Infinite Impulse Response (IIR)

Digital Filters (I)

<u>Mapping analog filters</u>

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Design objectives: IIR designs

Given that:

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Or equivalently:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Require that::

$$\left.\begin{array}{l} h[n] = 0 \quad n < 0 \\ \sum_{n=0}^{\infty} |h[n]| < \infty \end{array}\right\}$$

Realizable and stable

General design problem:

Find a_k and b_k's such that filter responses (time response, frequency response, group delay etc,) approximate the desired responses with some specified error criterion(ex:- MSE)

Design Techniques:

Indirect design:

Mapping analog filters

Direct design:

Direct placement of pole-zeros in the z-plane via closed form solutions.

Optimal designs:

Placement of pole-zeros via computer-aided techniques open form solutions

Mapping from analog filters:

Given an analog filter:

$$H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k}$$

Or equivalently:

$$\sum_{k=0}^{N} c_k \frac{d^k y_a(t)}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_a(t)}{dt^k}$$

Use an approximate mapping function to obtain an equivalent digital filter.

Mapping functions:

Mapping differentials

Mapping Integrals

Impulse - Invariant mapping

Bilinear mapping

The matched z-transform

Step-Invariant mapping

others

Mapping differentials:

Backward:

$$\left.\frac{dy_a(t)}{dt}\right|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n]-y[n-1]}{T}$$

Forward:

$$\left.\frac{dy_a(t)}{dt}\right|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n+1]-y[n]}{T}$$

$\nabla^{(1)}$ Backward or forward difference operator

Generalize:

$$\left.\frac{d^k y_a(t)}{dt^k}\right|_{t=nT} = \left.\frac{d}{dt}\left(\frac{d^{k-1} y_a(t)}{dt^{k-1}}\right)\right|_{t=nT}$$

Also:

$$\nabla^{(k)}[y[n]] = \nabla^{(1)}[\nabla^{(k-1)}[y[n]]]$$

with $\nabla^{(0)}(.) = y[n]$

Ex:-

$$\nabla^{(2)}[y[n]] = \nabla^{(1)}[\frac{y[n]-y[n-1]}{T}]$$

$$= \frac{1}{T}[\nabla^{(1)}[y[n]] - \nabla^{(1)}[y[n-1]]]$$

$$= \frac{1}{T}[\frac{1}{T}\{y[n]-y[n-1]\} - \frac{1}{T}\{y[n-1]-y[n-2]\}]$$

$$= \frac{1}{T^2}[y[n]-2y[n-1]+y[n-2]]$$

Analog filter differential equation becomes:

$$\sum_{k=0}^{N} c_k \nabla^{(k)}[y[n]] = \sum_{k=0}^{M} d_k \nabla^{(k)}[x[n]]$$

with
$$\begin{cases} y[n] = y_a(nT) \\ x[n] = x_a(nT) \end{cases}$$

Z-domain function:

Since:
$$Z[\nabla^{(1)}[y[n]]] = [\frac{1-z^{-1}}{T}]Y(z)$$

then:
$$Z[\nabla^{(k)}[y[n]]] = [\frac{1-z^{-1}}{T}]^{k} Y(z)$$

and
$$H(z) = \frac{\sum_{k=0}^{M} d_k [\frac{1-z^{-1}}{T}]^k}{\sum_{k=0}^{N} c_k [\frac{1-z^{-1}}{T}]^k}$$

is the digital filter function via mapping from the analog filter $H_a(s)$

Compare H(z) with H_a(s)

$$H_a(s) = \frac{\sum_{k=0}^{M} d_k s^k}{\sum_{k=0}^{N} c_k s^k}$$

Conclusion:

$$s = \frac{1 - z^{-1}}{T}$$

Backward difference mapping

$$s = \frac{z - 1}{T}$$

Forward difference mapping

$$\nabla^{(1)}[y[n]] = \frac{y[n+1] - y[n]}{T}$$

Mapping differentials: A summary

Given a desired analog filter $H_a(s)$, two mapping functions

$$s = \frac{1 - z^{-1}}{T} \quad \text{and} \quad s = \frac{z - 1}{T}$$

may then be substituted to yield a corresponding digital filter H(z). The procedure is simple and straightforward

Desirable properties of mapping functions:

(a) The $j\omega'$ axis ($s = j\omega'$) in the s-plane should be mapped on to the unit circle in the z-plane. This preserves the frequency selective properties of analog filters.

(b) Points in the LHS of the s-plane ($Re(s) < 0$) should be mapped to inside the unit circle ($|z| < 1$). Stability property will be preserved

Evaluation of difference mapping functions:

Backward difference: $s = \dfrac{1 - z^{-1}}{T}$ or $z = \dfrac{1}{1 - sT}$

Now we can show: let $s = j\omega'$

(1) $$z = \frac{1}{1 - j\omega' T} = \frac{1}{2}\left(1 + \frac{1 + j\omega' T}{1 - j\omega' T}\right)$$

$$= \frac{1}{2}\left(1 + e^{j 2 \tan^{-1} \omega' T}\right)$$

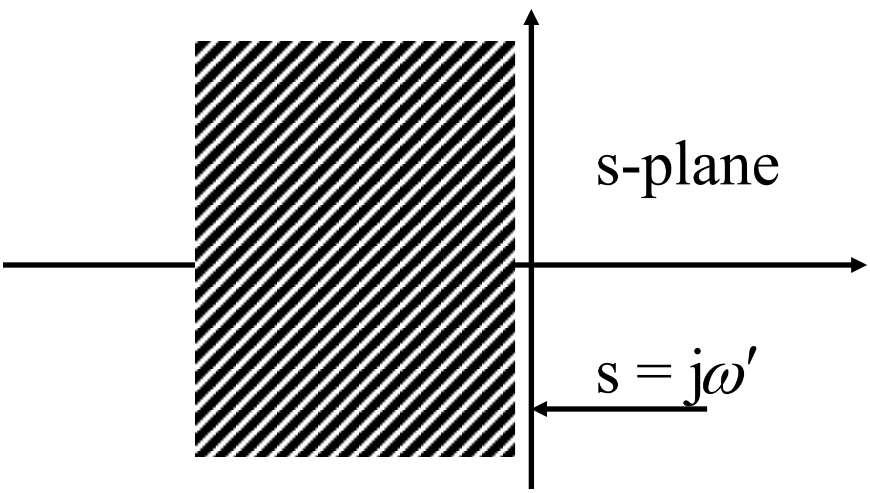
(2) $$\mathrm{Re}(z) = \frac{1}{2} + \frac{\cos(2 \tan^{-1}(\omega' T))}{2}$$

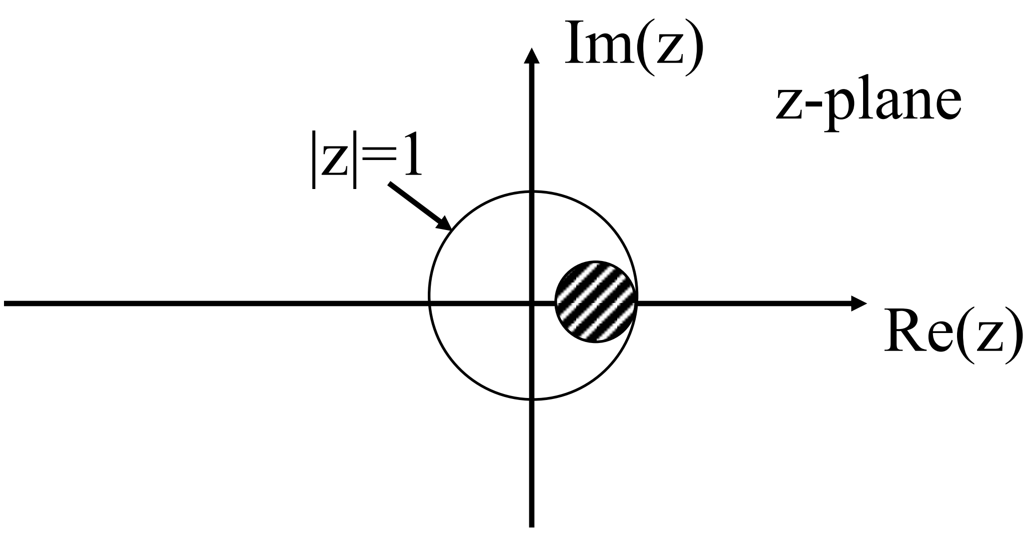
(3) $$\mathrm{Im}(z) = \frac{1}{2} \sin(2 \tan^{-1}(\omega' T))$$

(4) $$\{\mathrm{Re}(z) - \frac{1}{2}\}^2 + \{\mathrm{Im}(z)\}^2 = (\frac{1}{2})^2$$

(5) with $sT = \alpha + j\beta$, $\alpha < 0$,

$$|z| = \frac{1}{\sqrt{(1-\alpha)^2 + \beta^2}} < 1$$

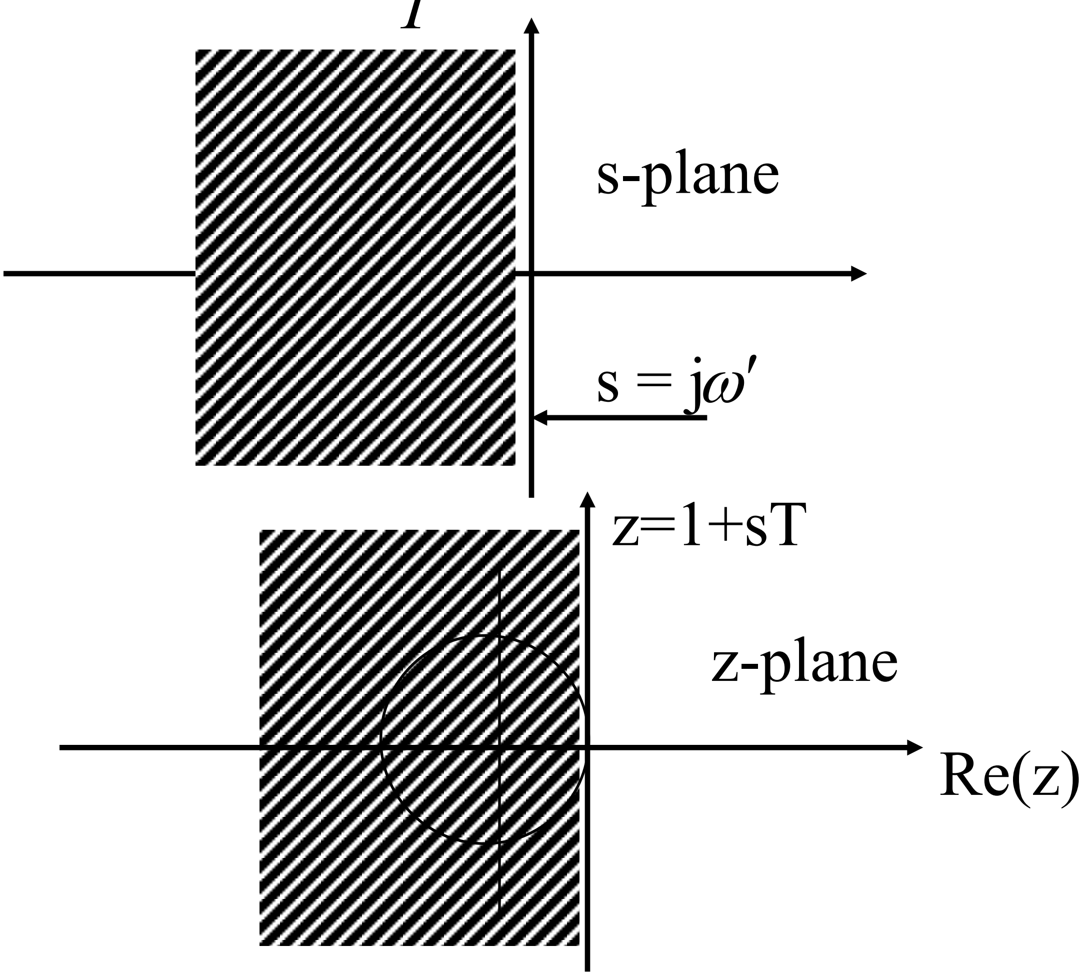




Conclusion: Mapping function does not in general satisfy the desired properties except for extremely small sampling times.

Forward difference operator:

$$s = \frac{z-1}{T} \qquad z = 1 + sT$$



Conclusion: Some pole locations in s-plane may be mapped to locations outside the unit circle. Generally this mapping is not recommended. Extremely small sampling times may provide satisfactory results