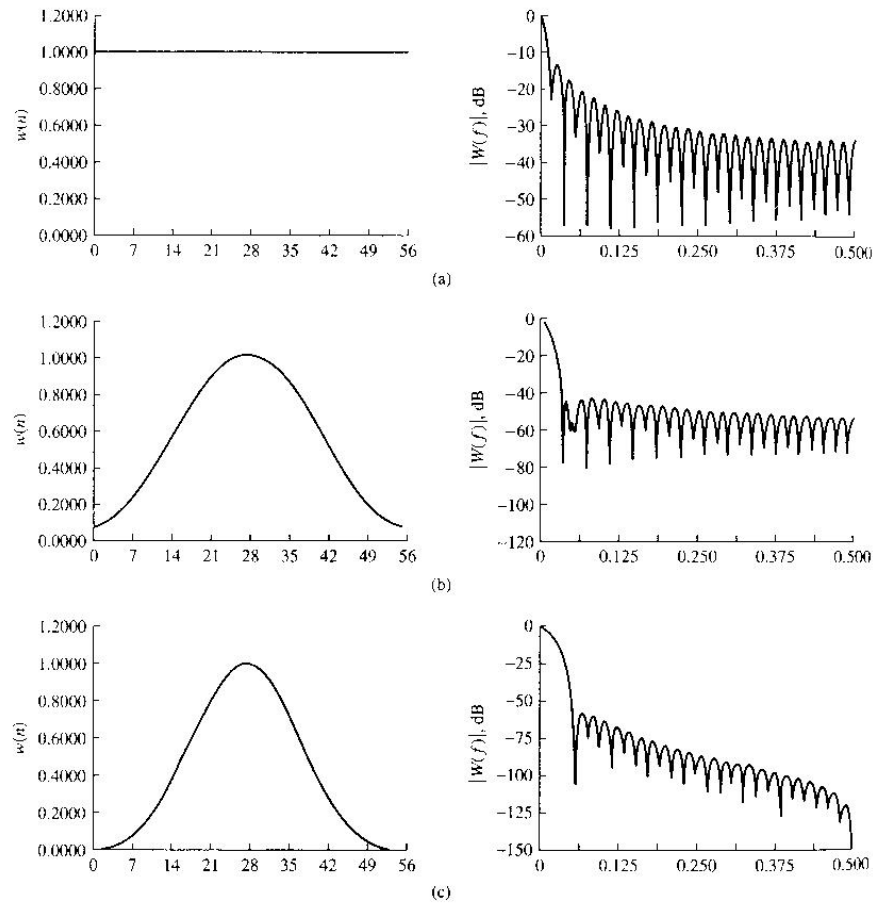


Fig 4<sup>1</sup>. Comparison of the time and frequency domain characteristics of common window functions (a) rectangular; (b) Hamming; (c) Blackman

- We observe that the window functions other than the Kaiser window, have fixed characteristics, such as transition width and stop band attenuation.

Example: Obtain the coefficients of an FIR lowpass filter to meet the specifications given below using the window method.

passband edge frequency	1.5 kHz
transition width	0.5 kHz
stopband attenuation	>50 dB
sampling frequency	8 kHz

Solution: From Table 2.1, we select  $h_D(n)$  for lowpass filter which is given by

$$h_D(n) = 2f_c \frac{\sin(n\omega_c)}{n\omega_c} \quad n \neq 0$$

$$h_D(n) = 2f_c \quad n = 0$$

Table 2.2 indicates that the Hamming, Blackman or Kaiser window will satisfy the stopband attenuation requirements. We will use the Hamming window for simplicity. Now  $\Delta f = 0.5/8 = 0.0625$ . From  $N = 3.3/\Delta f = 3.3/0.0625 = 52.8$ , let  $N = 53$ . The filter coefficients are obtained from

$$h_D(n)w(n) \quad -26 \leq n \leq 26$$

where

$$h_D(n) = \frac{2f_c \sin(n\omega_c)}{n\omega_c} \quad n \neq 0$$

$$h_D(n) = 2f_c \quad n = 0$$

$$w(n) = 0.54 + 0.46 \cos(2\pi n/53) \quad -26 \leq n \leq 26$$

Because of the smearing effect of the window on the filter response, the cutoff frequency of the resulting filter will be different from that given in the specifications. To account for this, we will use an  $f'_c$  that is centred on the transition band:

$$f'_c = f_c + \Delta f/2 = (1.5 + 0.25) \text{ kHz} = 1.75 \text{ kHz} \rightarrow 1.75/8 = 0.21875$$

Noting that  $h(n)$  is symmetrical, we need only compute values for  $h(0)$ ,  $h(1)$ , ...,  $h(26)$  and then use the symmetry property to obtain the other coefficients.

$$n = 0: \quad h_D(0) = 2f'_c = 2 \times 0.21875 = 0.4375$$

$$w(0) = 0.54 + 0.46 \cos(0) = 1$$

$$h(0) = h_D(0)w(0) = 0.4375$$

$$\begin{aligned} n = 1: \quad h_D(1) &= \frac{2 \times 0.21875}{2\pi \times 0.21875} \sin(2\pi \times 0.21875) \\ &= \frac{\sin(360^\circ \times 0.21875)}{\pi} = 0.31219 \end{aligned}$$

$$w(1) = 0.54 + 0.46 \cos(2\pi/53)$$

$$= 0.54 + 0.46 \cos(360^\circ/53) = 0.99677$$

$$h(1) = h(-1) = h_D(1)w(1) = 0.31118$$

$$\begin{aligned} n = 2: \quad h_D(2) &= \frac{2 \times 0.21875}{2 \times 2\pi \times 0.21875} \sin(2 \times 2\pi \times 0.21875) \\ &= \frac{\sin(157.5^\circ)}{2\pi} = 0.06013 \end{aligned}$$

$$w(2) = 0.54 + 0.46 \cos(2\pi \times 2/53)$$

$$= 0.54 + 0.46 \cos(720^\circ/53) = 0.98713$$

$$h(2) = h(-2) = h_D(2)w(2) = 0.06012$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$n = 26: \quad h_D(26) = \frac{2 \times 0.21875}{26 \times 2\pi \times 0.21875} \sin(26 \times 2\pi \times 0.21875)$$

$$= -0.01131$$

$$w(26) = 0.54 + 0.46 \cos(2\pi \times 26/53)$$

$$= 0.54 + 0.46 \cos(9360^\circ/53) = 0.08081$$

$$h(26) = h(-26) = h_D(26)w(26) = -0.000914$$

- We note that the indices of the filter coefficients run from -26 to 26. We make the filter causal by adding 26 to each index so that the indices start at zero.
- The filter coefficients, with indices adjusted, are listed in Table 3.

$h[0] =$	$-9.1399895e-04$	$= h[52]$
$h[1] =$	$2.1673690e-04$	$= h[51]$
$h[2] =$	$1.3270280e-03$	$= h[50]$
$h[3] =$	$3.2138355e-04$	$= h[49]$
$h[4] =$	$-1.9238177e-03$	$= h[48]$
$h[5] =$	$-1.4683633e-03$	$= h[47]$
$h[6] =$	$2.3627318e-03$	$= h[46]$
$h[7] =$	$3.4846558e-03$	$= h[45]$
$h[8] =$	$-1.9925839e-03$	$= h[44]$
$h[9] =$	$-6.2837232e-03$	$= h[43]$
$h[10] =$	$4.5320247e-09$	$= h[42]$
$h[11] =$	$9.2689460e-03$	$= h[41]$
$h[12] =$	$4.3430586e-03$	$= h[40]$
$h[13] =$	$-1.1271299e-02$	$= h[39]$
$h[14] =$	$-1.1402453e-02$	$= h[38]$
$h[15] =$	$1.0630714e-02$	$= h[37]$
$h[16] =$	$2.0964392e-02$	$= h[36]$
$h[17] =$	$-5.2583216e-03$	$= h[35]$
$h[18] =$	$-3.2156086e-02$	$= h[34]$
$h[19] =$	$-7.5449714e-03$	$= h[33]$
$h[20] =$	$4.3546153e-02$	$= h[32]$
$h[21] =$	$3.2593190e-02$	$= h[31]$
$h[22] =$	$-5.3413653e-02$	$= h[30]$
$h[23] =$	$-8.5682029e-02$	$= h[29]$
$h[24] =$	$6.0122145e-02$	$= h[28]$
$h[25] =$	$3.1118568e-01$	$= h[27]$
$h[26] =$	$4.3750000e-01$	$= h[26]$

Table 3<sup>1</sup>



## Kaiser Window – Adjustable Window Function

The most widely used *adjustable window function* is the *Kaiser* window function. Kaiser found that a near-optimal window function could be formed by using the zeroth-order modified Bessel function of the first kind.

The *Kaiser* window is given by:

$$w[n] = \begin{cases} \frac{I_0\{\beta\sqrt{1-(2n/N-1)^2}\}}{I_0(\beta)} & -(N-1)/2 \leq n \leq (N-1)/2 \\ 0 & \textit{otherwise} \end{cases}$$

where  $I_0(\cdot)$  represents the zeroth-order modified Bessel function of the first kind given by:

$$I_o(x) = 1 + \sum_{k=1}^L \left[ \frac{(x/2)^k}{k!} \right]^2 \quad \text{where typically, } L < 25.$$

## Design Using Kaiser Window Function contd....

- The *Kaiser* window has two parameters:
  - 1) Length ( $N = 2M + 1$ )
  - 2) Shape Parameter  $\beta$
- By varying these two parameters the window length and shape can be adjusted to trade sidelobe amplitude for mainlobe width.
- The parameter  $\beta$  controls the minimum attenuation  $\alpha_s$  i.e., ripple  $\delta_s$  in the stop band of the windowed filter response.

## Design Using Kaiser Window Function contd....

Estimation of  $\beta$  and the filter length  $N=2M+1$  for a given  $\alpha_s$  and transition bandwidth  $\Delta f$  is determined empirically.

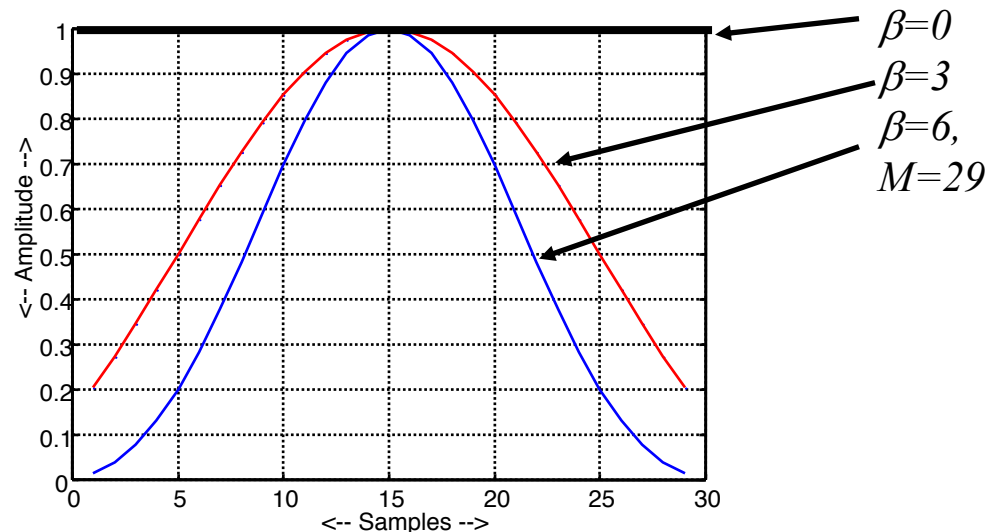
$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & \text{for } 21 \leq \alpha_s \leq 50 \\ 0 & \text{for } \alpha_s < 21 \end{cases}$$

$$N = \begin{cases} \frac{\alpha_s - 7.95}{14.36\Delta f} + 1 & \text{for } \alpha_s > 21 \\ \frac{0.9222}{\Delta f} + 1 & \text{for } \alpha_s \leq 21 \end{cases}$$

Note: The *kaiser* window does not provide independent control over the passband ripple  $\delta_p$ . However in practice,  $\delta_p$  is equal to  $\delta_s$ .

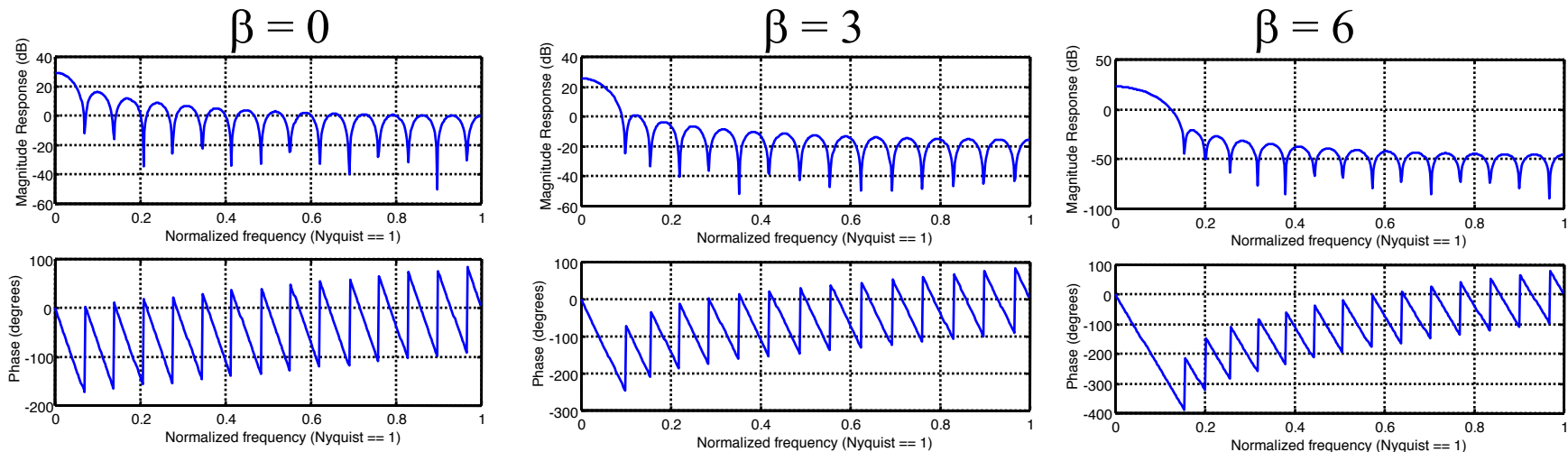
## Design Using Kaiser Window Function contd...

By varying the two parameters  $M$  and  $\beta$ , the window shape could be adjusted to trade sidelobe amplitude for the mainlobe width. The figure below shows continuous envelopes of *kaiser* window of length  $M=29$  and  $\beta=0, 3, 6$ . Notice the shape as  $\beta$  varies from 0 to higher values.



## Design Using Kaiser Window Function contd...

Taking the fourier transform of the obtained windows we see from the figures below we notice the tradeoff between the width of the mainlobe and the amplitude of the sidelobes. If the window is tapered more, the sidelobes of the fourier transform become smaller, but the main lobe becomes wider.



## Design Using *Kaiser* Window Function Example

- Specification of a Lowpass Filter:

$w_p = 0.3\pi$ , passband edge frequency

$w_s = 0.4\pi$ , stopband edge frequency

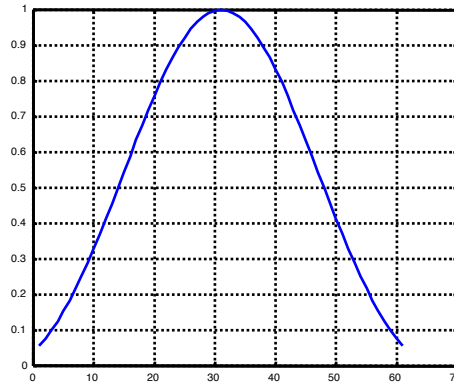
$\alpha_s = 50\text{dB}$  attenuation

Soln)

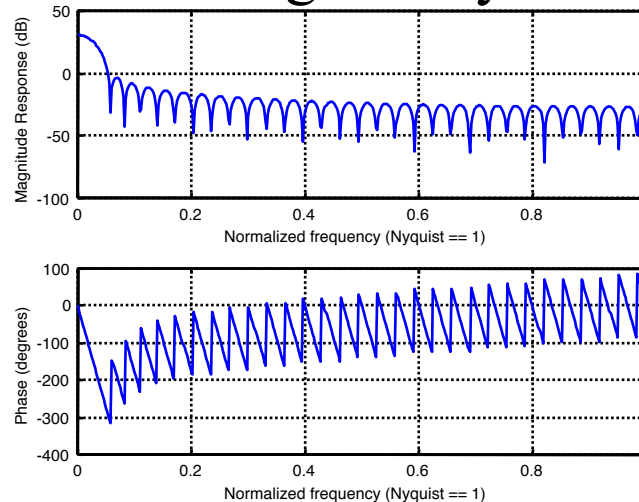
- We have  $w_c = (w_p + w_s)/2 = 0.35\pi$
- We know  $\alpha_s = -20\log_{10}(\delta_s)$  dB. Thus  $\delta_s = 0.00316$
- We know how to calculate  $\beta$  for a given  $\alpha_s$ . Thus  $\beta = 4.5335$ .
- The transition bandwidth  $\Delta f$  is given by  $\Delta f = (w_s - w_p)/2\pi = 0.05$
- Next we determine  $N$  using the empirical formula given in the earlier slide. We get  $N = 59.565$  we choose the next higher odd value i.e., 61. Thus  $M = 30$ .

# Design Using *Kaiser* Window Function *Example*

For the calculated window parameters the *kaiser* window is :



Its Fourier transform is given by:

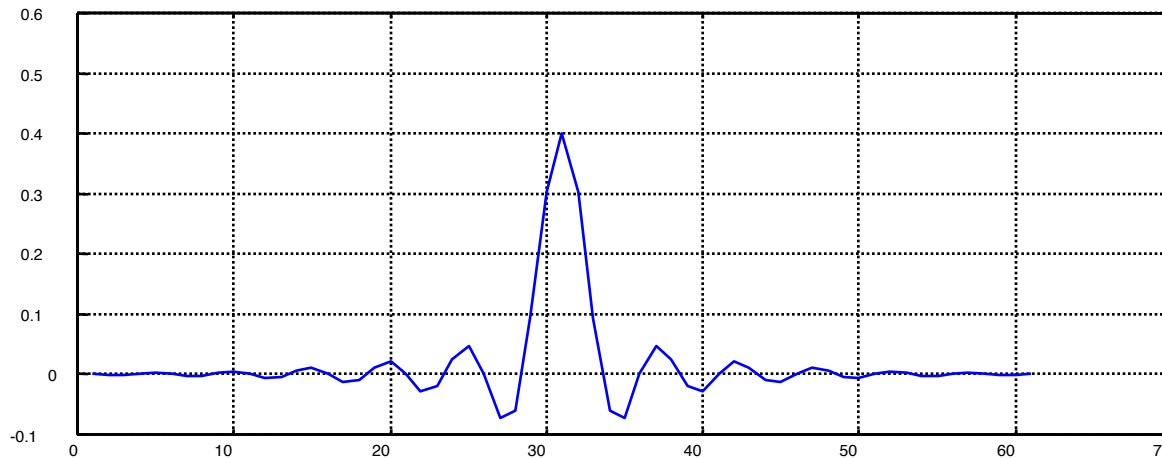


## Design Using *Kaiser* Window Function *Example*

We obtain the impulse-response coefficients of the required FIR filter by windowing the ideal filter function.

$$h[n] = \frac{2f_c \sin w_c n}{n\omega_c} \cdot w[n] \quad n \neq 0, -M \leq n \leq M$$
$$= 2f_c \quad n = 0$$

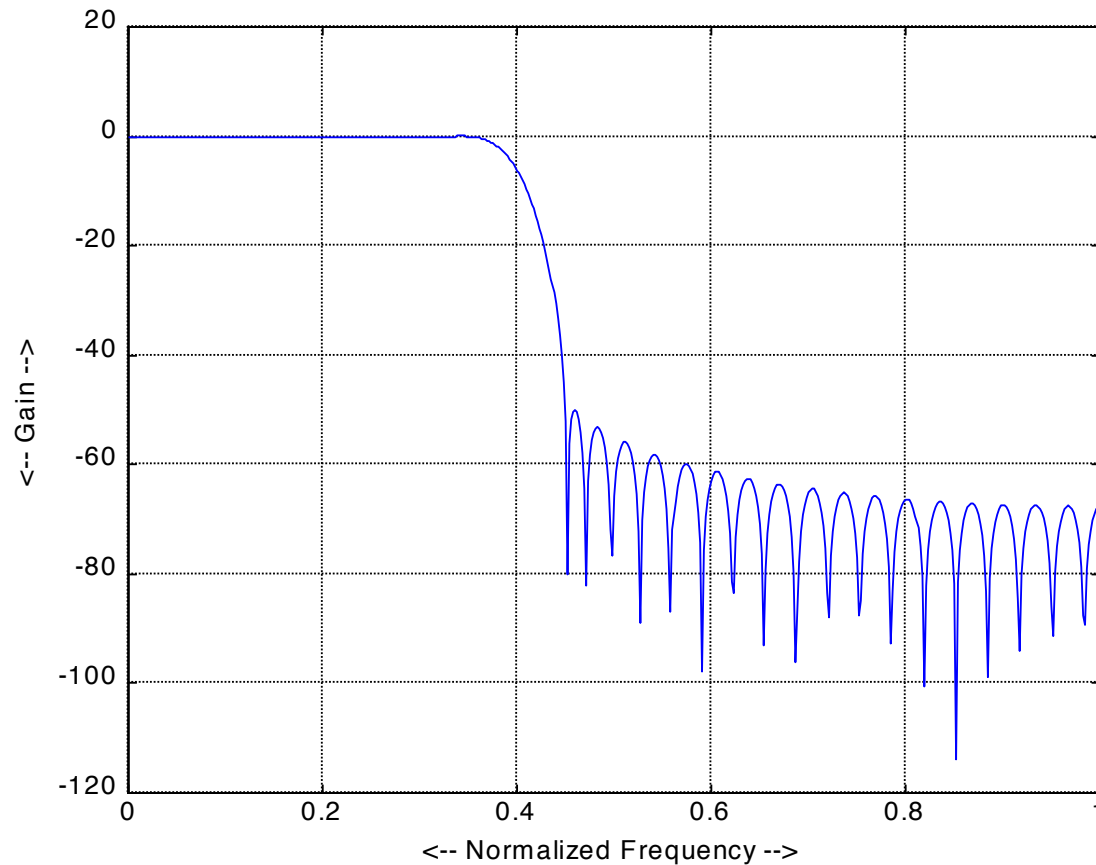
For  $M=30$  and  $w_c=0.35\pi$  we get the coefficients to be :





# Design Using *Kaiser* Window Function *Example*

The Magnitude Response of the obtained Filter is :



## Advantages and disadvantages of the window method

- The window method is simple to apply and simple to understand.
- The major disadvantage is lack of flexibility, for example equal *peak* pass band and stop band ripples, which is generally undesirable.
- Because of the effect of convolution of the spectrum of the window function and the desired response, the pass band and stop band edge frequencies cannot be precisely specified.

# References

1. “Digital Signal Processing – A Practical Approach” -  
Emmanuel C. Ifeakor and Barrie W. Jervis  
Second Edition