Analog Filters (III) Chebyshev Filters

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Chebyshev Lowpass Filter Design

Define :

$$|H(j\omega')|^2 = \frac{K}{1+\varepsilon^2 C_N^2(\omega'/\omega'_p)}$$

$$|H_{LP_p}(j\omega')|^2 = \frac{1}{1+\varepsilon^2 C_N^2(\omega')}$$

(normalized prototype, $\omega'_p = 1$)

Where :

$C_N(\omega')$ is the N^{th} order chebyshev polynomial.

$C_N(\omega') = \cos(N\cos^{-1}\omega') \quad 0 \le \omega' \le 1$

$C_N(\omega') = \cosh(N\cosh^{-1}\omega') \quad \omega' > 1$

ε is the ripple parameter $0 < \varepsilon < 1$ sets the ripple amplitude in the ripple passband $0 \le \varepsilon \le 1$

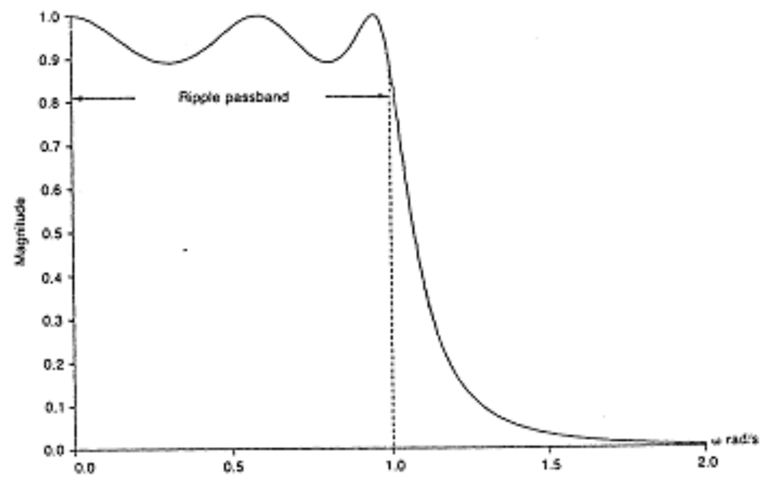
Example : N = 1

$$C_N(\omega') = \cos(\cos^{-1}\omega') \qquad 0 \le \omega' \le 1$$

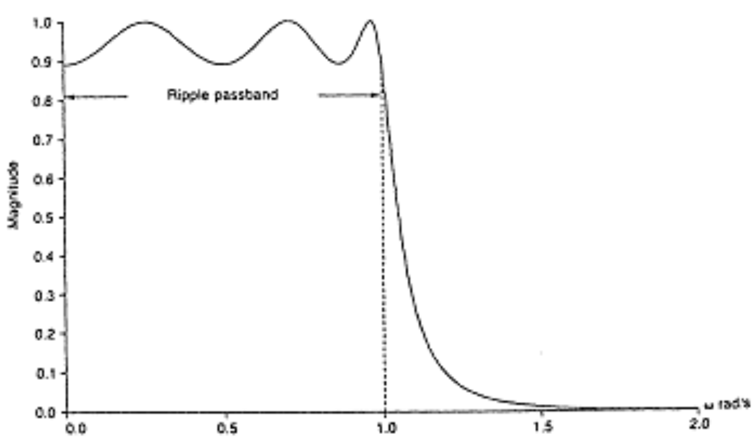
$$C_N(\omega') = \cosh(\cosh^{-1}\omega') \qquad \omega' > 1$$

i.e $C_N(\omega') = \omega'$.

$$|H_{LP_p}(j\omega')|^2 = \frac{1}{1+\varepsilon^2\omega'^2}$$



(a) Chebyshev prototype lowpass frequency response: $N = 5$ and $\epsilon = 0.5088$



(b) Chebyshev prototype lowpass frequency response: $N = 6$ and $\epsilon = 0.5088$

Generation of Higher order polynomials:

$C_N(\omega')$

N^{th} Polynomial :

$$C_N(\omega') = \cos(N\cos^{-1}\omega')$$

$(N+1)^{th}$ Polynomials

$$C_{N+1}(\omega') = \cos((N+1)\cos^{-1}\omega')$$

Let $\alpha = \cos^{-1}\omega'$

then $$C_{N+1}(\alpha) = \cos((N+1)\,\alpha)$$

$$= \cos(N\alpha + \alpha)$$

Follows:

1.) $C_{N+1}(\alpha) = \cos(N\,\alpha)\cos(\alpha) - \sin(N\,\alpha)\sin(\alpha)$

Also

2.) $C_{N-1}(\alpha) = \cos(N\alpha)\cos(\alpha) + \sin(N\alpha)\sin(\alpha)$

By adding $C_{N+1}(\alpha)$ & $C_{N-1}(\alpha)$ we get :

$$C_{N+1}(\alpha) + C_{N-1}(\alpha) = 2\cos(N\alpha)\cos(\alpha)$$

substituting back $\alpha = \cos^{-1}\omega'$

$$C_{N+1}(\alpha) + C_{N-1}(\alpha) = 2\cos(N \cos^{-1}\omega') \cos(\cos^{-1}\omega')$$

$$= 2\omega'\cos(N \cos^{-1}\omega')$$

Follows :

$$C_{N+1}(\omega') = 2\omega' C_N(\omega') - C_{N-1}(\omega')$$

Example: N = 1

$$C_2(\omega') = 2\,\omega'\, C_1(\omega') - C_0(\omega')$$

$$= 2\omega'.\omega' - 1$$

$$= 2\omega'^2 - 1$$

For N = 2

$$C_3(\omega') = 2\omega' C_2(\omega') - C_1(\omega')$$

$$= 2\omega'(2\omega'^2 - 1) - \omega'$$

$$= 4\omega'^3 - 3\omega'$$

N	$C_N(\omega)$
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$

Table of Chebyshev polynomials

Note :w=ω' in the table

Properties of C_N and $|H_{LP}(j\omega')|$

N = 2 : even

$$|H_{LP_p}(j\omega')| = \frac{1}{\left(1+\varepsilon^2 C_2^2(\omega')\right)^{1/2}}$$

$$\omega' = 0$$

$$C_2^2(0) = (2\omega'^2 - 1)^2\Big|_{\omega'=0} = 1$$

P1 $$|H_{LP_p}(j0)| = \frac{1}{\left(1+\varepsilon^2\right)^{1/2}}$$ N even

N = 3 : odd

$$|H_{LP_p}(j\omega')| = \frac{1}{\left(1+\varepsilon^2 C_3^2(\omega')\right)^{1/2}}$$

$$\omega' = 0$$

$$C_3^2(0) = (4\omega'^3 - 3\omega')^2\Big|_{\omega'=0} = 0$$

P2 $$|H_{LP_p}(j0)| = 1$$ N odd

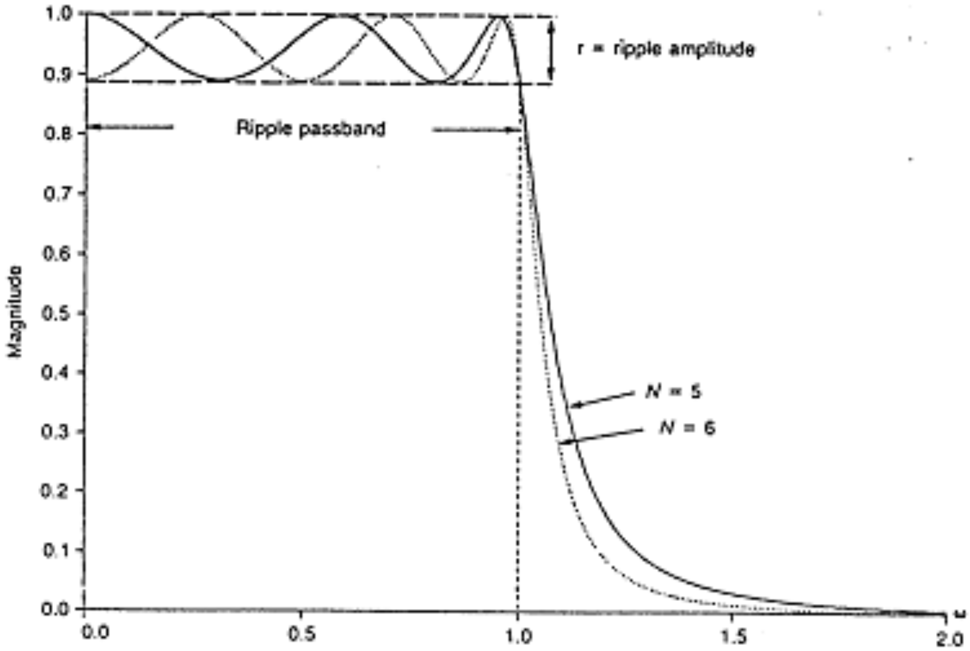


Fig: Chebyshev prototype low pass response

P3 $\omega' = 1$

$$C_N^2(\omega')\big|_{\omega'=1} = [\cos(N\cos^{-1}(1))]^2$$
$$= 1 \text{ all N}$$

$$|H(j1)| = \frac{1}{\left(1+\varepsilon^2\right)^{1/2}} \qquad \text{all N}$$

P4 $|H(j1)| > \dfrac{1}{\sqrt{2}}$ $\qquad \omega'_c = f(N,\varepsilon)$

P5 There are N maximum and minimum points between $\omega' = 0$ and $\omega' = 1$

P6 For $\omega' > 1$ $|H(j\omega')|$ is monotonically decreasing function and :

$$|H(j\omega')| \approx \frac{1}{\varepsilon 2^{N-1} \omega'^{N}}$$

$$\omega' >> 1$$

P7 Construction of stable and causal transfer function $H_{LP}(s)$

$$H_{LP}(s) H_{LP}(-s) = \left. \frac{1}{1 + \varepsilon^2 C_N^2(\omega')} \right|_{\omega' = -js}$$

Factor this part [$H_{LP}(s)$] by collecting all poles located in the left - half plane. Chebyshev poles lie on a ellipse

P8 The ripple amplitude in dB is given by :

$$\gamma_{dB} = -10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right)$$

$$= 10\log_{10}\left(1+\varepsilon^2\right)$$

N	1/2-dB ripple ($\epsilon = 0.3493$, $\epsilon^2 = 0.1220$)
1	$s + 2.863$
2	$s^2 + 1.425s + 1.516$
3	$s^3 + 1.253s^2 + 1.535s + 0.716$
4	$s^4 + 1.197s^3 + 1.717s^2 + 1.025s + 0.379$
5	$s^5 + 1.173s^4 + 1.937s^3 + 1.310s^2 + 0.753s + 0.179$

N	1-dB ripple ($\epsilon = 0.5088$, $\epsilon^2 = 0.2589$)
1	$s + 1.965$
2	$s^2 + 1.098s + 1.103$
3	$s^3 + 0.988s^2 + 1.238s + 0.491$
4	$s^4 + 0.953s^3 + 1.454s^2 + 0.743s + 0.276$
5	$s^5 + 0.937s^4 + 1.689s^3 + 0.974s^2 + 0.581s + 0.123$

N	2-dB ripple ($\epsilon = 0.7648$, $\epsilon^2 = 0.5849$)
1	$s + 1.308$
2	$s^2 + 0.804s + 0.823$
3	$s^3 + 0.738s^2 + 1.022s + 0.327$
4	$s^4 + 0.716s^3 + 1.256s^2 + 0.517s + 0.206$
5	$s^5 + 0.707s^4 + 1.500s^3 + 0.694s^2 + 0.459s + 0.082$

Source: M. S. Ghausi and K. R. Laker, *Modern Filter Design*, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1981.

Fig: Prototype Chebyshev denominator polynomials

Example :

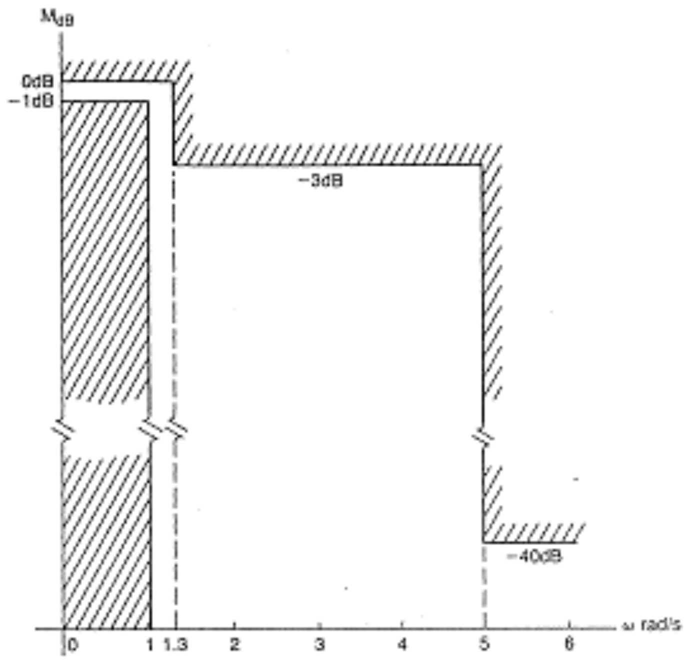


Fig: Low pass filter specifications

1 db ripple:

$$1 = 10\log_{10}(1 + \varepsilon^2)$$

$$1.2589 = 1 + \varepsilon^2$$

$$\varepsilon = 0.5088$$

$\omega' \geq 5$: -40 dB: $|H(j\omega')|^2 \leq 10^{-4}$

By trial and error: N = 3 will satisfy this requirement.

From the table giving the Chebhshev prototype denominator polynomial (Table 10.3) the prototype transfer function is:

$$H_{LP_p}(s) = \frac{k}{s^3 + 0.988s^2 + 1.2385s + 0.491}$$

N odd : $H_{LP}(0) = 1$

Therefore K =0.491

Butterworth / Chebyshev Filters

Let $\varepsilon = 1 \leftrightarrow 3$ dB

$$\underbrace{\frac{dM(\omega')}{d\omega'}\bigg|_{\omega'=\omega'_c=1}}_{\text{Chebyshev}} = \underbrace{\left(\frac{-N}{2\sqrt{2}}\right)}_{\text{Butterworth}} N$$

Stopband Attenuation (SBA)

$$\underbrace{\text{SBA(dB)}}_{\text{Chebyshev}} \cong \underbrace{6(\text{N-1})}_{\text{Additional dB}} + \underbrace{20\log\varepsilon + 20\log\omega'}_{\text{Butterworth}}$$

The order N for a Chebyshev approximation is determined as follows:

If the response at $\omega' = \omega'_s$ is required to be 1/A, then the filter order is determined from the following equation.

$$N \geq \frac{\cosh^{-1}(\sqrt{A^2 - 1} / \varepsilon)}{\cosh^{-1}(\omega'_s / \omega'_p)}$$

Where ω'_p is the pass band cutoff frequency and ε is the passband ripple.

The order can also be computed as:

$$N \geq \frac{\cosh^{-1}\left(\dfrac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1}\right)}{2\cosh^{-1}\left(\dfrac{\omega'_s}{\omega'_p}\right)}$$

where A_p and A_s are, respectively, the pass band ripple and stop band attenuation in dB, and ω'_s is the stop band edge frequency.

Previous Example:

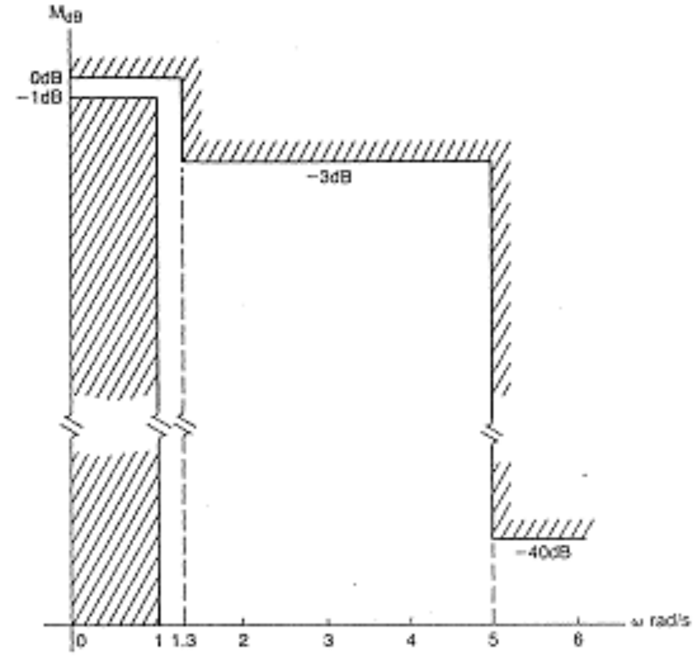


Fig: Low pass filter specifications

1 db ripple:

$$1 = 10\log_{10}(1 + \varepsilon^2)$$

$$1.2589 = 1 + \varepsilon^2$$

$$\varepsilon = 0.5088$$

$$10\log_{10}\left(\frac{1}{A^2}\right) = -40$$

$$A = 100, \quad \omega'_p = 1.0, \quad \omega'_s = 5.0$$

Putting in the values in the equation for N, we get

$$N \geq 2.6059, \quad \text{i.e } \mathrm{N} = 3$$