Analog Filters (II) Butterworth Filters

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Analog Butterworth Filters

Define : Lowpass Butterworth Filters (N[th] order)

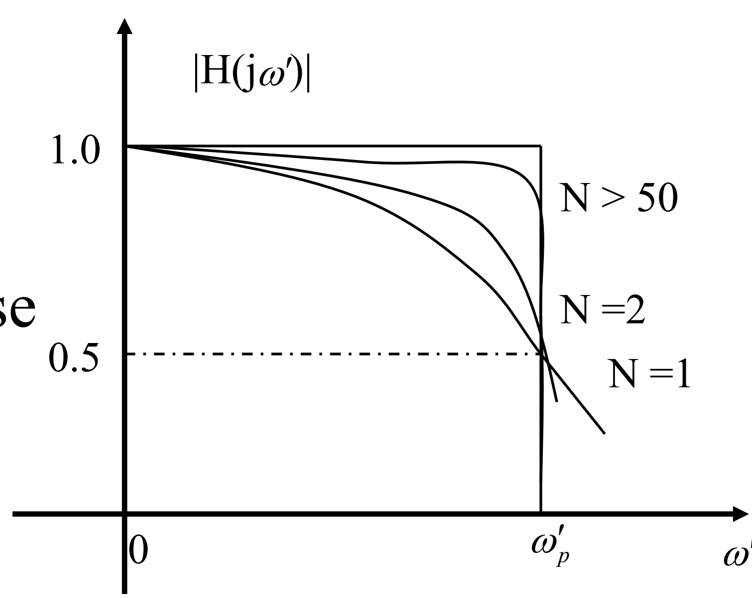
$$|H(j\omega')|^2 = \frac{1}{1 + (\frac{\omega'}{\omega'_p})^{2N}}$$

or

$$|H(j\omega')|^2 = \frac{1}{1 + (\omega')^{2N}}$$

(normalized prototype, $\omega'_p = 1$)

Maximally flat response



Notes :

Assume $\omega'_p = 1$

1. Magnitude

$$A = |H(j\omega')| = 1/\sqrt{2} \quad \text{at} \quad \omega' = 1$$

$$A^2 = |H(j\omega')|^2 = 1/2 \quad \text{at} \quad \omega' = 1$$

$$A_{dB} = -3\text{dB at } \omega' = \omega'_p = 1 \text{ all N}$$

2. $A \rightarrow 0 \quad \text{as} \quad \omega' \rightarrow \infty$

3. $A^2 = 1 \quad \text{at} \quad \omega' = 0 \quad \text{all N}$

$A = 1 \quad \text{at} \quad \omega' = 0 \quad \text{all N}$

4. Maximally Flat response

$$\left.\frac{d^n M}{d\omega'^n}\right|_{\omega'=0} = 0 \quad n = 1, 2, \ldots 2N-1$$

5. $|H(j\omega')|^2$ is a monotonically decreasing function of ω' .

$$|H(j\omega')|^2 = \frac{1}{1+\omega'^{2N}}$$

$$\frac{d}{d\omega'}(|H(j\omega')|^2) = 2|H(j\omega')|\frac{d}{d\omega'}(|H(j\omega')|)$$

$$\frac{d}{d\omega'}(|H(j\omega')|) = \frac{1}{2|H(j\omega')|}\frac{d}{d\omega'}(|H(j\omega')|^2)$$

$$\frac{d}{d\omega'}(|H(j\omega')|^2) = \frac{-2N\omega'^{2N-1}}{(1+\omega'^{2N})^2}$$

$$|H(j\omega')| = (\frac{1}{1+\omega'^{2N}})^{1/2}$$

$\therefore$

$$\frac{d}{d\omega'}(|H(j\omega')|) = \frac{-N\omega'^{2N-1}}{(1+\omega'^{2N})^{3/2}}$$

$Follows:$

$$\frac{d}{d\omega'}(|H(j\omega')|) < 0..\omega' > 0$$

6. $$\frac{d}{d\omega'}(|H(j\omega')|) = -0.354N$$
$$\omega' = \omega'_p = 1$$

7. High - Frequency roll - off of an N^{th} order butterworth filter is :

-20N dB/ Decade

let $\omega' \gg \omega'_p$

$$|H(j\omega')|^2 = \frac{1}{1+\left(\frac{\omega'}{\omega'_p}\right)^{2N}} \approx \frac{1}{\left(\frac{\omega'}{\omega'_p}\right)^{2N}}$$

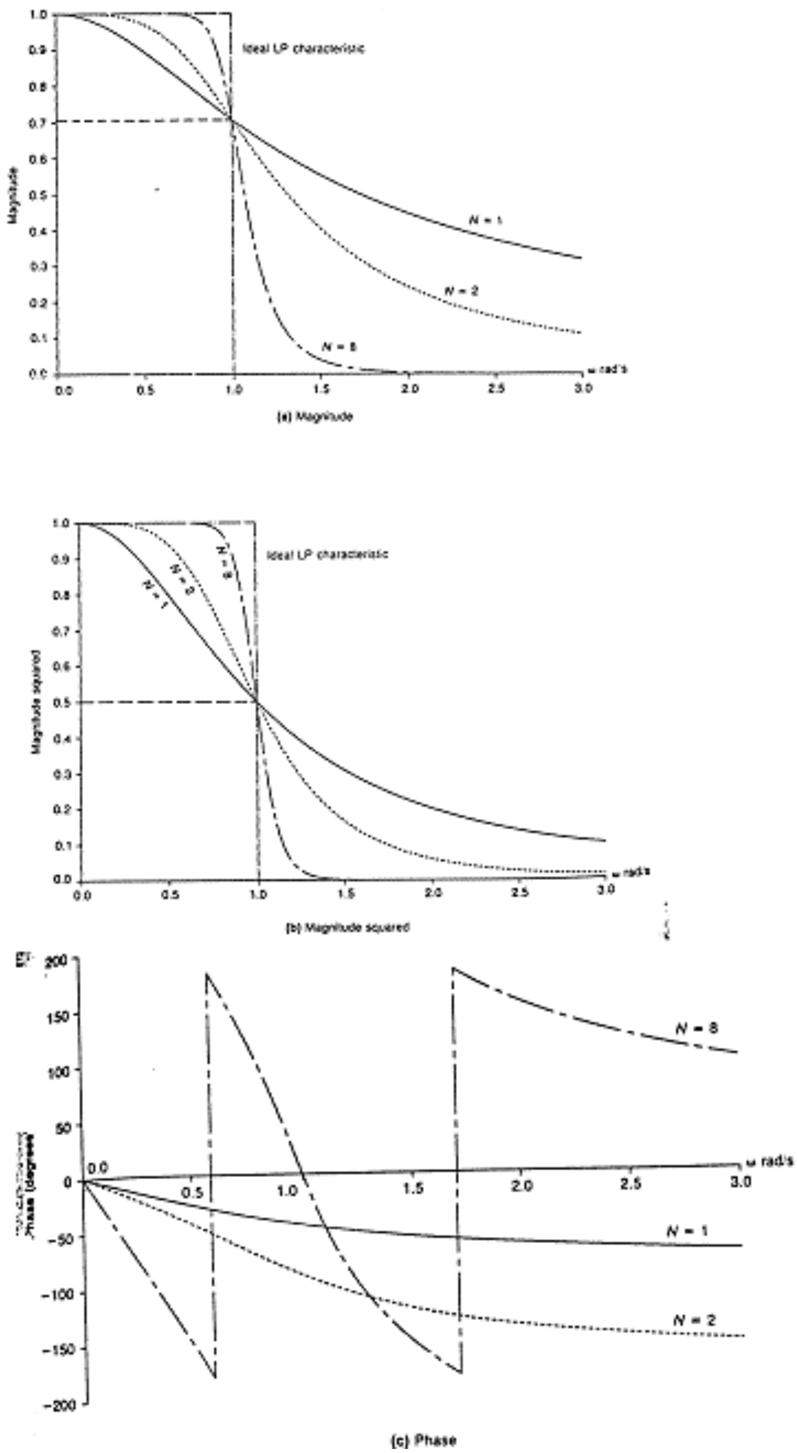
$$-10\log\left|\left(\frac{\omega'}{\omega'_p}\right)^{2N}\right| = -20N\log\left|\frac{\omega'}{\omega'_p}\right|$$

$\therefore$

$$20N\log_{10}\left|H(j\omega')\right| = A_{dB}$$

$$A_{dB}(\omega' \gg \omega'_p) = -20N\log\left|\frac{\omega'}{\omega'_p}\right|$$

Analog Filters



Poles of the Lowpass Prototype (Normalized $\omega'_p = 1$)

Butterworth Filter

Magnitude Square Function :

$$|H(j\omega')|^2 = \left.\frac{1}{1+\omega'^{2N}}\right._{\omega'_p=1}$$

Now :

$$H(s)H(-s) = \left.\frac{1}{1+\omega'^{2N}}\right|_{\omega'^2=-s^2\,(s=j\omega')}$$

$$H(s)H(-s) = \frac{1}{1+(-s^2)^N}$$

Poles: Solve: $1 + (-s^2)^N = 0$

a. N is odd

$$1 - s^{2N} = 0$$

$$s^{2N} = 1$$

Implies : Roots are :

$$s_k = 1.e^{j(\frac{k2\pi}{2N})} = 1.e^{j(\frac{k\pi}{N})}$$

$$k = 0,1,2,.........2N-1$$

b. N is even :

$$1 + s^{2N} = 0 \quad s^{2N} = -1$$

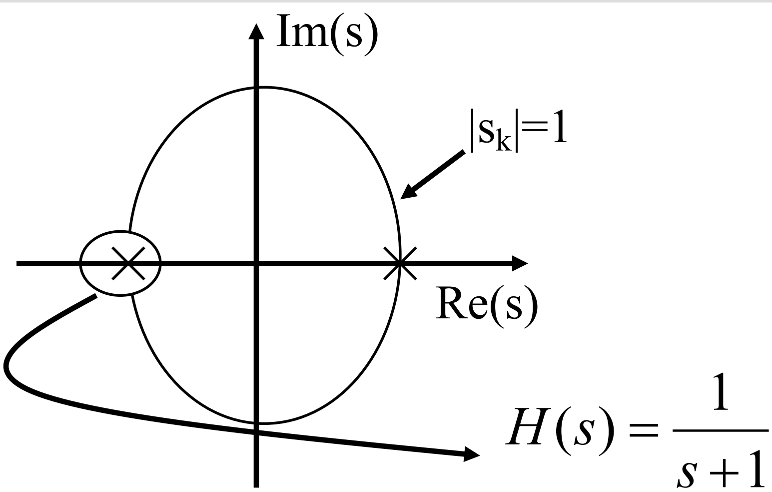
Roots :

$$s_k = 1.e^{j(\frac{\pi + k2\pi}{2N})}$$

$$k = 0,1,2,.........2N-1$$

Example: N = 1

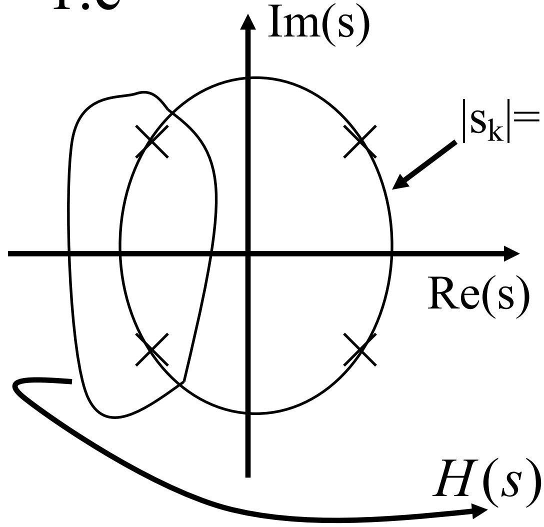
$s_0 = 1 \quad s_1 = 1.e^{j\pi}$



Example: N = 2

$s_0 = 1.e^{j\pi/4}$, $s_1 = 1.e^{j3\pi/4}$, $s_2 = 1.e^{j5\pi/4}$

$s_3 = 1.e^{j7\pi/4}$

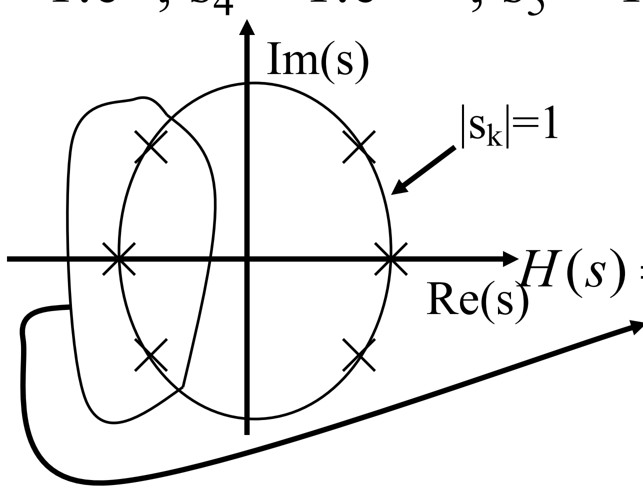


$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Example: N = 3

$s_0 = 1$, $s_1 = 1.e^{j\pi/3}$, $s_2 = 1.e^{j2\pi/3}$

$s_3 = 1.e^{j\pi}$, $s_4 = 1.e^{j4\pi/3}$, $s_5 = 1.e^{j5\pi/3}$



$$H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

Conclusion :

Poles of a normalized ($\omega'_p = 1$) Butterworth (LPF) filter lie uniformly spaced on a unit circle in the s-plane at the following locations:

$$s_k = e^{j\pi(2k+N-1)/2N} = \cos\left[\frac{(2k+N-1)\pi}{2N}\right] + j\sin\left[\frac{(2k+N-1)\pi}{2N}\right]$$

$$k = 1,2,\ldots,N$$

The poles occur in complex conjugate pairs and lie on the left-hand side of the s-plane.

TABLE 10.1 BUTTERWORTH PROTOTYPE COEFFICIENTS

N	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1.0000							
2	1.4141	1.0000						
3	2.0000	2.0000	1.0000					
4	2.6131	3.4142	2.6131	1.0000				
5	3.2361	5.2361	5.2361	3.2361	1.0000			
6	3.8637	7.4641	9.1416	7.4641	3.8637	1.0000		
7	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	1.0000	
8	5.1258	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258	1.0000

$$H_{LP_p}(s) = \frac{1}{1 + a_1 s + a_2 s^2 + \cdots + a_N s^N}$$

Source: M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*. New York, John Wiley and Sons, 1960.

The Butterworth Low pass filter order N is given by:

$$N \geq \frac{\log\left(\dfrac{10^{\frac{A_s}{10}}-1}{10^{\frac{A_p}{10}}-1}\right)}{2\log\left(\dfrac{\omega'_s}{\omega'_p}\right)}$$

where A_p and A_s are, respectively, the pass band ripple and stop band attenuation in dB, and ω'_s is the stop band edge frequency.

Find N : the order of the Butterworth Filter for normalized (ω'_p=1) specifications.

Let :

$$A = |H(j\omega')| = \frac{1}{\left(1+\omega'^{2N}\right)^{1/2}}$$

$$A_{dB} = 20\log_{10} A$$

$$= 10\log_{10} A^2$$

Specify $\omega' = \omega'_a$ if different from ω'_p

$$A_{dB} = 10\log_{10} |H(j\omega')|^2$$

$$= 10\log_{10} \frac{1}{1+\omega_a'^{2N}}$$

$$= 10\log_{10}(1) - 10\log_{10}(1+\omega_a'^{2N})$$

$$A_{dB} = -10\log_{10}(1+\omega_a'^{2N})$$

$$-A_{dB}/10 = \log_{10}(1+\omega_a'^{2N})$$

$$10^{-[A_{dB}/10]} = 1+\omega_a'^{2N}$$

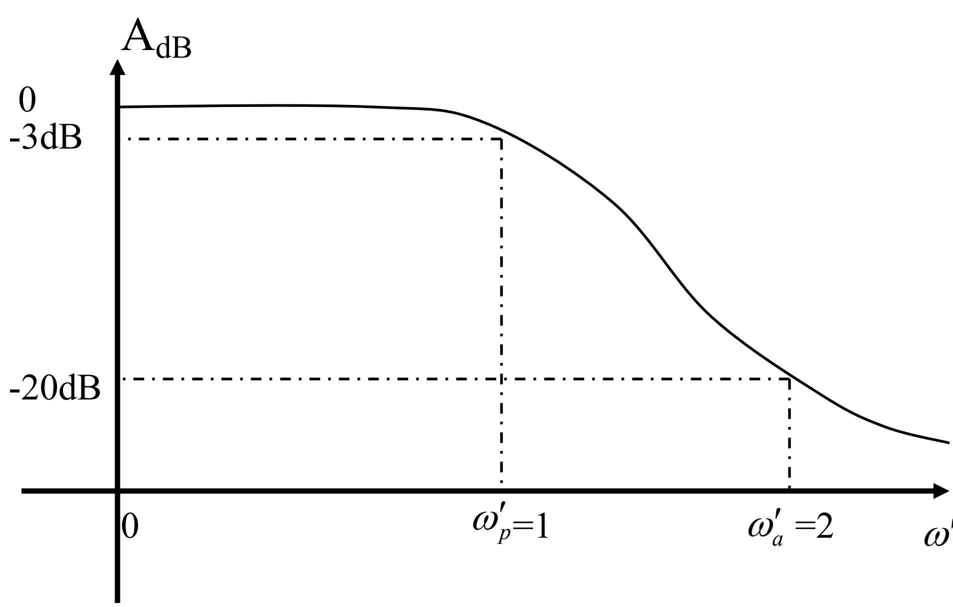
$$10^{-[A_{dB}/10]} - 1 = \omega_a'^{2N}$$

Follows

$$\log_{10}\left(10^{-[A_{dB}/10]}-1\right)=2N\log_{10}\omega'_a$$

$$N=\frac{\log_{10}\left(10^{-[A_{dB}/10]}-1\right)}{2\log_{10}\omega'_a}$$

Example :



For $\omega' = \omega'_a \geq 2$ rad, $A_{dB} \leq$ -20 dB

$$N = \frac{\log_{10}\left(10^{-[-20/10]} - 1\right)}{2\log_{10} 2} \approx 3.31$$

Choose : N = 4

From table having the Butterworth Prototype Coefficients (table 10.1)

$$H(s) = \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}$$

$$|H(j2)|_{db} = -24dB$$

More than satisfies the given specification