Infinite Impulse Response (IIR)

Digital Filters (II)

Impulse-Invariance mapping

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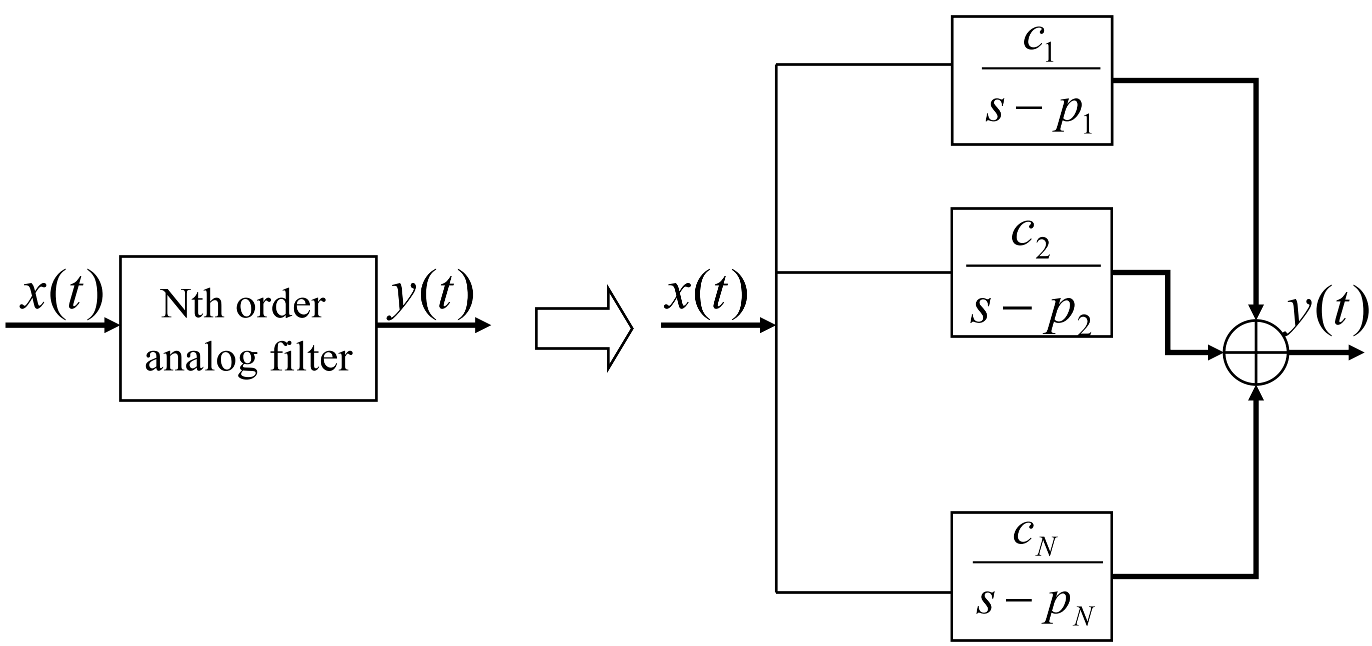
Impulse-invariance design problem

Given the impulse response of a desired analog filter: $h(t) \longleftrightarrow H(s)$, design a digital filter $H(z)$ such that:

$$h[n] = h(t)\Big|_{t=nT} = h(nT)$$

$$\text{and} \quad h[n] \leftrightarrow H(z)$$

Consider an analog filter: H(s)



Follows:

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

Then:

$$h(t) = \sum_{k=1}^{N} c_k e^{p_k t}$$

<u>Digital Filter</u>

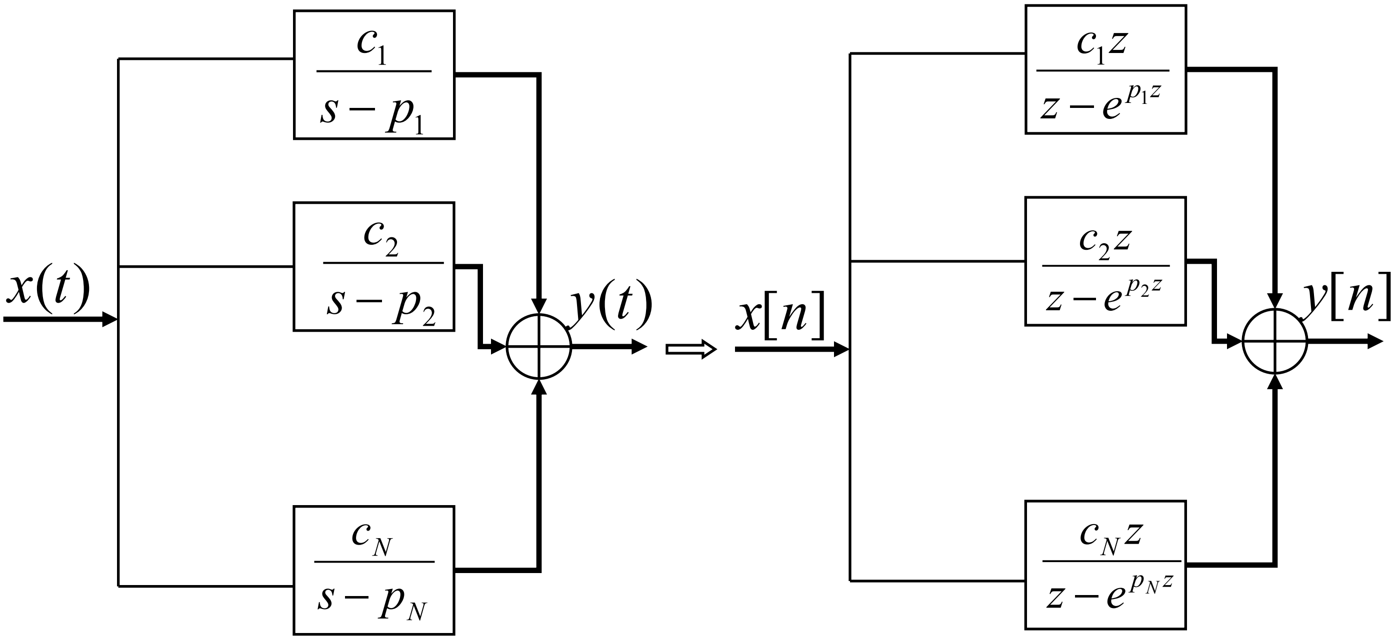
$$h[n] = h(nT) = \sum_{k=1}^{N} c_k e^{p_k nT}$$

Then the Z-transform is given as:

$$H(z) = Z\{h[n]\} = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

or

$$H(z) = \sum_{k=1}^{N} \frac{c_k z}{z - e^{p_k T}}$$



Analog filter

Digital filter

Implies:

$$\sum_{k=1}^{N} \frac{c_k}{s-p_k} \leftrightarrow H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}$$

Pole mapping:

Pole $s = p_k$ in the s-plane $\longrightarrow$

Pole $z = e^{p_k T}$ in the z-plane

Example: Approximate an analog integrator with a digital filter. T = 0.01.

$$H(s) = \frac{1}{s}$$

s-plane pole: $s = 0$

z-plane pole: $z = e^{(-0)(0.001)} = 1$

Follows: $$H(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$y[n] = x[n] + y[n-1]$$

<u>Impulse-invariance procedure</u>:

Impulse Invariant Design Method

Third order Butterworth low-pass filter's transfer function:

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

The partial fraction expansion is :

$$H(s) = \frac{1}{(s+1)(s+0.5 - j0.866)(s+0.5+j0.866)}$$

$$= \frac{C_1}{(s+1)} + \frac{C_2}{(s+0.5-j0.866)} + \frac{C_3}{(s+0.5+j0.866)}$$

Algebra gives the coefficients as :

$$= \frac{1}{(s+1)} + \frac{0.577e^{-j2.62}}{(s+0.5-j0.866)} + \frac{0.577e^{j2.62}}{(s+0.5+j0.866)}$$

The three poles are :

$$p_1 = -1, p_2 = -0.5 + j0.866, p_3 = -0.5 - j0.866$$

Therefore, the transfer function can be written as :

$$H(z) \;=\; \frac{1}{(1-e^{-T}z^{-1})}+\frac{1}{(1-e^{(-0.5+j0.866)T}z^{-1})}+\frac{1}{(1-e^{(-0.5-j0.866)T}z^{-1})}$$

Simplifying, we get

$$H(z) \;=\; \frac{z}{(z-e^{-T})}+\frac{-z^{2}-1.154e^{-0.5T}\cos(5\pi/6+0.866T)}{(1-e^{(-0.5+j0.866)T}z^{-1})}$$

This can be further simplified and expressed as :

$$H(z) \;=\; \frac{b_0z^2+b_1z}{z^3+a_1z^2+a_2z+a_3}$$

where

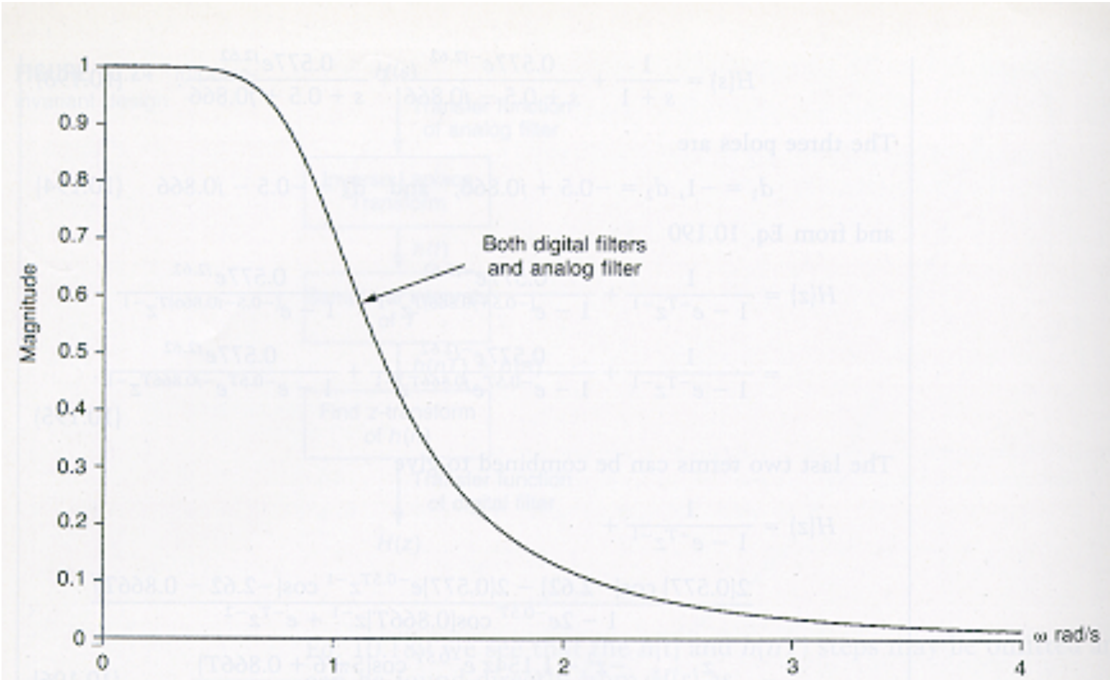
$$b_0=-2e^{-0.5T}\cos(0.866T)+e^{-T}+1.154e^{-0.5T}\cos(5\pi/6+0.866T)$$

$$b_1=e^{-T}+1.154e^{-1.5T}\cos(5\pi/6+0.866T)$$

$$a_1=-(e^{-T}+2e^{-0.5T}\cos(0.866T))$$

$$a_2=e^{-T}+2e^{-0.5T}\cos(0.866T)$$

$$a_3=-e^{-2T}.$$



Example: Pole-mapping illustration
Consider a three-pole low pass Butterworth filter with ω'_p=10 rad/s. Sampling rate of 20 samples/s

This three-pole Butterworth filter has poles at radius 10 and angles of 120, 180 and 240 degrees.

$$p_1 : 10\angle 120^0 = -5 + j8.66$$

$$p_2 : 10\angle 180^0 = -10$$

$$p_3 : 10\angle 240^0 = -5 - j8.66$$

First write:

$$H(s) = \frac{c_1}{s-(-5+j8.66)} + \frac{c_2}{s-(-10)} + \frac{c_3}{s-(-5-j8.66)}$$

Verify:

$$c_1 : 0.00577\angle 210^0$$

$$c_2 : 0.01$$

$$c_3 : 0.00577\angle 150^0$$

Poles:

$$p_1 \text{ in s-plane} \rightarrow e^{(-5+j8.66)(0.05)}$$

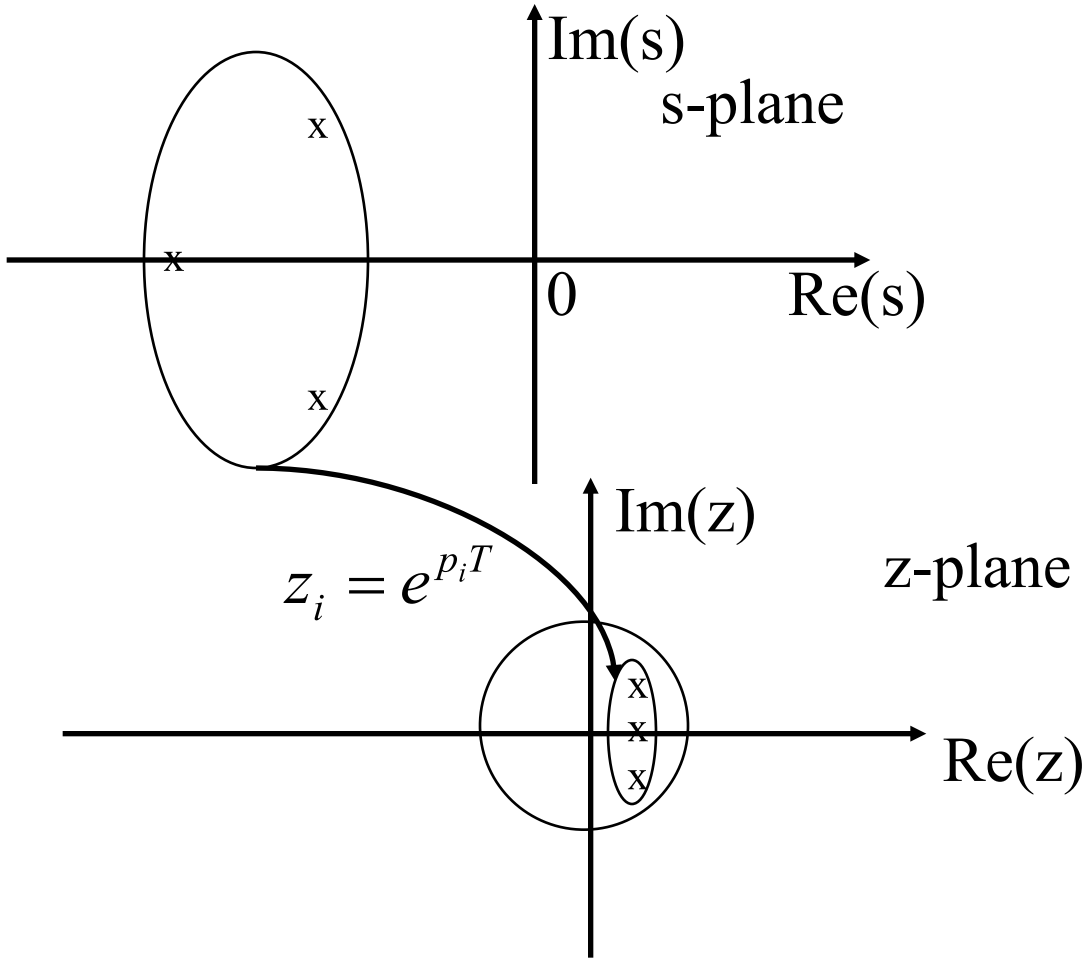
$$= 0.7788\angle 24.8^0 \text{ in z-plane}$$

$$p_2 \text{ in s-plane} \rightarrow e^{(-10)(0.05)}$$

$$= 0.6065 \text{ in z-plane}$$

$$p_3 \text{ in s-plane} \rightarrow e^{(-5-j8.66)(0.05)}$$

$$= 0.7788\angle -24.8^0 \text{ in z-plane}$$



$$H(z) = \frac{0.006\angle 210^0}{1 - z^{-1}(0.779\angle 24.8^0)} + \frac{0.01}{1 - z^{-1}(0.607)} + \frac{0.006\angle 150^0}{1 - z^{-1}(0.779\angle -24.8^0)}$$

Example: Multiple pole

Let

$$H(s) = \frac{c}{(s-p)^2}$$

then $$h(t) = cte^{pt} \qquad t \geq 0$$

$$= 0 \qquad t < 0$$

Follows:
(Digital) $$h[n] = cnTe^{pt} \qquad n \geq 0$$

Now: $$H(z) = \sum_{n=0}^{\infty} cnTe^{pnT} z^{-n}$$

$$= cT \sum_{n=0}^{\infty} n(e^{pT} z^{-1})^n$$

Then $$H(z) = \frac{cTe^{pT} z^{-1}}{(1 - e^{pT} z^{-1})^2}$$

If Re(p) < 0, analog filter is stable, and the pole e^{pT} in the z-plane is inside the unit circle and thus the digital filter is also stable

Evaluation of impulse-invariance mapping:

Pole-mapping: $z_i = e^{p_i T}$

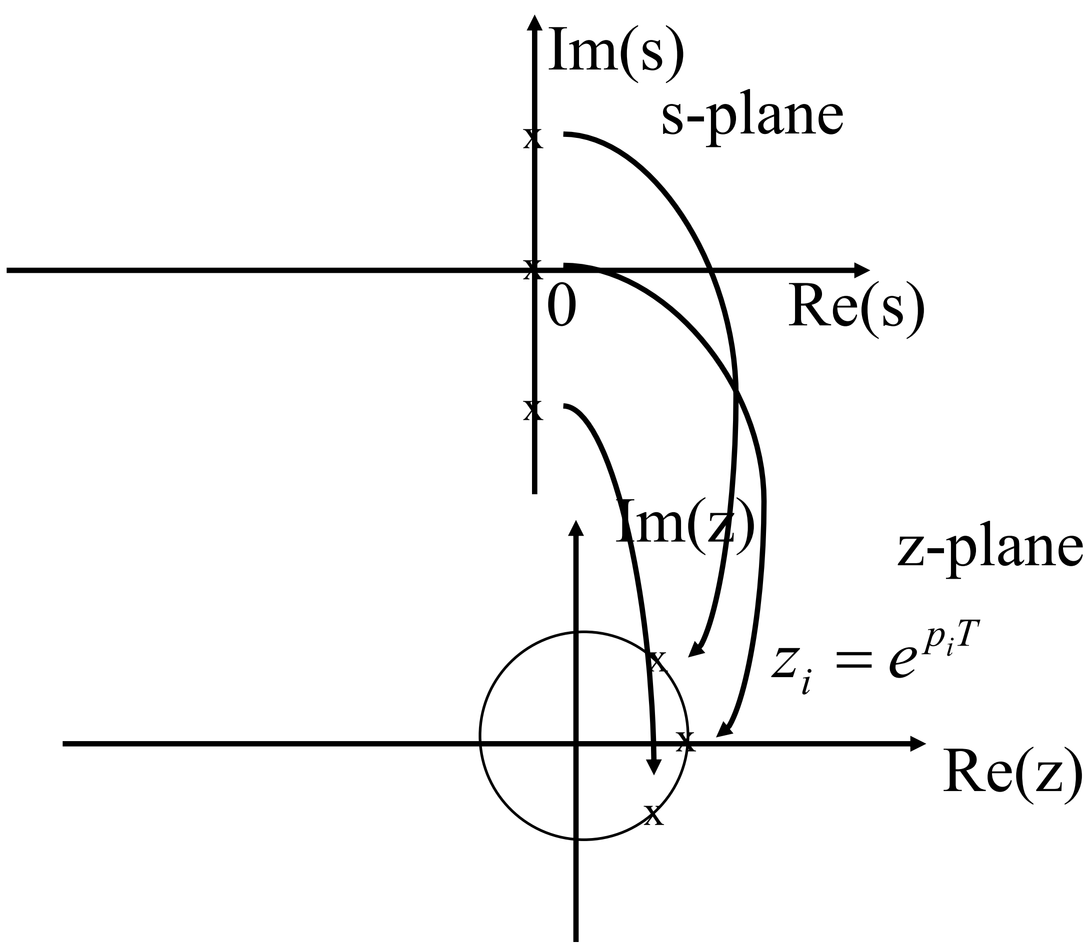
Suppose we consider poles on the $j\omega'$ axis of the s-plane. Let

$$p_i = \mathrm{Re}(p_i) + j\,\mathrm{Im}(p_i)$$
$$= \sigma_i + j\omega_i'$$

If σ= 0 (imaginary axis), then

$$z_i = e^{\sigma_i T} e^{j\omega_i' T} = e^{0T} e^{j\omega_i' T}$$

$$\text{then } \left| z_i \text{ (pole in the z - plane)} \right| = \left| e^{j\omega_i' T} \right| = 1 \quad \text{all } \omega_i'$$

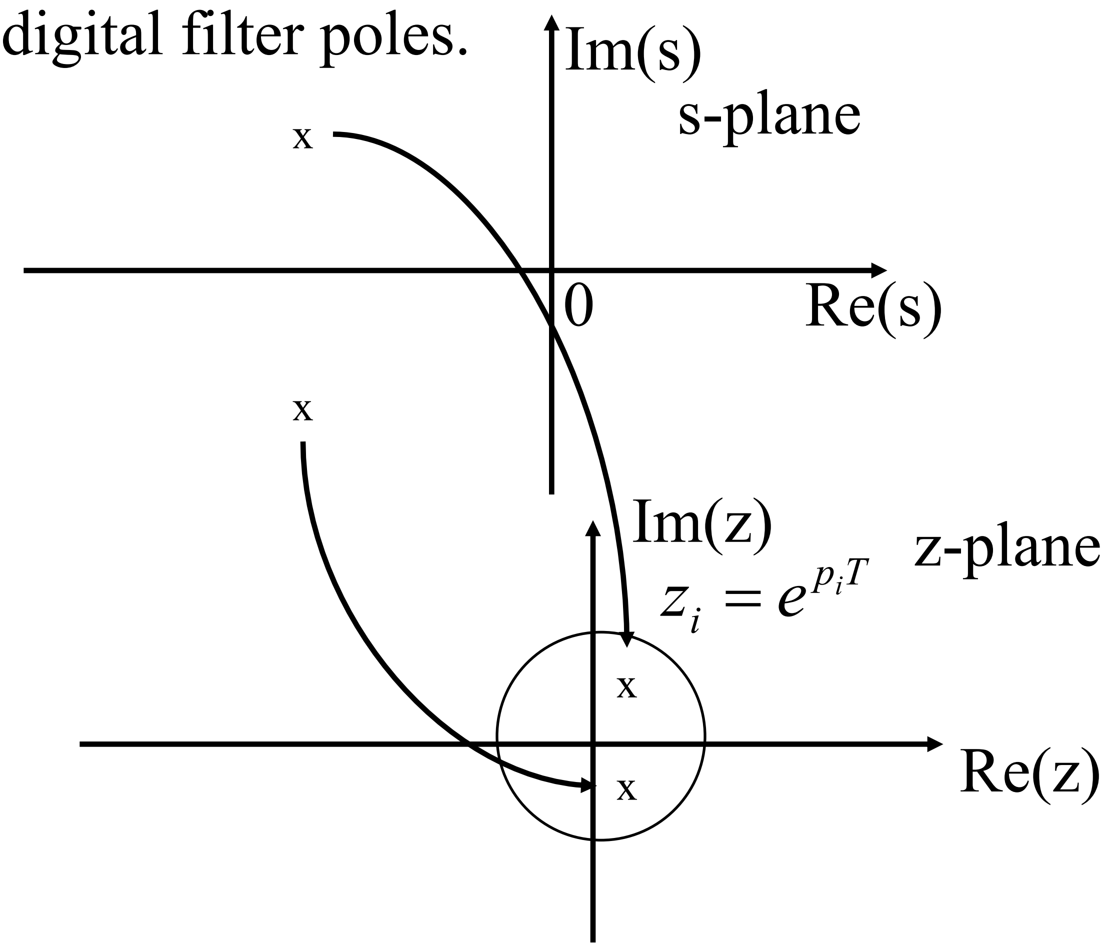


Thus, the impulse invariant mapping maps poles from the s-plane's $j\omega'$ axis to the z-plane's unit circle.

Consider stable analog poles:

$$\text{If } \mathrm{Re}(p_i) = \sigma'_i < 0$$
$$\text{then}: \left|z_\mathrm{i}\right| = \left|e^{\sigma_i T}\right| \left|\cancelto{1}{e^{j\omega'_i T}}\right| < 1$$

All s-plane poles with negative real parts map to z-plane poles inside the unit circle. In other words, stable analog poles are mapped to stable digital filter poles.



If $\sigma > 0$, $e^{\sigma T} > 1$, all poles on the right half of the s-plane map to digital poles outside the unit circle.

Conclusion: The impulse invariant mapping preserves the stability of the filter.

Deficiency of the impulse-invariance mapping:

Consider two poles in the s-plane,

$$\text{s-pole}: p_1 = \sigma + j\omega'$$

$$\text{s-pole}: p_2 = \sigma + j(\omega' + \frac{2\pi}{T}) \quad \text{same real part}$$

$$\text{z-pole}: z_1 = e^{\sigma T} e^{j\omega' T}$$

$$\text{z-pole}: z_2 = e^{\sigma T} e^{j(\omega' + \frac{2\pi}{T})T} = e^{\sigma T} e^{j\omega' T}$$

Two s-plane poles map to the same location in the z-plane. If s-plane poles have the same real parts and imaginary parts that differ by some integer multiples of $2\pi/T$, then there are an infinite number of s-plane poles that map to the same location in the z-plane. This will result in aliased poles in the z-plane.

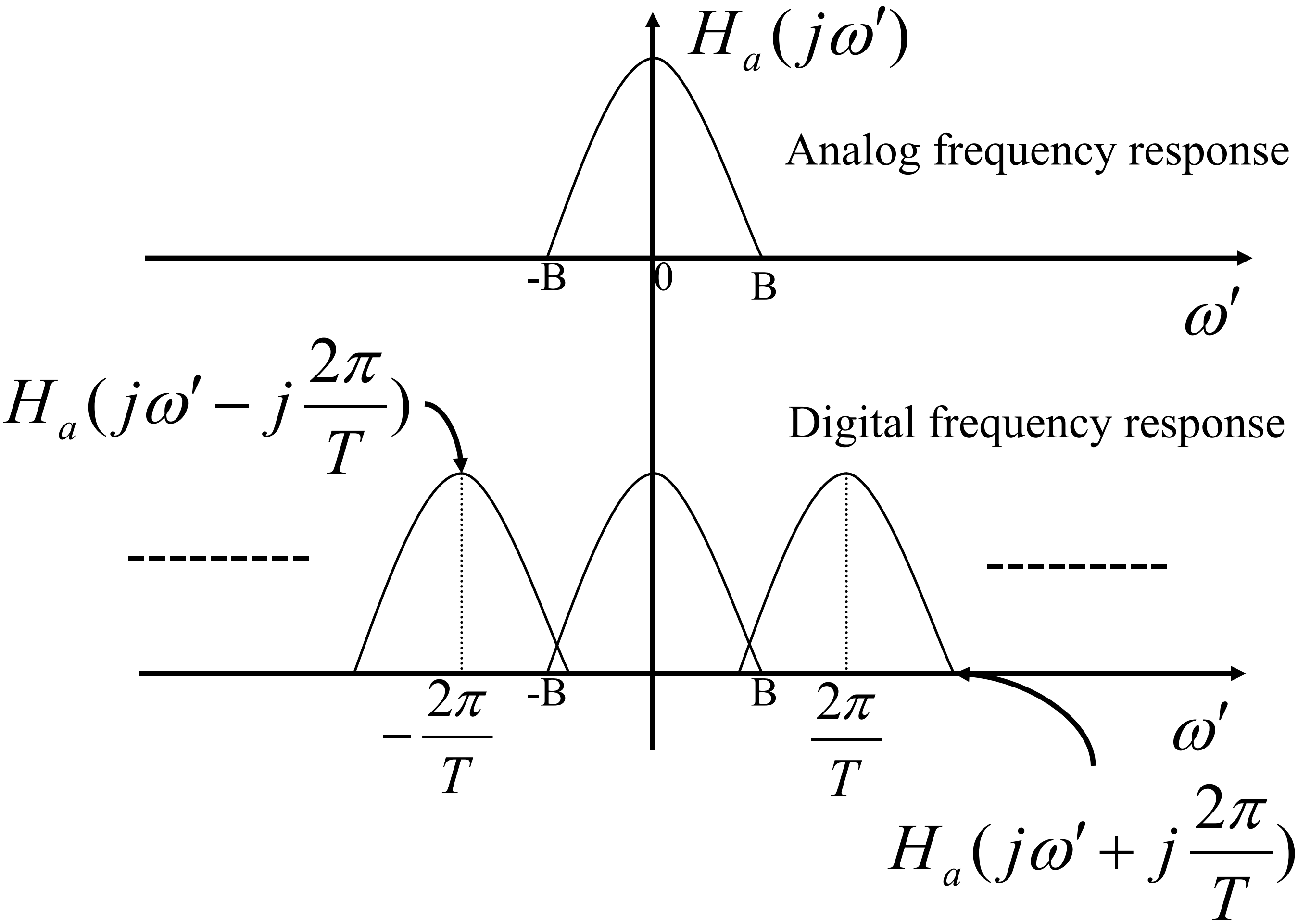
Relate analog frequency response $H_a(j\Omega)$ with digital frequency response $H(e^{j\Omega T})$

$$h[n] = h(nT) = h_a(t)\Big|_{t=nT}$$

$$H(e^{j\omega' T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(j\omega' + jk\frac{2\pi}{T})$$

Digital frequency response reproduces the analog frequency response every 2π/T.

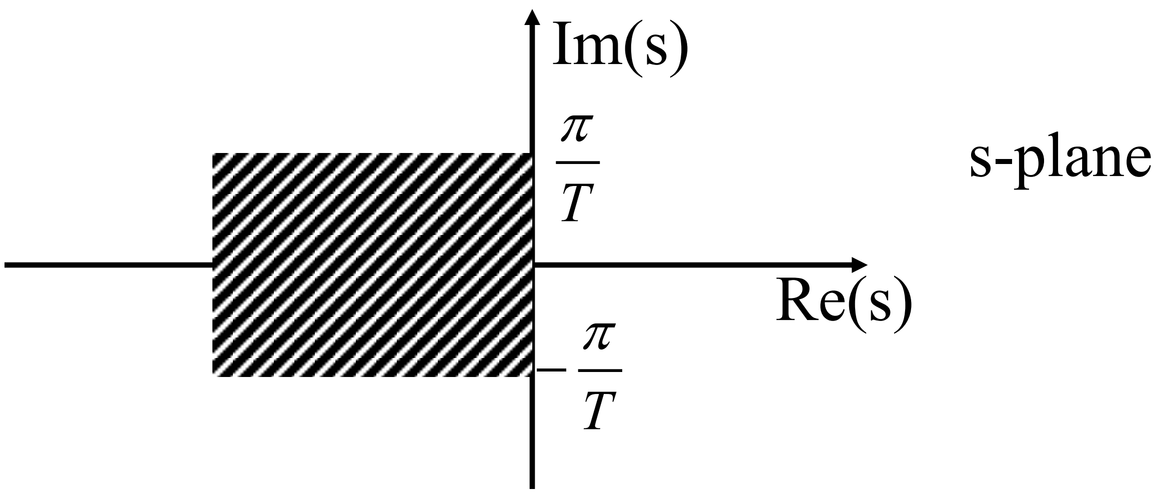
$$\text{Rewrite: } H(e^{j\omega' T}) = \frac{1}{T}H_a(j\omega') + \frac{1}{T}H_a(j\omega' \pm j\frac{2\pi}{T}) + ...$$

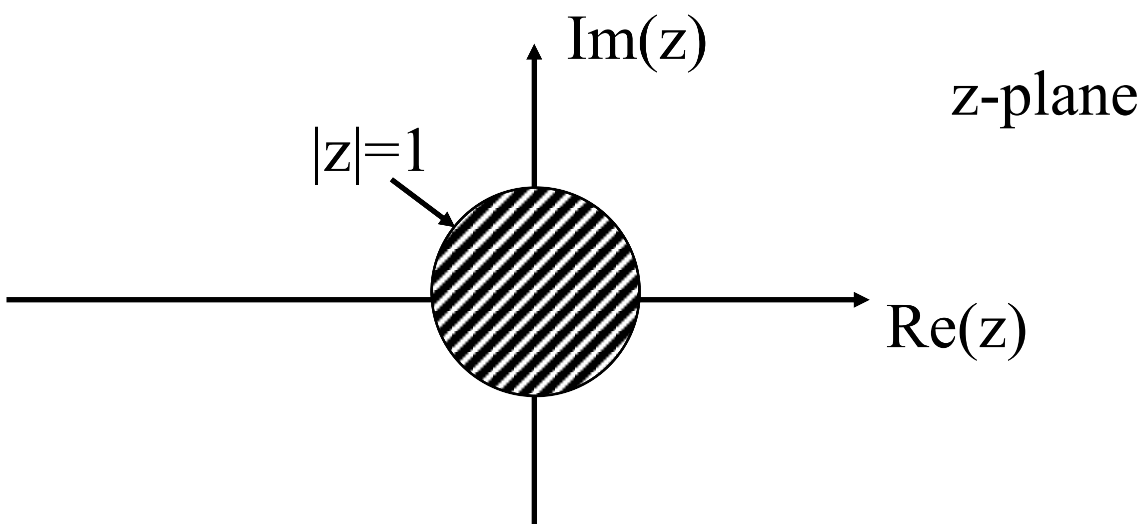


For lower sampling rates (larger T's) the digital frequency response can be an aliased version of the analog frequency response.

To avoid aliasing, restrict the pole mapping to the primary strip

$$-\frac{\pi}{T} \leq \omega' \leq \frac{\pi}{T}$$





One can prevent the aliasing by making the primary strip wide enough (small enough T) to include all the analog filter poles. Thus each horizontal strip of the s-plane of width $2\pi/T$ is mapped onto the entire z-plane.

To prevent significant aliasing:

$$H_a(j\omega') \approx 0 \quad \text{for} \quad |\omega'| > \pi / T$$

Bandlimit the analog filter.

Examples

Problem 1.

Let $$H(s) = \frac{A}{s+\alpha}$$

be a causal first-order analog transfer function. Show that the Causal first-order Digital transfer function H(z) obtained from H(s) via the impulse invariance method is given by:

$$H(z) = \frac{A}{1-e^{-\alpha T} z^{-1}}$$

Solution:

Given,

$$H(s) = \frac{A}{s+\alpha}$$

From H(s), $h_a(t) = Ae^{-\alpha t}\mu(t)$

where $\mu(t)$ is unit step function

Hence,

$$h[n] = h_a(nT) = Ae^{-\alpha nT}\mu(n).$$

$\Rightarrow$

$$H(z) = A\sum_{n=0}^{\infty} e^{-\alpha nT} z^{-n} = \frac{A}{1-e^{-\alpha T} z^{-1}} \quad ; \quad |e^{-\alpha T}| < 1$$

Problem 2.

Let $$H(s) = \frac{s+\beta}{\lambda^2 + (s+\beta)^2}$$

be a causal second-order analog transfer function. Show that the Causal second-order Digital transfer function H(z) obtained from H(s) via the impulse invariance method is given by:

$$H(z) = \frac{z^2 - ze^{-\beta T}\cos(\lambda T)}{z^2 - 2ze^{-\beta T}\cos(\lambda T) + e^{-2\beta T}}$$

Solution:

Given,

$$H(s) = \frac{s+\beta}{\lambda^2 + (s+\beta)^2} = \frac{1}{2}\frac{1}{(s+\beta+j\lambda)} + \frac{1}{2}\frac{1}{(s+\beta-j\lambda)}$$

$$H(z) = \frac{1}{2}\left(\frac{1}{(1 - e^{-(\beta+j\lambda)T}z^{-1})} + \frac{1}{(1 - e^{-(\beta-j\lambda)T}z^{-1})}\right)$$

$$= \frac{1}{2}\left(\frac{1 - e^{-\beta T}e^{j\lambda T}z^{-1} + 1 - e^{-\beta T}e^{-j\lambda T}z^{-1}}{(1 - 2z^{-1}e^{-\beta T}\cos(\lambda T) + e^{-2\beta T}z^{-2})}\right)$$

$$= \frac{1}{2}\left(\frac{1 - z^{-1}e^{-\beta T}\cos(\lambda T)}{(1 - 2z^{-1}e^{-\beta T}\cos(\lambda T) + e^{-2\beta T}z^{-2})}\right)$$

$$= \frac{z^2 - ze^{-\beta T}\cos(\lambda T)}{z^2 - 2ze^{-\beta T}\cos(\lambda T) + e^{-2\beta T}}$$

Problem 3.

Given causal IIR digital transfer functions. Determine their respective parent causal analog transfer functions, T = 0.3 sec.

a) $H(z) = \dfrac{2}{1-e^{-0.9}z^{-1}} + \dfrac{3}{1-e^{-1.2}z^{-1}}$ and $T = 0.3$

Solution:

We have, from problem 1,

$$H(\mathrm{z}) = A\sum_{n=0}^{\infty} e^{-\alpha nT} z^{-n} = \frac{A_1}{1-e^{-\alpha T}z^{-1}} + \frac{A_2}{1-e^{-\beta T}z^{-1}} \quad (\text{For } n = 2)$$

Comparing $G_a(z)$ with :

$$H(z) = \frac{2}{1-e^{-\alpha T}z^{-1}} + \frac{3}{1-e^{-\beta T}z^{-1}} \text{ where } T = 0.3$$

we get,

$\alpha = 3$ and $\beta = 4$

Therefore,

$$H(s) = \frac{2s}{s+3} + \frac{3s}{s+4}$$

b) $$H(z) = \frac{z^2 - ze^{-0.6}\cos(0.9)}{z^2 - 2ze^{-0.6}\cos(0.9) + e^{-1.2}}$$

From problem 2, we have that when

$$H(z) = \frac{z^2 - ze^{-\beta T}\cos(\lambda T)}{z^2 - 2ze^{-\beta T}\cos(\lambda T) + e^{-2\beta T}}$$

then H(s) is given as:

$$H(s) = \frac{s + \beta}{\lambda^2 + (s + \beta)^2}$$

Comparison gives that:

$$\beta T = 0.6 \quad \text{and} \quad \lambda T = 0.9$$

With $T = 0.3$, we get

$$\beta = 1/3 \quad \text{and} \quad \lambda = 3$$

Therefore,

$$H(s) = \frac{s + \frac{1}{3}}{(s + \frac{1}{3})^2 + 9}$$