Infinite Impulse Response (IIR)

Digital Filters (III)

Bilinear mapping

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- Mapping of integrals

Consider an analog integrator:

$$H(s) = \frac{1}{s}$$

$$\Longrightarrow \quad y(t) = \int_0^t x(\tau)d\tau$$

Consider the nth sample:

$$y(nT) = \int_0^{nT} x(\tau)d\tau$$

Consider now the (n+1)th sample:

$$y(nT + T) = \int_0^{nT+T} x(\tau)d\tau$$

$$= \underbrace{\int_0^{nT} x(\tau)d\tau}_{\text{nth sample}} + \int_{nT}^{nT+T} x(\tau)d\tau$$
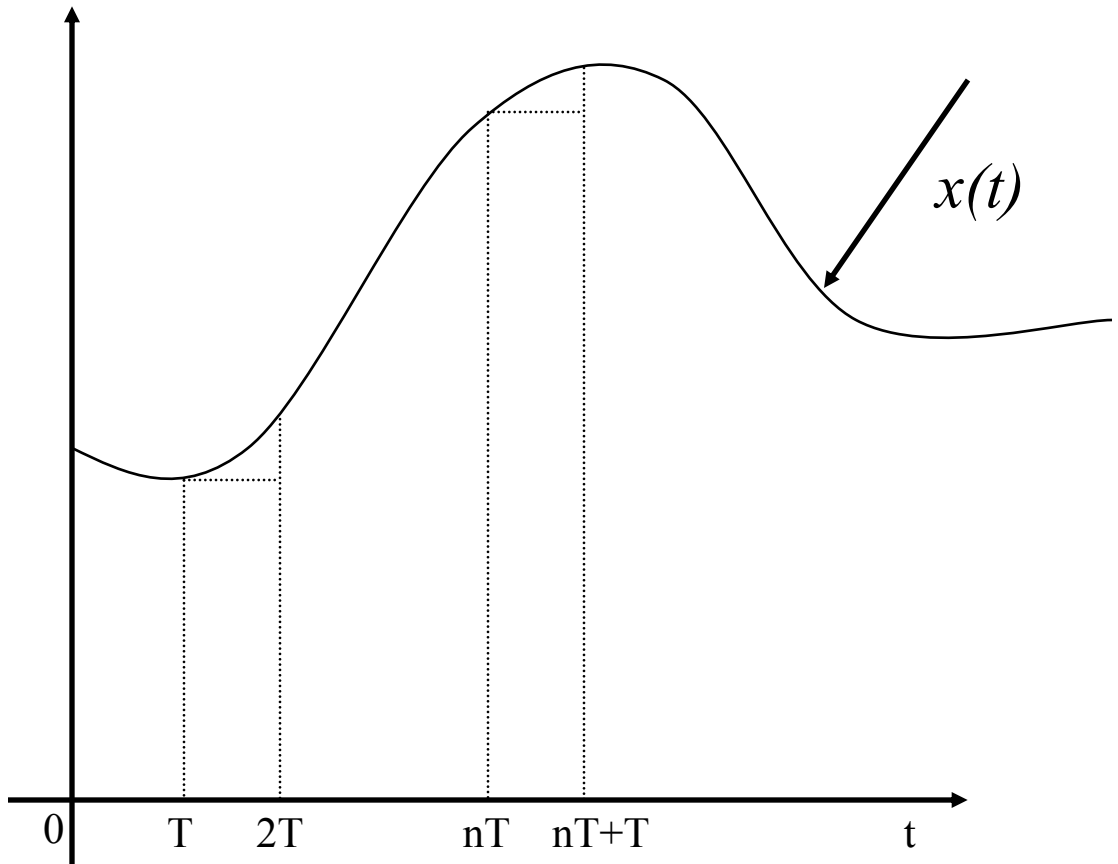
nth sample

We now have a recursion formula:

$$y(nT + T) = y(nT) + \int\limits_{nT}^{nT+T} x(\tau)d\tau$$

(n+1)th sample   nth sample

Consider the term: $\displaystyle\int\limits_{nT}^{nT+T} x(\tau)d\tau$

Assume trapezoidal integration:

x(t)

0   T   2T   nT   nT+T   t

Follows:

$$\int_{nT}^{nT+T} x(\tau)d\tau = \left[\text{Triangle Area}\right] + \left[\text{Rectangle Area}\right]$$

$$= \frac{1}{2}T\left[x(nT+T) - x(nT)\right] + \left[Tx(nT)\right]$$

$$= \frac{T}{2}\left[x(nT) + x(nT+T)\right]$$

Therefore,

$$y(nT+T) = y(nT) + \frac{T}{2}\left[x(nT) + x(nT+T)\right]$$

Digital Integrator

Z-transform:

$$zY(z) = Y(z) + \frac{T}{2}\left[X(z) + zX(z)\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\frac{z+1}{z-1}$$

- **Conclusion**:

$$H(s) = \frac{1}{s} \qquad \longrightarrow \qquad H(z) = \frac{T}{2}\frac{z+1}{z-1}$$

Implies: s $\longrightarrow$ z mapping

$$s^{-1} = \frac{T}{2}\frac{z+1}{z-1}$$

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

Bilinear mapping

Bilinear transformation: General approach:

$$s = \frac{z-1}{z+1} \qquad \text{(T is ignored)}$$

$$z = \frac{1+s}{1-s}$$

Analog filter mapping:

$$H(z) = H(s)\Big|_{s=\frac{z-1}{z+1}}$$

Evaluation of mapping:

(a) Let $s = \sigma + j\omega'$ then

$$z = \frac{(1+\sigma) + j\omega'}{(1-\sigma) - j\omega'}$$

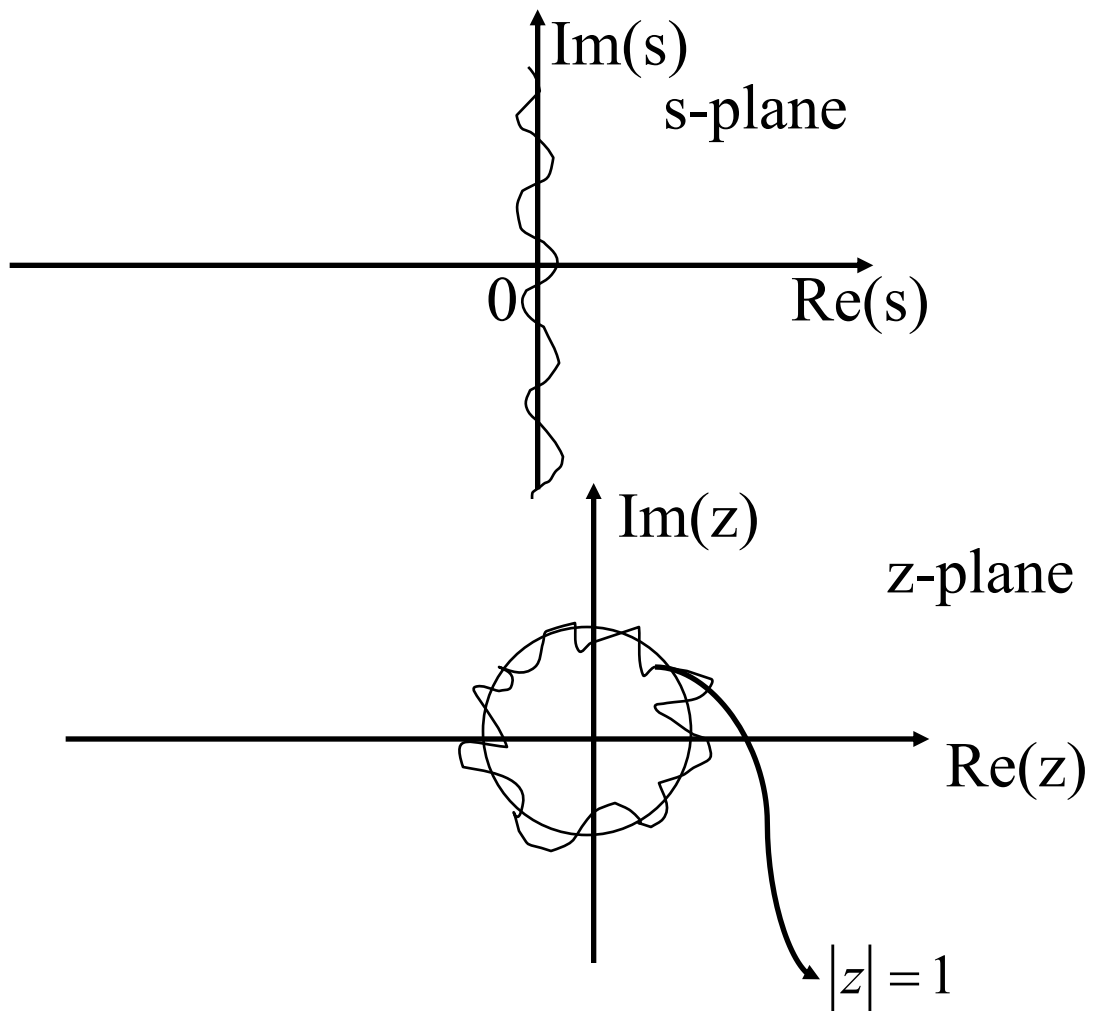Let also $z = re^{j\psi}$ polar form in the z-plane

then

$$|z| = r = \frac{\sqrt{(1+\sigma)^2 + \omega'^2}}{\sqrt{(1-\sigma)^2 + \omega'^2}}$$

$$\psi = \tan^{-1}(\frac{\omega'}{1+\sigma}) - \tan^{-1}(\frac{-\omega'}{1-\sigma})$$

$$\psi = \tan^{-1}(\frac{\omega'}{1+\sigma}) + \tan^{-1}(\frac{\omega'}{1-\sigma})$$

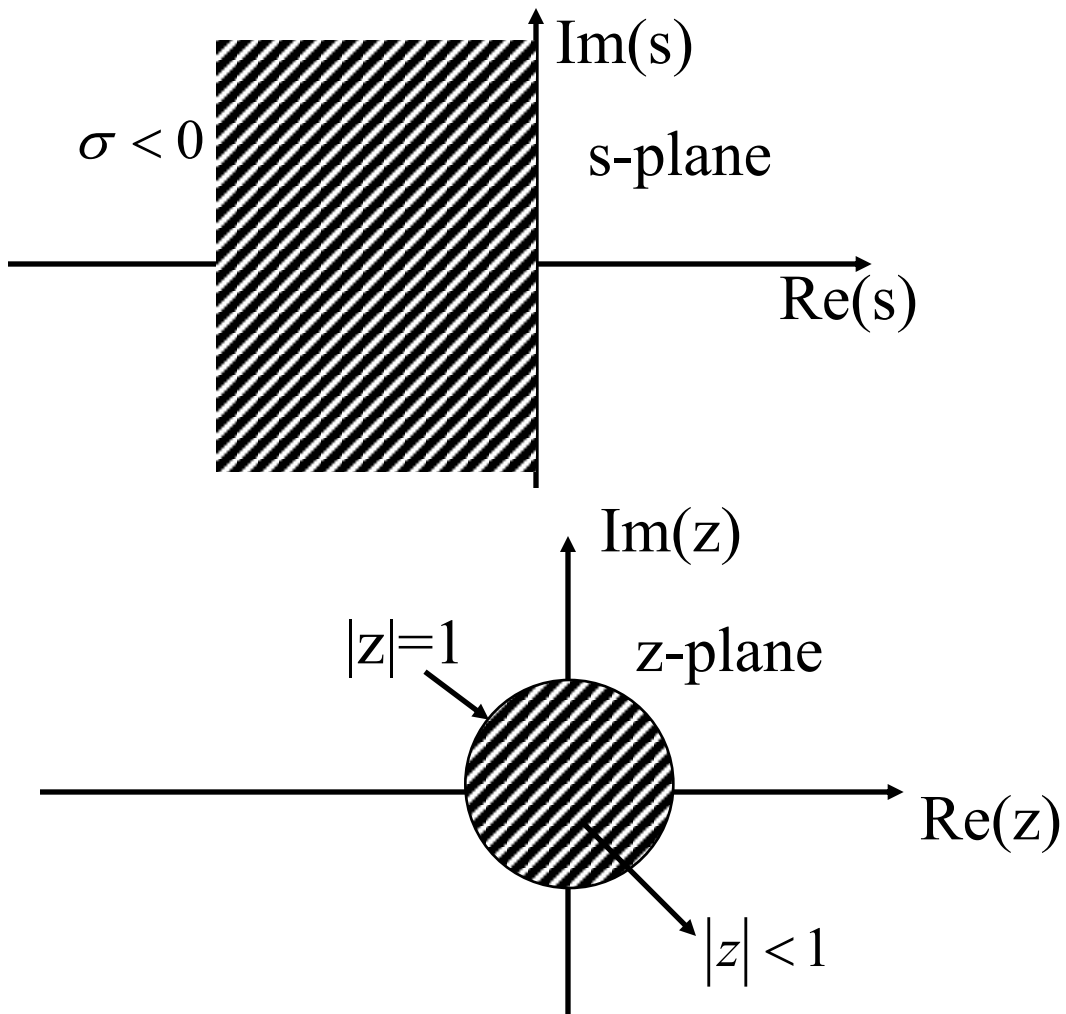(b) Consider the points on the imaginary axis of the s-plane for all ω'.

$$\sigma = 0 \Rightarrow |z| = r = 1 \qquad \text{all } \omega'$$

(c) Consider the points in the left-half of the s-plane.

$$\text{Re}(s) = \sigma < 0$$

then $|z| < 1$ all ω'
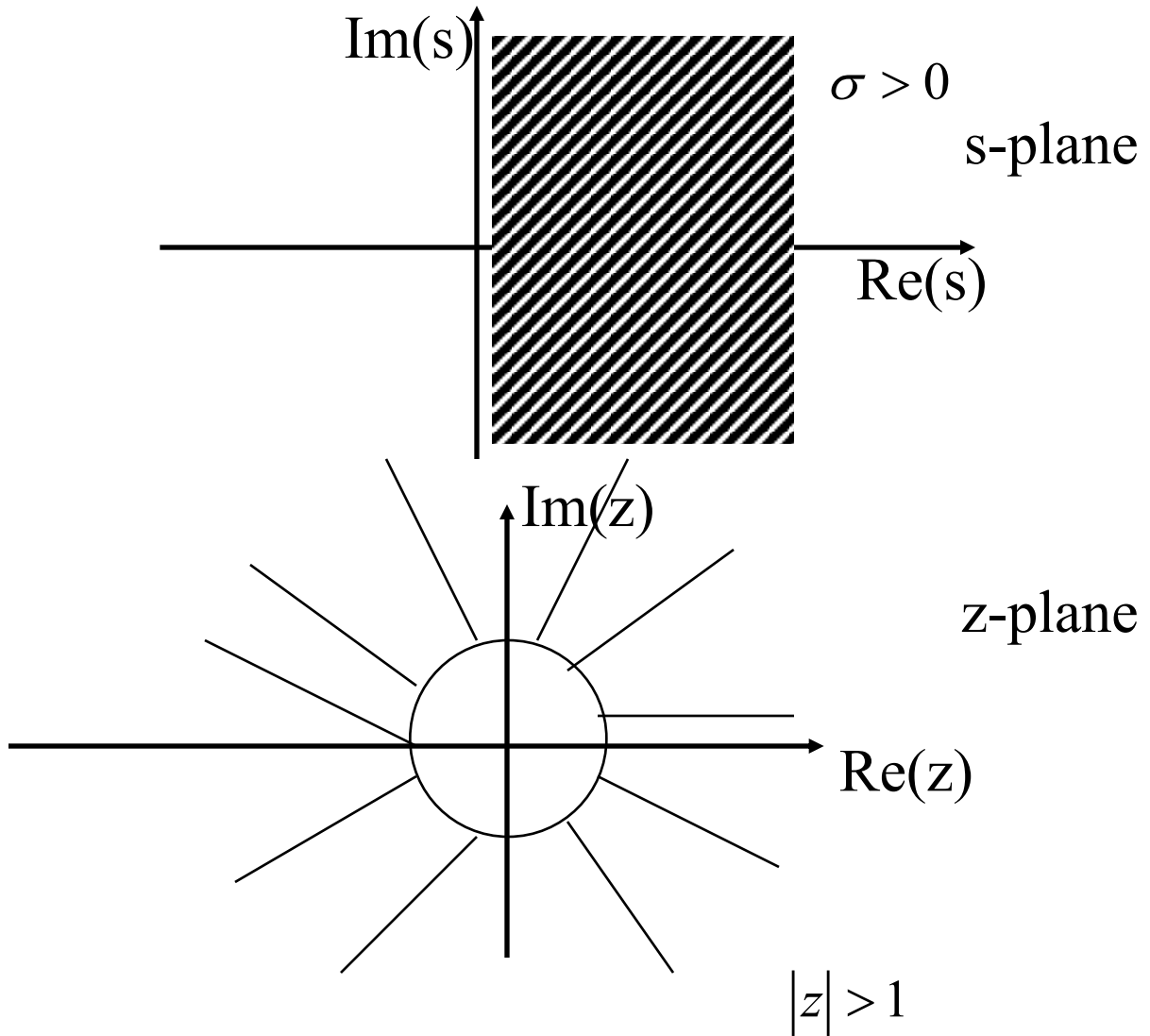

Stable poles are preserved.

(d) Consider the points in the right-half of the s-plane.

$$\text{Re}(s) = \sigma > 0$$

then $|z| > 1$ all ω'

Im(s)

$\sigma > 0$

s-plane

Re(s)

Im(z)

z-plane

Re(z)

$|z| > 1$

Conclusion: Stable analog filters are mapped from s-plane to z-plane as stable digital filters. Aliasing problem is eliminated.

Analog and Digital frequencies relationship.

Consider

$$s = \frac{z-1}{z+1}$$

$$j\omega'_p = \frac{e^{j\omega_p T} - 1}{e^{j\omega_p T} + 1} = \frac{e^{j\omega_p T/2}(e^{j\omega_p T/2} - e^{-j\omega_p T/2})}{e^{j\omega_p T/2}(e^{j\omega_p T/2} + e^{-j\omega_p T/2})}$$

$$= \frac{j\sin(\omega_p T/2)}{\cos(\omega_p T/2)} = j\tan(\omega_p T/2)$$

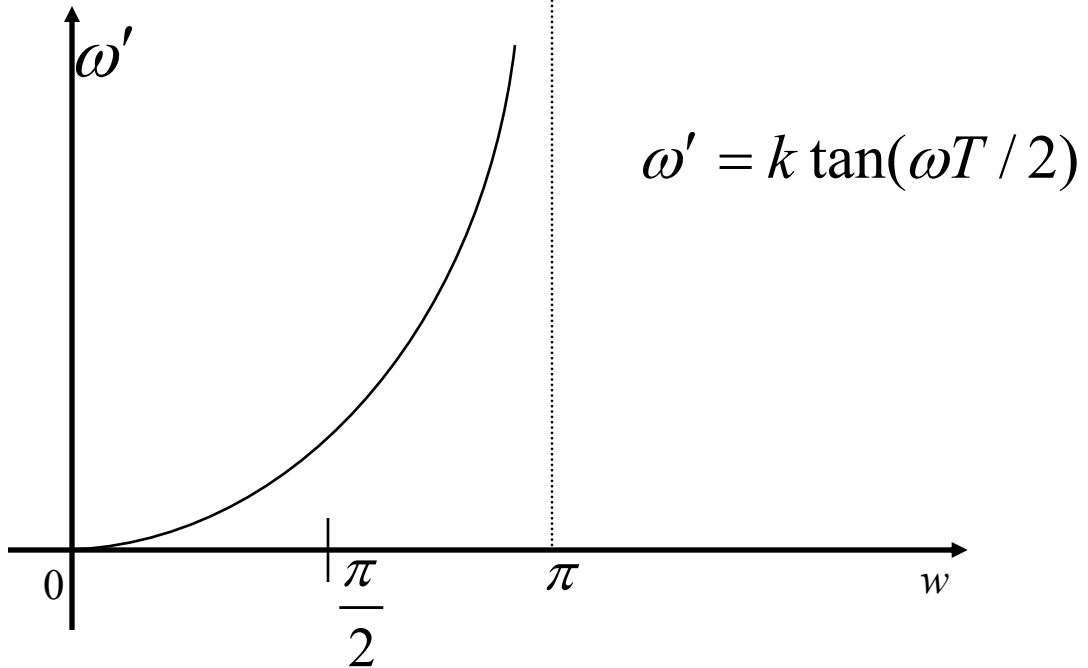$$\boxed{\omega'_p = k\tan(\omega_p T/2)}$$

where ω_p = specified cutoff frequency

ω'_p = prewarped cutoff frequency

k = 1 or 2/T

T = sampling period

Frequency scale distortion:

$$\omega' = k \tan(\omega T / 2)$$

Digital frequencies may result into scaled values different than the desired or specified values except where analog amplitude functions are piecewise constant. This problem is eliminated or at least minimized by prewarping frequency specifications

Summary of Bilinear Transformation method[1]

For standard, frequency-selective IIR filters, the steps for using the BZT may be summarized as follows:

(1) Use the digital filter specifications to find a suitable normalized, prototype, analog lowpass filter, $H(s)$.

(2) Determine and prewarp the bandedge or critical frequencies of the desired filter. For lowpass or highpass filters there is just one bandedge or cutoff frequency (ω_p, say). For bandpass and bandstop filters, we have the lower and upper passband edge frequencies, ω_{p1} and ω_{p2}, each of which needs to be prewarped (the stopband edge frequencies may also be specified):

$$\omega'_p = \tan\left(\frac{\omega_p T}{2}\right) \tag{8.20a}$$

$$\omega'_{p1} = \tan\left(\frac{\omega_{p1} T}{2}\right); \quad \omega'_{p2} = \tan\left(\frac{\omega_{p2} T}{2}\right) \tag{8.20b}$$

(3) Denormalize the analog prototype filter by replacing s in the transfer function, $H(s)$, using one of the following transformations, depending on the type of filter required:

$$s = \frac{s}{\omega'_p} \qquad \text{lowpass to lowpass} \tag{8.21a}$$

$$s = \frac{\omega'_p}{s} \qquad \text{lowpass to highpass} \tag{8.21b}$$

$$s = \frac{s^2 + \omega_0^2}{Ws} \qquad \text{lowpass to bandpass} \tag{8.21c}$$

$$s = \frac{Ws}{s^2 + \omega_0^2} \qquad \text{lowpass to bandstop} \tag{8.21d}$$

where

$$\omega_0^2 = \omega'_{p2}\omega'_{p1}, \quad W = \omega'_{p2} - \omega'_{p1}$$

(4) Apply the BZT to obtain the desired digital filter transfer function, $H(z)$, by replacing s in the frequency-scaled (i.e. denormalized) transfer function, $H'(s)$, as follows:

$$s = \frac{z - 1}{z + 1}$$

Example 1:

Low pass filter: Obtain the transfer function H(z) of the digital low pass filter to approximate the following transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Use Bilinear Transformation method and assume a 3 dB cut off frequency of 150Hz and a sampling frequency of 1.28kHz.

Soln. The critical frequency, $\omega_p = 2\pi \times 150$ rad/s, and F_s =1/T = 1.28kHz, giving a prewarped critical frequency of

$$\omega_p' = \tan(\omega_p T/2) = 0.3857$$

The frequency-scaled analog filter is given by

$$H'(s) = H(s)|_{s=s/\omega_p'} = \frac{1}{(s/\omega_p')^2 + \sqrt{2}s/\omega_p' + 1}$$

$$= \frac{\omega_p'^2}{s^2 + \sqrt{2}\omega_p's + \omega_p's} = \frac{0.1488}{s^2 + 0.5455s + 0.1488}$$

Applying the BZT gives

$$H(z) = H'(s)|_{s=\frac{z-1}{z+1}} = \frac{0.0878z^2 + 0.1756z + 0.0878}{z^2 - 1.0048z + 0.3561}$$

$$= \frac{0.0878(1 - 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$

Example 2:

Highpass filter The normalized transfer function of a simple, analog lowpass, resistance–capacitance (RC) filter is given by

$$H(s) = \frac{1}{s+1}$$

Starting from the s-plane equation, determine, using the BZT method, the transfer function of an equivalent discrete-time highpass filter. Assume a sampling frequency of 150 Hz and a cutoff frequency of 30 Hz.

Soln.

The critical frequency for the digital filter is $\omega_p = 2\pi \times 30$ rad/s. The cutoff frequency, after prewarping, is $\omega'_p = \tan(\omega_p T/2)$. With $T = 1/150$ Hz, $\omega'_p = \tan(\pi/5) = 0.7265$.

Using the lowpass-to-highpass transformation of Equation 8.21a, the denormalized analog transfer function is obtained as

$$H'(s) = H(s)\big|_{s = \omega'_p/s} = \frac{1}{\omega'_p/s + 1} = \frac{s}{s + 0.7265}$$

The z-plane transfer function is obtained by applying the BZT:

$$H(z) = H'(s)\big|_{s = (z-1)/(z+1)} = \frac{(z-1)/(z+1)}{(z-1)/(z+1) + 0.7265}$$

Simplifying, we have

$$H(z) = 0.5792\, \frac{1 - z^{-1}}{1 + 0.1584 z^{-1}}$$

The coefficients of the discrete-time filter are

$$b_0 = 0.5792, \quad a_1 = 0.1584$$

$$b_1 = -0.5792$$

Example 3:

A simple LRC notch filter has the following normalized, s-plane transfer function:

$$H(s) = \frac{s^2 + 1}{s^2 + s + 1}$$

Determine the transfer function of an equivalent discrete-time filter using the BZT method. Assume a notch frequency of 50 Hz and a sampling frequency of 500 Hz.

Soln.

Since we already have the s-plane transfer function, it is inappropriate to apply a lowpass to bandstop transformation as this would amount to a double transformation. The critical frequency is

$$\omega_p' = \tan\left(\frac{\omega_p T}{2}\right) = \tan\left(\frac{2\pi \times 50}{500 \times 2}\right) = 0.324\,919\,6$$

The frequency-scaled s-plane transfer function is

$$H'(s) = H(s)\Big|_{s \to \frac{s}{\omega_p'}}$$

$$= \frac{\left(\dfrac{s}{\omega_p'}\right)^2 + 1}{\left(\dfrac{s}{\omega_p'}\right)^2 + \dfrac{s}{\omega_p'} + 1}$$

$$= \frac{s^2 + 0.105\,572}{s^2 + 0.324\,919\,6s + 0.105\,572}$$

Apply the BZT:

$$H(z) = H'(s)\Big|_{s \to \frac{z-1}{z+1}}$$

$$= \frac{\left(\dfrac{z-1}{z+1}\right)^2 + 0.105\,572}{\left(\dfrac{z-1}{z+1}\right)^2 + 0.324\,919\,6\left(\dfrac{z-1}{z+1}\right) + 0.105\,572}$$

Use of BZT and Classical Analog Filters

Analog Filters Review

1. Low pass Butterworth filter of order N

Frequency response
$$|H(j\omega')|^2 = \frac{1}{1+(\frac{\omega'}{\omega'_p})^{2N}} \quad (8.24)$$

Poles of a normalized ($\omega'_p = 1$) Butterworth LPF

$$s_k = e^{j\pi(2k+N-1)/2N} = \cos\left[\frac{(2k+N-1)\pi}{2N}\right] + j\sin\left[\frac{(2k+N-1)\pi}{2N}\right]$$
$$k = 1,2,...,N$$

(8.25)

Filter order N
$$N \geq \frac{\log\left(\frac{10^{\frac{A_s}{10}}-1}{10^{\frac{A_p}{10}}-1}\right)}{2\log\left(\frac{\omega'_s}{\omega'_p}\right)} \quad (8.26)$$

Analog Filters Review contd

2. Low pass Chebyshev Type 1 Filter of order N

Frequency response

$$| H(j\omega') |^2 = \frac{K}{1 + \varepsilon^2 C_N^2(\omega'/\omega'_p)}$$

(8.27)

Poles of a normalized ($\omega'_p = 1$) Chebyshev LPF

$$s_k = \sinh(\alpha)\cos(\beta_k) + j\cosh(\alpha)\sin(\beta_k),$$

$$\text{where} \quad \alpha = \frac{1}{N}\sinh^{-1}(\frac{1}{\varepsilon});$$

$$\beta_k = \frac{(2k + N - 1)\pi}{2N}, \, k = 1,2,...,N$$

(8.28)

Filter order N

$$N \geq \frac{\cosh^{-1}\left(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1}\right)}{2\cosh^{-1}\left(\frac{\omega'_s}{\omega'_p}\right)}$$

(8.29)

Analog Filters Review contd

3. Low pass Elliptic Filter of order N

Frequency response

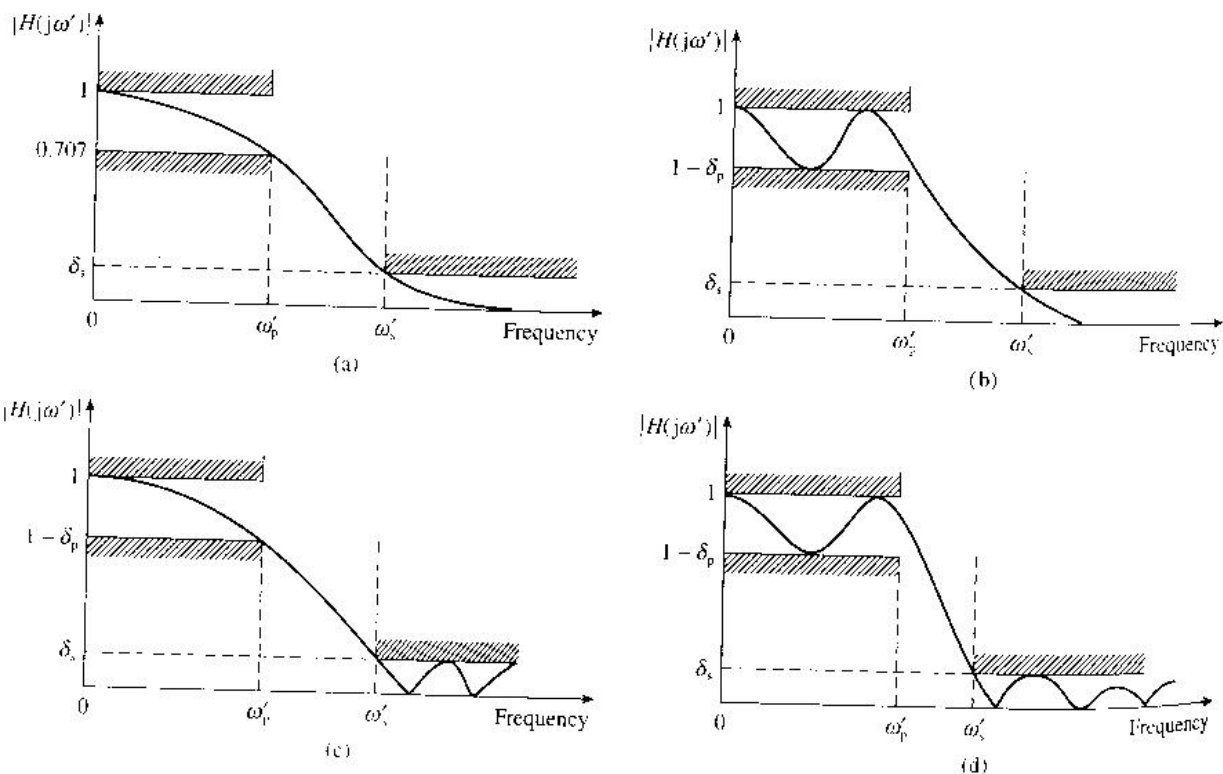$$|H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 G_N^2(\omega')}$$

(8.30)

Fig: Sketches of frequency response of some classical analog filters

(a) Butterworth response (b) Chebyshev type I © Chebyshev type II

(d) Elliptic

Analog Frequency Transformations

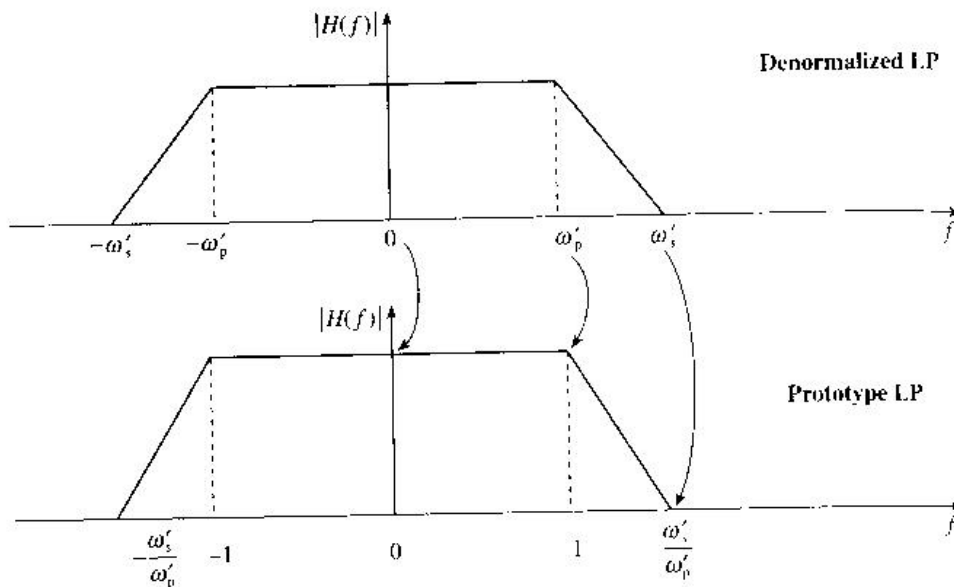1. *Desired Low pass* $\longrightarrow$ *Low pass prototype*

The low pass to low pass transformation is:

$$s = \frac{s}{\omega'_p} \qquad \text{From 8.21 (a)}$$

Denote frequencies for

LPF $\longrightarrow$ ω_{lp} and Prototype $\longrightarrow$ ω^p

$$j\omega^p = j\frac{\omega_{lp}}{\omega'_p} \quad \text{i.e.} \quad \omega^p = \frac{\omega_{lp}}{\omega'_p} \quad (8.31)$$

2. *Desired High pass* → *Low pass prototype*

The low pass to high pass transformation is:

$$s = \frac{\omega'_p}{s} \qquad \text{From 8.21 (b)}$$

Denote frequencies for

HPF ⟶ ω_{hp} and Prototype ⟶ ω^p

$$j\omega^p = -j\frac{\omega'_p}{\omega_{hp}} \quad \text{i.e.} \quad \omega^p = -\frac{\omega'_p}{\omega_{hp}} \qquad (8.32)$$

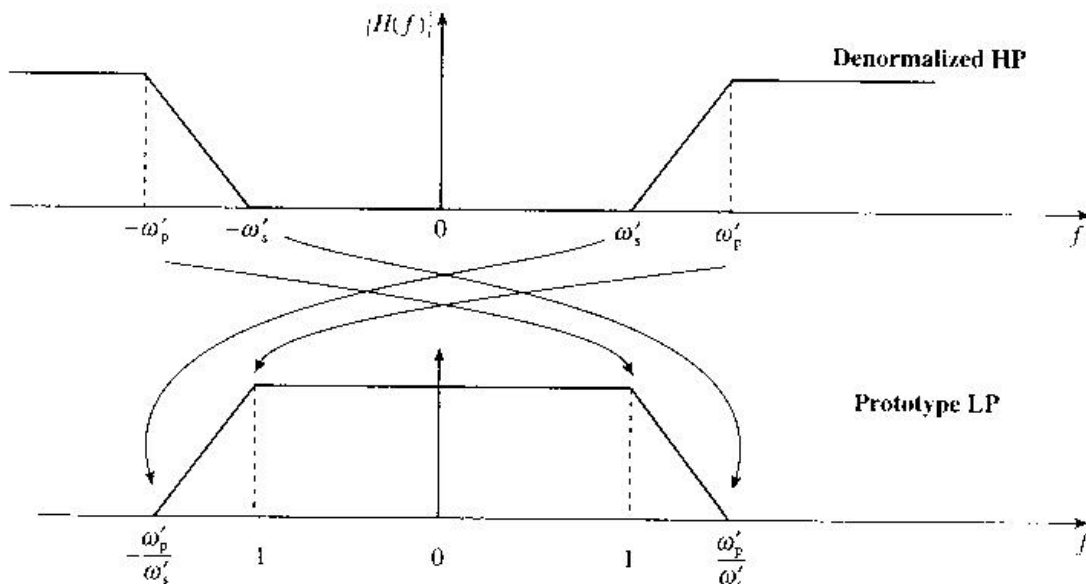## 2. *Desired Band pass* $\longrightarrow$ *Low pass prototype*

The low pass to band pass transformation is:

$$s = \frac{s^2 + \omega_o^2}{Ws} \quad \text{From 8.21 (c)}$$

Denote frequencies for

BPF $\longrightarrow$ ω_{bp} and Prototype $\longrightarrow$ ω^p

$$j\omega^p = \frac{(j\omega_{bp})^2 + \omega_o^2}{jW\omega_{bp}} \quad \text{i.e.} \quad \omega^p = \frac{\omega_{bp}^2 - \omega_o^2}{W\omega_{bp}} \quad (8.33)$$

For a band pass filter,

$\omega'_{p1}, \omega'_{p2}$ = lower and upper pass band edge frequencies

$\omega'_{s1}, \omega'_{s2}$ = lower and upper stop band edge frequencies

ω_0 = center frequency ($\omega_0^2 = \omega'_{p1}\omega'_{p2}$)

From eqn (8.33)

(1) when $\omega_{bp} = \omega'_{s1}$, $\omega^p = \omega'^p_{s1} = \dfrac{\omega'^2_{s1} - \omega^2_0}{W\omega'_{s1}}$

(2) when $\omega_{bp} = \omega'_{p1}$, $\omega^p = \dfrac{\omega'^2_{p1} - \omega^2_0}{W\omega'_{p1}} = \dfrac{\omega'^2_{p1} - \omega'_{p1}\omega'_{p2}}{(\omega'_{p2} - \omega'_{p1})\omega'_{p1}} = -1$

(3) when $\omega_{bp} = \omega'_{p2}$, $\omega^p = \dfrac{\omega'^2_{p2} - \omega^2_0}{W\omega'_{p2}} = \dfrac{\omega'^2_{p2} - \omega'_{p1}\omega'_{p2}}{(\omega'_{p2} - \omega'_{p1})\omega'_{p2}} = 1$
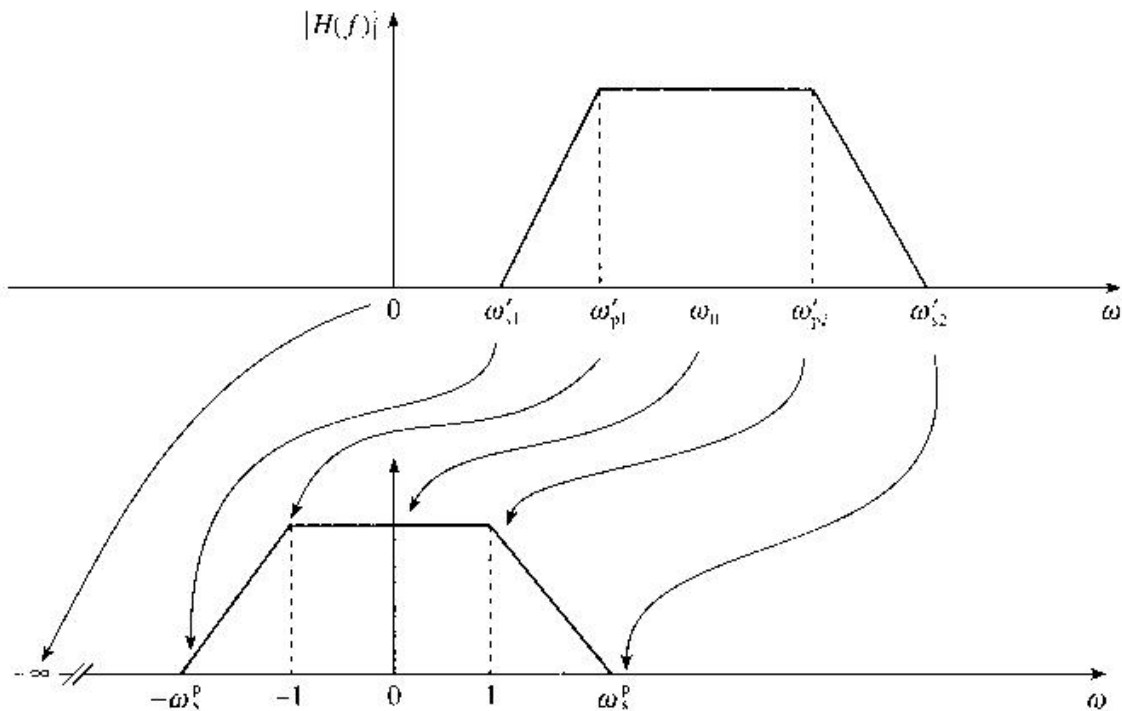
(4) when $\omega_{bp} = \omega'_{s2}$, $\omega^p = \omega'^p_{s2} = \dfrac{\omega'^2_{s2} - \omega^2_0}{W\omega'_{s2}}$

(5) $\omega_{bp} = \omega_0$, $\omega^p = \dfrac{\omega'^2_0 - \omega^2_0}{W\omega^2_0} = 0$

(6) $\omega^p_s = \min(\omega'^p_{s1}, \omega'^p_{s2})$.

Thus, the critical frequencies of interest for the prototype LP filter are

$0, 1, \min(\omega^p_{s1}, |\omega^p_{s2}|)$

2. *Desired Band stop* $\longrightarrow$ *Low pass prototype*

The low pass to band stop transformation is:

$$s = \frac{Ws}{s^2 + \omega_o^2} \qquad \text{From 8.21 (d)}$$

Denote frequencies for

BSF $\longrightarrow$ ω_{bs} and Prototype $\longrightarrow$ ω^p

$$j\omega^p = \frac{jW\omega_{bs}}{(j\omega_{bs})^2 + \omega_o^2} \quad \text{i.e.} \quad \omega^p = \frac{W\omega_{bs}}{\omega_o^2 - \omega_{bs}^2} \quad (8.34)$$

For a band stop filter,

$-\omega'_{p1}, \omega'_{p2}$ = lower and upper pass band edge frequencies

$\omega'_{s1}, \omega'_{s2}$ = lower and upper stop band edge frequencies

ω_0 $\qquad$ = center frequency $(\omega_0^2 = \omega'_{p1}\omega'_{p2})$

$$(1) \quad \text{when } \omega_{bs} = \omega'_{p1}, \ \omega^p = \frac{W\omega'_{p1}}{\omega'^2_0 - \omega'^2_{p1}} = \frac{(\omega'_{p2} - \omega'_{p1})\omega'_{p1}}{\omega'_{p1}\omega'_{p2} - \omega'^2_{p1}} = 1$$

$$(2) \quad \text{when } \omega_{bs} = \omega'_{s1}, \ \omega^p = \omega^{p(1)}_s = \frac{W\omega'_{s1}}{\omega^2_0 - \omega^2_{s1}}$$

$$(3) \quad \text{when } \omega_{bs} = \omega'_{s2}, \ \omega^p = \omega^{p(2)}_s = \frac{W\omega'_{s2}}{\omega^2_0 - \omega^2_{s2}}$$

$$(4) \quad \text{when } \omega_{bs} = \omega_0, \ \omega^p = \frac{W\omega_0}{\omega^2_0 - \omega^2_0} = \infty$$

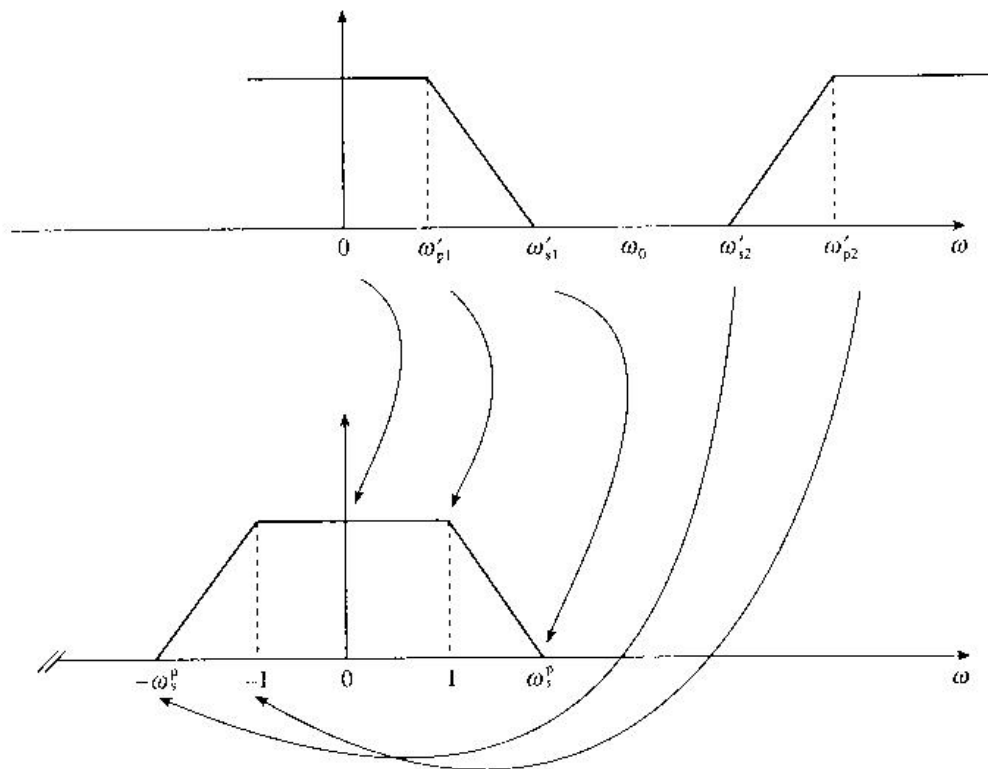$$(5) \quad \text{when } \omega_{bs} = \omega'_{p2}, \ \omega^p = \frac{W\omega'_{p2}}{\omega^2_0 - \omega'^2_{p2}} = \frac{(\omega'_{p2} - \omega'_{p1})\omega'_{p2}}{\omega'_{p1}\omega'_{p2} - \omega'^2_{p2}} = -1.$$

The BZT methodology using classical analog filters

In cases where the prototype lowpass filter does not exist, the stages of the BZT method are:

(1) Prewarp the bandedge or critical frequencies of the digital filter as described previously.

(2) Find a suitable lowpass prototype analog filter, based on the digital filter specifications and one of the classical filter characteristics. This involves using one of the frequency transformation equations (depending on the type of digital filter – LP, HP, BP or BS) in reverse to determine the specifications of the prototype LP filter. From this the order of the prototype filter and hence its transfer function, $H(s)$, are found.

(3) Denormalize the prototype analog LP filter, $H(s)$, by frequency transformation and scaling to obtain a new transfer function, $H'(s)$, as described previously.

(4) Apply the BZT to obtain the desired digital filter transfer function, $H(z)$, by replacing s in the frequency scaled transfer function, $H'(s)$, as described previously.

Design Examples

Lowpass filter   A lowpass digital filter meeting the following specifications is required:

passband	0–500 Hz
stopband	2–4 kHz
passband ripple	3 dB
stopband attenuation	20 dB
sampling frequency	8 kHz

Determine the following:

(1) pass- and stopband edge frequencies for a suitable analog prototype lowpass filter;

(2) order, N, of the prototype lowpass filter;

(3) coefficients, and hence the transfer function, of the discrete-time filter using the bilinear z-transform.

Assume a Butterworth characteristic for the filter.

Solution   (1)   From the specifications, the prewarped frequencies are

$$\omega_p' = \tan\left(\frac{2\pi \times 500}{2 \times 8000}\right) = 0.198\,912$$

$$\omega_s' = \tan\left(\frac{2\pi \times 2000}{2 \times 8000}\right) = 1$$

$$\omega_s^p = \frac{\omega_s'}{\omega_p'} = 1/0.198\,912 = 5.0273$$

Thus, the prewarped pass- and stopband edge frequencies for the prototype LP filter are: 0, 1, 5.0273.

(2)   Next, we use Equation 8.25 and the values of the parameters above to determine the order of the filter.

Now

$$10^{A_s/10} - 1 = 10^{20/10} - 1 = 99; \quad 10^{A_p/10} - 1 = 10^{3/10} - 1 = 0.995\,262\,3;$$

$$\log\left(\frac{99}{0.995\,262\,3}\right) = 1.997\,697$$

For the prototype LPF

$$\omega_p^p = 1; \quad \omega_s^p = 5.0273; \quad \log\left(\frac{\omega_s^p}{\omega_p^p}\right) = 2\log(5.0273) = 1.402\,66.$$

$$N \geqslant \frac{1.997\,697}{1.402\,66} = 1.424. \text{ Let } N = 2$$

(3) The poles of the prototype filter are (from Equation 8.26)

$$s_{p,1} = \cos\left[\frac{(2 + 2 - 1)\pi}{4}\right] + j\sin\left[\frac{(2 + 2 - 1)\pi}{4}\right] = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

$$s_{p,2} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$

The s-plane transfer function, $H(s)$, is

$$H(s) = \frac{1}{(s - s_{p,1})(s - s_{p,2})} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The frequency scaled s-plane transfer function is

$$H'(s) = H(s)\Big|_{\frac{s}{\omega'_p}} = \frac{1}{\left(\frac{s}{\omega'_p}\right)^2 + \sqrt{2}\frac{s}{\omega'_p} + 1}$$

$$= \frac{\omega'^2_p}{s^2 + \sqrt{2}s\omega'_p + \omega'^2_p}$$

Applying the BZT:

$$H(z) = H'(s)\Big|_{\frac{z-1}{z+1}} = \frac{\omega'^2_p}{\left(\frac{z-1}{z+1}\right)^2 + \sqrt{2}\omega'_p\left(\frac{z-1}{z+1}\right) + \omega'^2_p}$$

$$= \frac{\omega'^2_p(z + 1)^2}{(z - 1)^2 + \sqrt{2}\omega'_p(z - 1)(z + 1) + \omega'^2_p(z + 1)^2}$$

After simplification and dividing top and bottom by z^2, we have

$$H(z) = \frac{\omega'^2_p}{1 + \sqrt{2}\omega'_p + \omega'^2_p} \times \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{2(\omega'^2_p - 1)z^{-1}}{1 + \sqrt{2}\omega'_p + \omega'^2_p} + \frac{(1 - \sqrt{2}\omega'_p + \omega'^2_p)z^{-2}}{1 + \sqrt{2}\omega'_p + \omega'^2_p}}$$

Using the values of the parameters

$$1 + \sqrt{2}\omega'_p + \omega'^2_p = 1.320\,87;\ \omega'^2_p - 1 = -0.960\,43$$

$$1 - \sqrt{2}\omega'_p + \omega'^2_p = 0.758\,285\,8;\ \omega'^2_p = 0.039\,565\,9$$

and substituting in the equation above and simplifying, we have

$$H(z) = \frac{0.029\,95(1 + 2z^{-1} + z^{-2})}{1 - 1.4542z^{-1} + 0.574\,08z^{-2}}$$

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Highpass filter   A highpass digital filter meeting the following specifications is required:

passband	2–4 kHz
stopband	0–500 Hz
passband ripple	3 dB
stopband attenuation	20 dB
sampling frequency	8 kHz

Determine the following:

(1) pass- and stopband edge frequencies for a suitable analog prototype lowpass filter;

(2) order, N, of the prototype lowpass filter;

(3) coefficients, and hence the transfer function, of the discrete-time filter using the bilinear z-transform.

Assume a Butterworth characteristic for the filter.

Solution   (1)   From the specifications, the prewarped frequencies are

$$\omega_s' = \tan\left(\frac{2\pi \times 500}{2 \times 8000}\right) = 0.198\,912$$

$$\omega_p' = \tan\left(\frac{2\pi \times 2000}{2 \times 8000}\right) = 1$$

$$\omega_s^p = \frac{\omega_p'}{\omega_s'} = 1/0.198\,912 = 5.0273$$

Thus, the pass- and stopband edge frequencies for the prototype LP filter are: 0, 1, 5.0273.

(2)   Next, we use Equation 8.25 and the values of the parameters above to determine the order of the filter.

Now

$$10^{A_s/10} - 1 = 10^{20/10} - 1 = 99; \quad 10^{A_c/10} - 1 = 10^{3/10} - 1 = 0.995\,262\,3$$

$$\log\left(\frac{99}{0.995\,262\,3}\right) = 1.997\,697$$

Let $N = 2$.

The poles of the prototype filter are (from Equation 8.26)

$$s_{p,1} = \cos\left[\frac{(2+2-1)\pi}{4}\right] + j\sin\left[\frac{(2+2-1)\pi}{4}\right] = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

$$s_{p,2} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$

The s-plane transfer function, $H(s)$, is

$$H(s) = \frac{1}{(s - s_{p,1})(s - s_{p,2})} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The frequency scaled s-plane transfer function is

$$H'(s) = H(s)\bigg|_{\frac{\omega_p'}{s}} = \frac{1}{\left(\frac{\omega_p'}{s}\right)^2 + \sqrt{2}\,\frac{\omega_p'}{s} + 1}$$

$$= \frac{s^2}{s^2 + \sqrt{2}s\omega_p' + \omega_p'^2}$$

Applying the BZT:

$$H(z) = H'(s)\bigg|_{s=\frac{z-1}{z+1}} = \frac{\left(\frac{z-1}{z+1}\right)^2}{\left(\frac{z-1}{z+1}\right)^2 + \sqrt{2}\omega_p'\left(\frac{z-1}{z+1}\right) + \omega_p'^2}$$

$$= \frac{(z-1)^2}{(z-1)^2 + \sqrt{2}\omega_p'(z-1)(z+1) + \omega_p'^2(z+1)^2}$$

After simplification and dividing top and bottom by z^2, we have

$$H(z) = \frac{1}{1 + \sqrt{2}\omega_p' + \omega_p'^2} \times \frac{1 - 2z^{-1} + z^{-2}}{1 + \frac{2(\omega_p'^2 - 1)z^{-1}}{1 + \sqrt{2}\omega_p' + \omega_p'^2} + \frac{(1 - \sqrt{2}\omega_p' + \omega_p'^2)z^{-2}}{1 + \sqrt{2}\omega_p' + \omega_p'^2}}$$

Using the values of the parameters

$$1 + \sqrt{2}\omega_p' + \omega_p'^2 = 3.414\,21; \quad \omega_p'^2 - 1 = 0$$

$$1 - \sqrt{2}\omega_p' + \omega_p'^2 = 0.585\,786; \quad \omega_p'^2 = 1$$

and substituting in the equation above and simplifying, we have

$$H(z) = \frac{0.292\,89(1 - 2z^{-1} + z^{-2})}{1 + 0.171\,57z^{-2}}$$

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Problem

Design a low-pass digital filter operating at the rate of 20 kHz. And having pass-band extending to 3 kHz. with a maximum Pass-band attenuation of 0.5 dB, and stop-band starting at 5 kHz. with a minimum stop-band attenuation of 10 dB.
a) Use Butter-worth design   b) Using Chebyshev design

Solution:

Digital frequencies in radians per sample:

$$\omega_p = \frac{2\pi f_{pass}}{f_s} = \frac{2\pi}{20} \times 3 = 0.3\pi$$

$$\omega_s = \frac{2\pi f_{stop}}{f_s} = \frac{2\pi}{20} \times 5 = 0.5\pi$$

For the bilinear transformation design,

$$\omega'_p = \tan(\omega_p/2) = 0.5095$$
$$\omega'_s = \tan(\omega_s/2) = 1.0$$

Since the filter specifications are not normalized,

$$1/k = \omega'_s / \omega'_p = 1/0.5095 = 1.9627.$$

$$1/k_1 = \frac{\sqrt{10^{-[A_{sdB}/10]} - 1}}{\sqrt{10^{-[A_{pdB}/10]} - 1}} = \frac{\sqrt{10^{10/10} - 1}}{\sqrt{10^{0.5/10} - 1}} = 8.588$$

a)   Using the Butter-worth design:

Order of the filter is:

$$N \quad = \quad \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} \quad = \quad \frac{\log_{10}(8.588)}{\log_{10}(1.9627)} \quad = \quad 3.1896 \cong 4$$

Therefore,

$$H(s) \quad = \quad \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}$$

Hence, the digital low - pass filter's transfer function is :

$$H(z) \quad = \quad H(s)\Big|_{s=\frac{z-1}{z+1}}$$

$$= \quad \frac{1}{\left(\dfrac{z-1}{z+1}\right)^4 + 2.6131\left(\dfrac{z-1}{z+1}\right)^3 + 3.4142\left(\dfrac{z-1}{z+1}\right)^2 + 2.6131\left(\dfrac{z-1}{z+1}\right) + 1}$$

b)   Using the Chebyshev design:

Order of the filter is:

$$N \quad = \quad \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} \quad = \quad 2.1929 \cong 3$$

Therefore,

$$H(s) \quad = \quad \frac{1}{s^3 + 1.253s^2 + 1.535s + 0.716}$$

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Hence, the digital low - pass filter's transfer function is :

$$H(z) \quad = \quad H(s)\Big|_{s=\frac{z-1}{z+1}}$$

$$= \quad \frac{1}{\left(\dfrac{z-1}{z+1}\right)^3 + 1.253\left(\dfrac{z-1}{z+1}\right)^2 + 1.535\left(\dfrac{z-1}{z+1}\right) + 0.716}$$