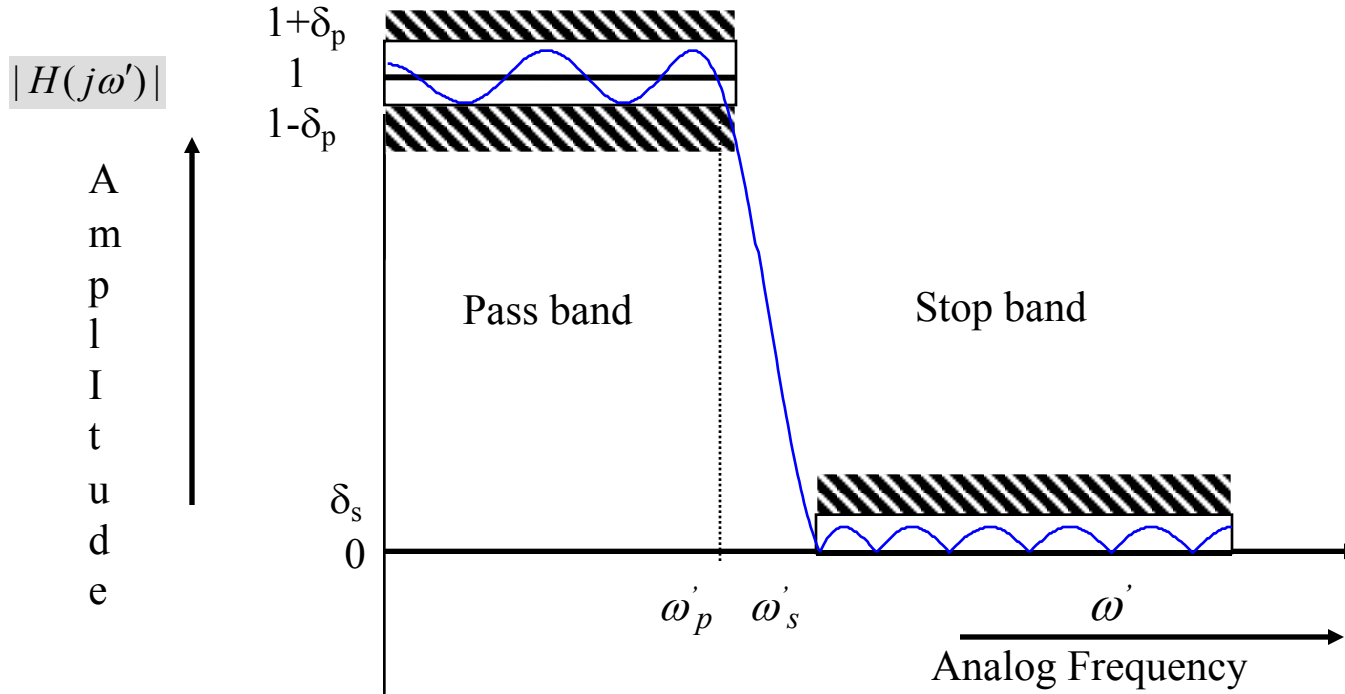


SUMMARY OF IIR FILTER THEORY

Yogananda Isukapalli

Analog Filter Specifications



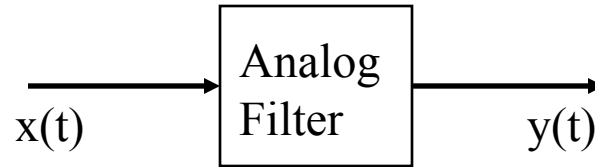
$$1 - \delta_p \leq |H(j\omega')| \leq 1 + \delta_p \quad \text{for } |\omega'| \leq \omega'_p,$$

$$|H(j\omega')| \leq \delta_s \quad \text{for } \omega'_s \leq |\omega'| \leq \infty,$$

$$\alpha_p = -20 \log_{10}(1 - \delta_p) \text{ dB},$$

$$\alpha_s = -20 \log_{10}(\delta_s) \text{ dB},$$

where α_p and α_s are the peak pass band ripple and minimum stop band attenuation in dB respectively



$$y(t) = \sum_{k=0}^N a_k \frac{d^k x(t)}{dt^k} + \sum_{k=0}^M b_k \frac{d^k y(t)}{dt^k}$$

Transfer function

$$H(s) = \frac{\sum_{i=0}^N a_i s^i}{1 - \sum_{i=0}^M b_i s^i}$$

Impulse response

$$h(t) = L^{-1} H(s)$$

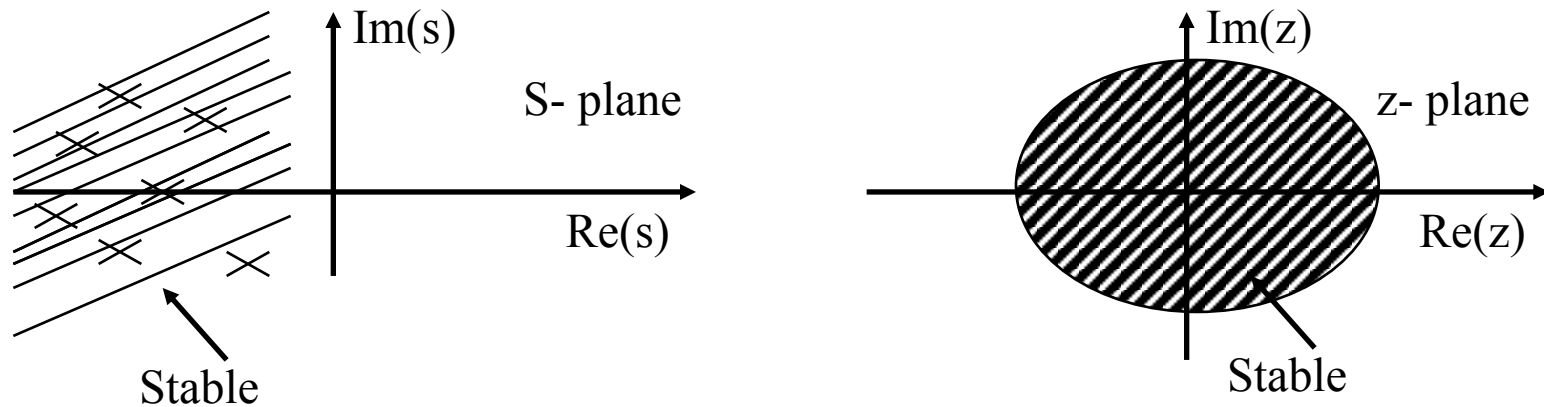
Example :

$$H(s) = \frac{a_0}{1 - b_1 s}$$

$$h(t) = \begin{cases} \frac{a_0}{b_1} e^{\frac{1}{b_1} t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Stability

Analog filters are stable when their poles are in the left half of the s-plane, whereas digital filters are stable when their poles are confined within the unit circle.



Analog Filter : Magnitude

$$\text{Function : } H(j\omega') = H(s) |_{s=j\omega'} = |H(j\omega')| < H(j\omega')$$

$$\text{Magnitude Square Function : } |H(j\omega')|^2 = H(s)H(-s) |_{s=j\omega'}$$

Analog Butterworth Filters

Define : Lowpass Butterworth Filters (N^{th} order)

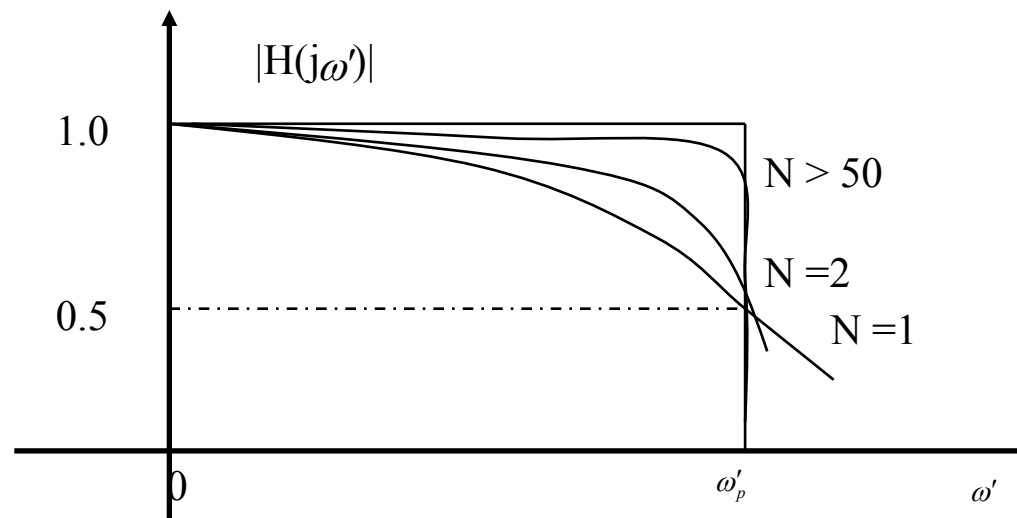
$$|H(j\omega')|^2 = \frac{1}{1 + \left(\frac{\omega'}{\omega'_p}\right)^{2N}}$$

or

$$|H(j\omega')|^2 = \frac{1}{1 + (\omega')^{2N}}$$

(normalized prototype,
 $\omega'_p = 1$)

Maximally flat
response



Poles of the Lowpass Prototype (Normalized $\omega'_p = 1$)

Butterworth Filter

Magnitude Square Function :

$$|H(j\omega')|^2 = \frac{1}{1 + \omega'^{2N}} \quad \omega'_p=1$$
$$H(s)H(-s) = \frac{1}{1 + \omega'^{2N}} \Big|_{\omega'^2 = -s^2 (s=j\omega')}$$
$$H(s)H(-s) = \frac{1}{1 + (-s^2)^N}$$

Poles: Solve: $1 + (-s^2)^N = 0$

a. N is odd $1 - s^{2N} = 0$

$$s^{2N} = 1$$

Roots :

$$s_k = 1.e^{j(\frac{k2\pi}{2N})} = 1.e^{j(\frac{k\pi}{N})}$$
$$k = 0,1,2,\dots,2N-1$$

b. N is even :

$$1 + s^{2N} = 0 \quad s^{2N} = -1$$

Roots :

$$s_k = 1.e^{j(\frac{\pi+k2\pi}{2N})}$$
$$k = 0,1,2,\dots,2N-1$$

Poles of a normalized ($\omega'_p = 1$) Butterworth (LPF) filter lie uniformly spaced on a unit circle in the s-plane at the following locations:

$$s_k = e^{j\pi(2k+N-1)/2N} = \cos\left[\frac{(2k+N-1)\pi}{2N}\right] + j \sin\left[\frac{(2k+N-1)\pi}{2N}\right]$$

$$k = 1, 2, \dots, N$$

The poles occur in complex conjugate pairs and lie on the left-hand side of the s-plane.

TABLE 10.1 BUTTERWORTH PROTOTYPE COEFFICIENTS								
N	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1	1.0000							
2	1.4141	1.0000						
3	2.0000	2.0000	1.0000					
4	2.6131	3.4142	2.6131	1.0000				
5	3.2361	5.2361	5.2361	3.2361	1.0000			
6	3.8637	7.4641	9.1416	7.4641	3.8637	1.0000		
7	4.4940	10.0978	14.5918	14.5918	10.0978	4.4940	1.0000	
8	5.1258	13.1371	21.8462	25.6884	21.8462	13.1371	5.1258	1.0000

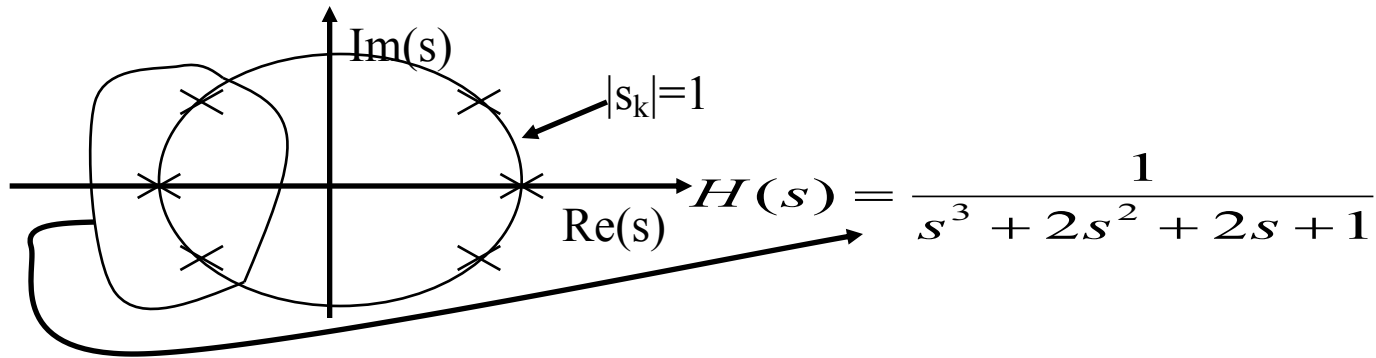
$$H_{LP_p}(s) = \frac{1}{1 + a_1s + a_2s^2 + \dots + a_Ns^N}$$

Source: M. E. Van Valkenburg, *Introduction to Modern Network Synthesis*. New York, John Wiley and Sons, 1960.

Example: $N = 3$

$$s_0 = 1, s_1 = 1.e^{j\pi/3}, s_2 = 1.e^{j2\pi/3}$$

$$s_3 = 1.e^{j\pi}, s_4 = 1.e^{j4\pi/3}, s_5 = 1.e^{j5\pi/3}$$



Butterworth
Filter order N

$$N \geq \frac{\log\left(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1}\right)}{2 \log\left(\frac{\omega'_s}{\omega'_p}\right)}$$

Analog Chebyshev Filters

Define :

$$|H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 C_N^2(\omega' / \omega'_p)}$$

$$|H_{LP_p}(j\omega')|^2 = \frac{1}{1 + \varepsilon^2 C_N^2(\omega')} \quad (\text{normalized prototype, } \omega'_p = 1)$$

Where :

$C_N(\omega')$ is the N^{th} order chebyshev polynomial.

$$C_N(\omega') = \cos(N \cos^{-1} \omega') \quad 0 \leq \omega' \leq 1$$

$$C_N(\omega') = \cosh(N \cosh^{-1} \omega') \quad \omega' > 1$$

ε is the ripple parameter $0 < \varepsilon < 1$ sets the ripple amplitude in the ripple passband $0 \leq \varepsilon \leq 1$

The ripple amplitude in dB is given by :

$$\begin{aligned}\gamma_{dB} &= -10 \log_{10} \left(\frac{1}{1 + \epsilon^2} \right) \\ &= 10 \log_{10} (1 + \epsilon^2)\end{aligned}$$

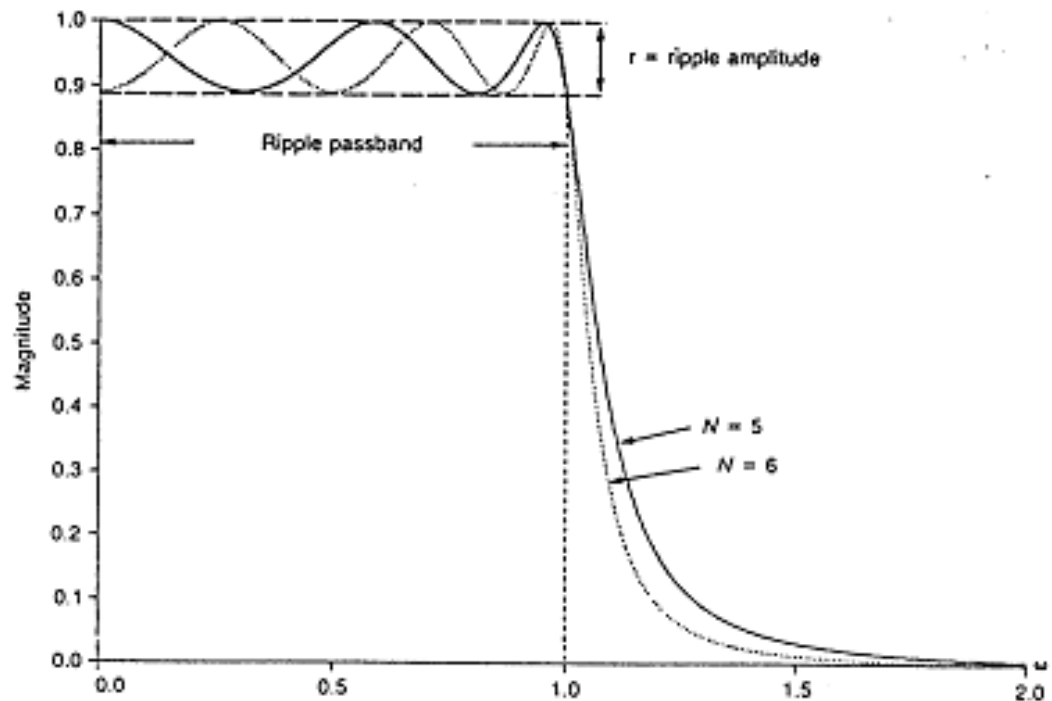


Fig: Chebyshev prototype low pass response

Also,

$$C_{N+1}(\omega') = 2\omega' C_N(\omega') - C_{N-1}(\omega')$$

N	$C_N(\omega)$
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$

Chebyshev
Filter order N

$$N \geq \frac{\cosh^{-1} \left(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1} \right)}{2 \cosh^{-1} \left(\frac{\omega'_s}{\omega'_p} \right)}$$

1/2-dB ripple ($\epsilon = 0.3493$, $\epsilon^2 = 0.1220$)	
N	
1	$s + 2.863$
2	$s^2 + 1.425s + 1.516$
3	$s^3 + 1.253s^2 + 1.535s + 0.716$
4	$s^4 + 1.197s^3 + 1.717s^2 + 1.025s + 0.379$
5	$s^5 + 1.173s^4 + 1.937s^3 + 1.310s^2 + 0.753s + 0.179$
1-dB ripple ($\epsilon = 0.5088$, $\epsilon^2 = 0.2589$)	
N	
1	$s + 1.965$
2	$s^2 + 1.098s + 1.103$
3	$s^3 + 0.988s^2 + 1.238s + 0.491$
4	$s^4 + 0.953s^3 + 1.454s^2 + 0.743s + 0.276$
5	$s^5 + 0.937s^4 + 1.689s^3 + 0.974s^2 + 0.581s + 0.123$
2-dB ripple ($\epsilon = 0.7648$, $\epsilon^2 = 0.5849$)	
N	
1	$s + 1.308$
2	$s^2 + 0.804s + 0.823$
3	$s^3 + 0.738s^2 + 1.022s + 0.327$
4	$s^4 + 0.716s^3 + 1.256s^2 + 0.517s + 0.206$
5	$s^5 + 0.707s^4 + 1.500s^3 + 0.694s^2 + 0.459s + 0.082$

Source: M. S. Ghausi and K. R. Laker, *Modern Filter Design*, Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1981.

Poles of a normalized ($\omega_p = 1$) Chebyshev LPF

$$s_k = \sinh(\alpha) \cos(\beta_k) + j \cosh(\alpha) \sin(\beta_k),$$

where $\alpha = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right)$;

$$\beta_k = \frac{(2k + N - 1)\pi}{2N}, \quad k = 1, 2, \dots, N$$

Fig: Prototype Chebyshev denominator polynomials

Mapping analog filters to digital domain:

- Mapping differentials:

$$\text{Backward: } \left. \frac{dy_a(t)}{dt} \right|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n] - y[n-1]}{T}$$

$$\text{Forward: } \left. \frac{dy_a(t)}{dt} \right|_{t=nT} = \nabla^{(1)}[y[n]] = \frac{y[n+1] - y[n]}{T}$$

$\nabla^{(1)}$  Backward or forward difference operator

$$\text{Also, } \nabla^{(k)}[y[n]] = \nabla^{(1)}[\nabla^{(k-1)}[y[n]]]$$

$$\text{with } \nabla^{(0)}(.) = y[n]$$

Analog Filters

$$H_a(s) = \frac{\sum_{k=0}^M d_k s^k}{\sum_{k=0}^N c_k s^k}$$

Mapping Functions



Digital Filters

$$H(z) = \frac{\sum_{k=0}^M d_k \left[\frac{1 - z^{-1}}{T} \right]^k}{\sum_{k=0}^N c_k \left[\frac{1 - z^{-1}}{T} \right]^k}$$

$$s = \frac{1 - z^{-1}}{T}$$

Backward

Difference Mapping

$$s = \frac{z - 1}{T}$$

Forward

Difference Mapping

- Both the mapping functions may produce satisfactory results only for extremely small sampling times and are generally not recommended.

Impulse Invariance Mapping:

$$H_a(s) = \sum_{k=1}^N \frac{c_k}{s - p_k} \quad \leftrightarrow \quad H(z) = \sum_{k=1}^N \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

- Evaluation – Pole Mapping:
 - (a) The impulse invariant mapping maps poles from the s -plane's $j\omega'$ axis to the z -plane's unit circle.
 - (b) All s -plane poles with negative real parts map to z -plane poles inside the unit circle. In other words, stable analog poles are mapped to stable digital filter poles.
 - (c) All poles on the right half of the s -plane, i.e. with positive real parts, map to digital poles outside the unit circle.

The impulse invariant mapping, thus, preserves the stability of the filter.

- Deficiency of the impulse-invariance mapping:

If s-plane poles have the same real parts and imaginary parts that differ by some integer multiples of $2\pi/T$, then there are an infinite number of s-plane poles that map to the same location in the z-plane. This will result in aliased poles in the z-plane.

- Aliasing in digital frequency response:

$$H(e^{j\omega'T}) = \frac{1}{T} H_a(j\omega') + \frac{1}{T} H_a(j\omega' \pm j\frac{2\pi}{T}) + \dots$$

Digital frequency response reproduces the analog frequency response every $2\pi/T$.

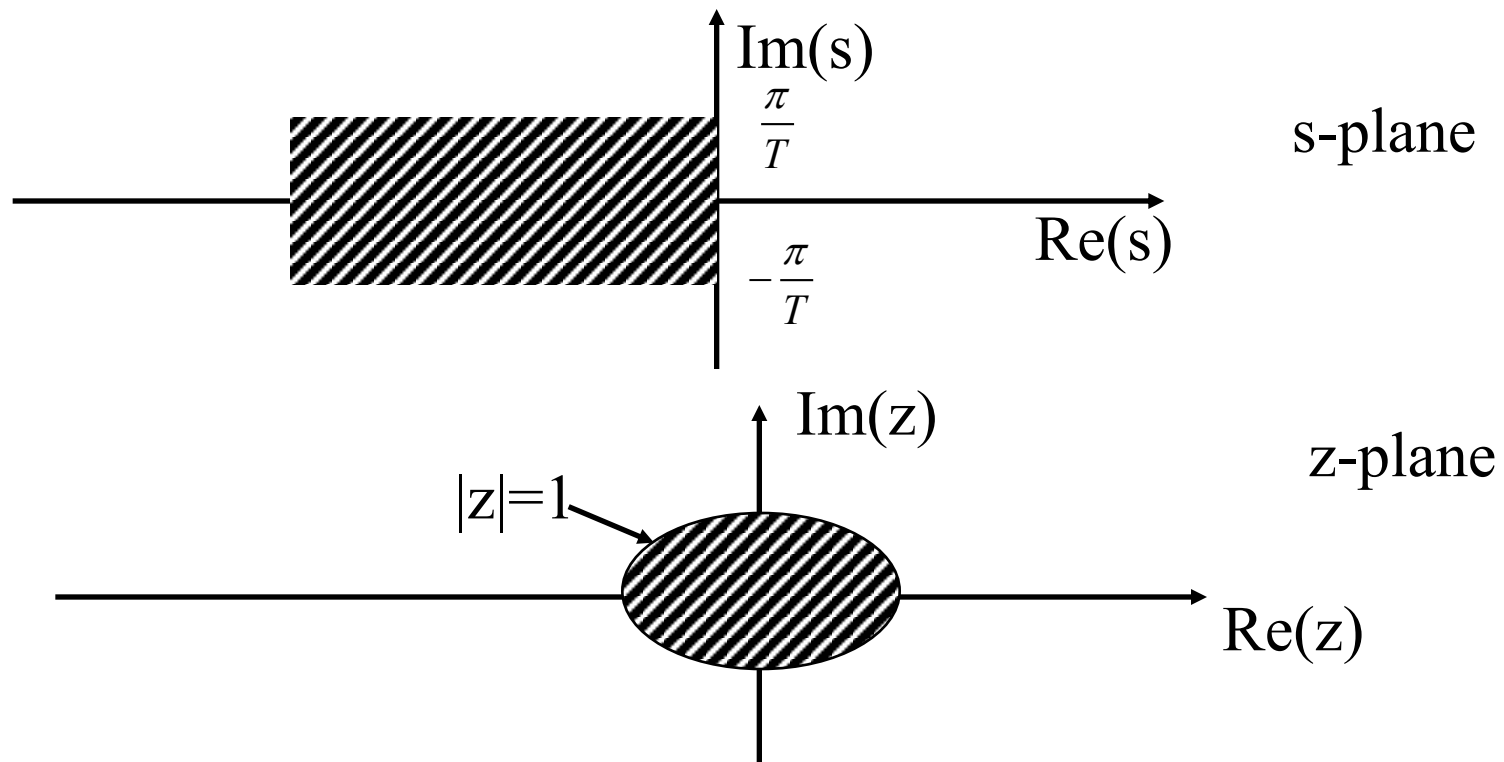
To avoid aliasing for lower sampling rates, the pole mapping is restricted to the primary strip $-\frac{\pi}{T} \leq \omega' \leq \frac{\pi}{T}$

Thus each horizontal strip of the s-plane of width $2\pi/T$ is mapped onto the entire z-plane.

To prevent significant aliasing:

$$H_a(j\omega') \approx 0 \quad \text{for} \quad |\omega'| > \pi / T$$

Band limit the analog filter.



- Example of the impulse-invariance mapping:

Third order Butterworth low-pass filter's transfer function:

$$H(s) = \frac{1}{(s+1)(s^2 + s + 1)}$$

The partial fraction expansion is :

$$\begin{aligned} H(s) &= \frac{1}{(s+1)(s+0.5-j0.866)(s+0.5+j0.866)} \\ &= \frac{C_1}{(s+1)} + \frac{C_2}{(s+0.5-j0.866)} + \frac{C_3}{(s+0.5+j0.866)} \end{aligned}$$

Algebra gives the coefficients as :

$$= \frac{1}{(s+1)} + \frac{0.577e^{-j2.62}}{(s+0.5-j0.866)} + \frac{0.577e^{j2.62}}{(s+0.5+j0.866)}$$

The three poles are :

$$p_1 = -1, p_2 = -0.5 + j0.866, p_3 = -0.5 - j0.866$$

Therefore, the transfer function can be written as :

$$H(z) = \frac{1}{(1 - e^{-T} z^{-1})} + \frac{1}{(1 - e^{(-0.5 + j0.866)T} z^{-1})} + \frac{1}{(1 - e^{(-0.5 - j0.866)T} z^{-1})}$$

Simplifying, we get

$$H(z) = \frac{z}{(z - e^{-T})} + \frac{-z^2 - 1.154e^{-0.5T} \cos(5\pi / 6 + 0.866T)}{(1 - e^{(-0.5 + j0.866)T} z^{-1})}$$

This can be further simplified and expressed as :

$$H(z) = \frac{b_0 z^2 + b_1 z}{z^3 + a_1 z^2 + a_2 z + a_3}$$

where

$$b_0 = -2e^{-0.5T} \cos(0.866T) + e^{-T} + 1.154e^{-0.5T} \cos(5\pi / 6 + 0.866T)$$

$$b_1 = e^{-T} + 1.154e^{-1.5T} \cos(5\pi / 6 + 0.866T)$$

$$a_1 = -(e^{-T} + 2e^{-0.5T} \cos(0.866T))$$

$$a_2 = e^{-T} + 2e^{-0.5T} \cos(0.866T)$$

$$a_3 = -e^{-2T}.$$

Bilinear Mapping:

$$H(s) = \frac{1}{s} \longrightarrow H(z) = \frac{T}{2} \frac{z+1}{z-1}$$

$$s^{-1} = \frac{T}{2} \frac{z+1}{z-1} \quad \text{or} \quad s = \frac{2}{T} \frac{z-1}{z+1}$$

- Evaluation – Pole Mapping:

Stable analog filters are mapped from s-plane to z-plane as stable digital filters. Aliasing problem is eliminated.

- Analog and Digital frequencies relationship for prewarping:

$$\omega' = k \tan(\omega T / 2)$$

$\omega' \rightarrow$ Analog frequency

$\omega \rightarrow$ Digital frequency

Summary of Bilinear Transformation method¹

For standard, frequency-selective IIR filters, the steps for using the BZT may be summarized as follows:

- (1) Use the digital filter specifications to find a suitable normalized, prototype, analog lowpass filter, $H(s)$.
- (2) Determine and prewarp the bandedge or critical frequencies of the desired filter. For lowpass or highpass filters there is just one bandedge or cutoff frequency (ω_p , say). For bandpass and bandstop filters, we have the lower and upper passband edge frequencies, ω_{p1} and ω_{p2} , each of which needs to be prewarped (the stopband edge frequencies may also be specified):

$$\omega'_p = \tan\left(\frac{\omega_p T}{2}\right) \quad (8.20a)$$

$$\omega'_{p1} = \tan\left(\frac{\omega_{p1} T}{2}\right); \quad \omega'_{p2} = \tan\left(\frac{\omega_{p2} T}{2}\right) \quad (8.20b)$$

- (3) Denormalize the analog prototype filter by replacing s in the transfer function, $H(s)$, using one of the following transformations, depending on the type of filter required:

$$s = \frac{s}{\omega'_p} \quad \text{lowpass to lowpass} \quad (8.21a)$$

$$s = \frac{\omega'_p}{s} \quad \text{lowpass to highpass} \quad (8.21b)$$

$$s = \frac{s^2 + \omega_0^2}{W_s} \quad \text{lowpass to bandpass} \quad (8.21c)$$

$$s = \frac{W_s}{s^2 + \omega_0^2} \quad \text{lowpass to bandstop} \quad (8.21d)$$

where

$$\omega_0^2 = \omega'_{p2} \omega'_{p1}, \quad W = \omega'_{p2} - \omega'_{p1}$$

- (4) Apply the BZT to obtain the desired digital filter transfer function, $H(z)$, by replacing s in the frequency-scaled (i.e. denormalized) transfer function, $H'(s)$, as follows:

$$s = \frac{z - 1}{z + 1}$$

Example 1:

Low pass filter: Obtain the transfer function $H(z)$ of the digital low pass filter to approximate the following transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Use Bilinear Transformation method and assume a 3 dB cut off frequency of 150Hz and a sampling frequency of 1.28kHz.

Soln. The critical frequency, $\omega_p = 2\pi \times 150$ rad/s, and $F_s = 1/T = 1.28$ kHz, giving a prewarped critical frequency of $\omega'_p = \tan(\omega_p T/2) = 0.3857$

The frequency-scaled analog filter is given by

$$\begin{aligned} H'(s) &= H(s)|_{s=s/\omega'_p} = \frac{1}{(s/\omega'_p)^2 + \sqrt{2}s/\omega'_p + 1} \\ &= \frac{\omega_p'^2}{s^2 + \sqrt{2}\omega'_p s + \omega_p'^2} = \frac{0.1488}{s^2 + 0.5455s + 0.1488} \end{aligned}$$

Applying the BZT gives

$$\begin{aligned} H(z) &= H'(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{0.0878z^2 + 0.1756z + 0.0878}{z^2 - 1.0048z + 0.3561} \\ &\approx \frac{0.0878(1 - 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}} \end{aligned}$$

Use of BZT and Classical Analog Filters

Analog Filters Review

- Low pass Butterworth filter of order N

Frequency response

$$|H(j\omega')|^2 = \frac{1}{1 + \left(\frac{\omega'}{\omega'_p}\right)^{2N}} \quad (8.24)$$

Poles of a normalized ($\omega'_p = 1$) Butterworth LPF

$$s_k = e^{j\pi(2k+N-1)/2N} = \cos\left[\frac{(2k+N-1)\pi}{2N}\right] + j \sin\left[\frac{(2k+N-1)\pi}{2N}\right] \quad (8.25)$$

$k = 1, 2, \dots, N$

Filter order N

$$N \geq \frac{\log\left(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1}\right)}{2 \log\left(\frac{\omega'_s}{\omega'_p}\right)} \quad (8.26)$$

Analog Filters Review contd

2. Low pass Chebyshev Type 1 Filter of order N

Frequency response

$$|H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 C_N^2(\omega' / \omega'_p)} \quad (8.27)$$

Poles of a normalized ($\omega'_p = 1$) Chebyshev LPF

$$s_k = \sinh(\alpha) \cos(\beta_k) + j \cosh(\alpha) \sin(\beta_k),$$

where $\alpha = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\varepsilon}\right);$

$$\beta_k = \frac{(2k + N - 1)\pi}{2N}, k = 1, 2, \dots, N \quad (8.29)$$

Filter order N

$$N \geq \frac{\cosh^{-1}\left(\frac{10^{\frac{A_s}{10}} - 1}{10^{\frac{A_p}{10}} - 1}\right)}{2 \cosh^{-1}\left(\frac{\omega'_s}{\omega'_p}\right)} \quad (8.28)$$

Analog Filters Review contd

3. Low pass Elliptic Filter of order N

Frequency
response

$$|H(j\omega')|^2 = \frac{K}{1 + \varepsilon^2 G_N^2(\omega')} \quad (8.30)$$

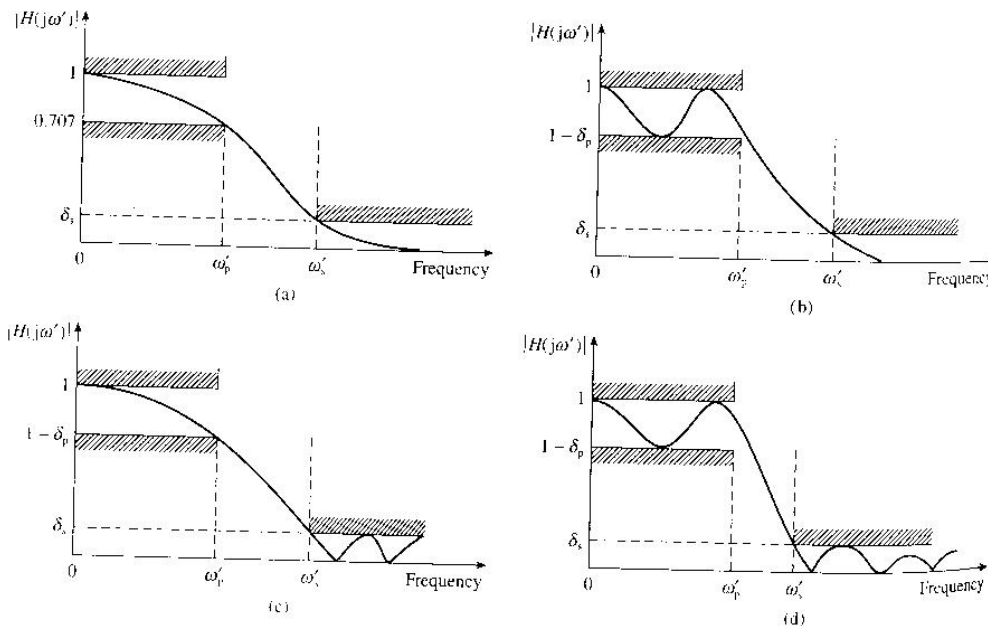


Fig: Sketches of frequency response of some classical analog filters

(a) Butterworth response (b) Chebyshev type I © Chebyshev type II

(d) Elliptic

Analog Frequency Transformations

1. *Desired Low pass* \longrightarrow *Low pass prototype*

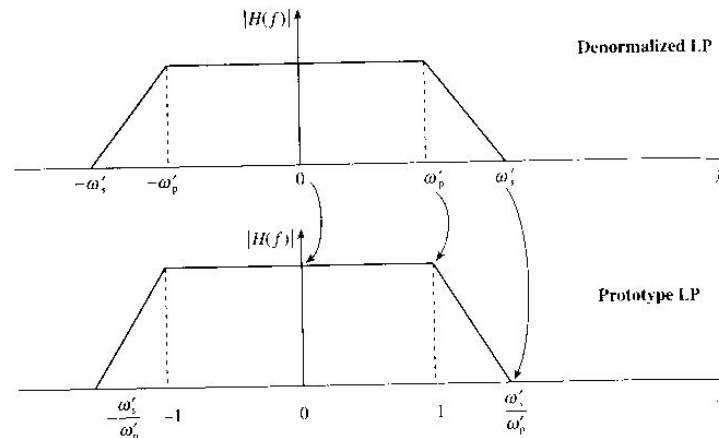
The low pass to low pass transformation is:

$$s = \frac{s}{\omega'_p} \quad \text{From 8.21 (a)}$$

Denote frequencies for

LPF $\longrightarrow \omega_{lp}$ and Prototype $\longrightarrow \omega^p$

$$j\omega^p = j \frac{\omega_{lp}}{\omega'_p} \quad \text{i.e.} \quad \omega^p = \frac{\omega_{lp}}{\omega'_p} \quad (8.31)$$



2. Desired High pass \longrightarrow Low pass prototype

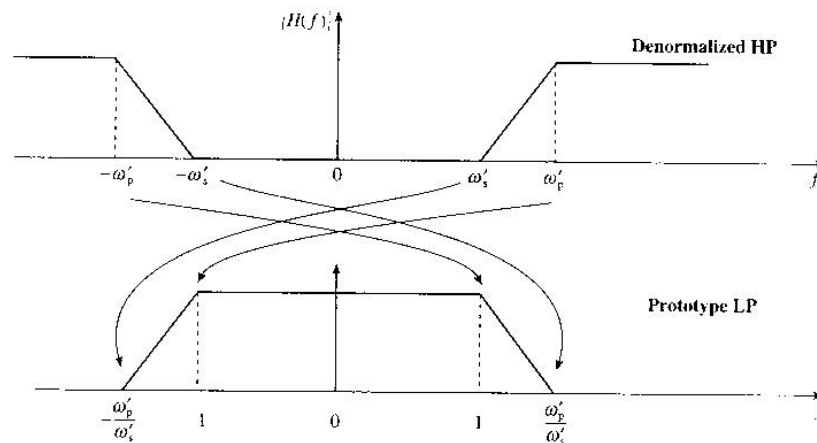
The low pass to high pass transformation is:

$$s = \frac{\omega'_p}{s} \quad \text{From 8.21 (b)}$$

Denote frequencies for

HPF $\longrightarrow \omega_{hp}$ and Prototype $\longrightarrow \omega^p$

$$j\omega^p = -j \frac{\omega'_p}{\omega_{hp}} \quad \text{i.e.} \quad \omega^p = -\frac{\omega'_p}{\omega_{hp}} \quad (8.32)$$



The BZT methodology using classical analog filters

In cases where the prototype lowpass filter does not exist, the stages of the BZT method are:

- (1) Prewarp the bandedge or critical frequencies of the digital filter as described previously.
- (2) Find a suitable lowpass prototype analog filter, based on the digital filter specifications and one of the classical filter characteristics. This involves using one of the frequency transformation equations (depending on the type of digital filter – LP, HP, BP or BS) in reverse to determine the specifications of the prototype LP filter. From this the order of the prototype filter and hence its transfer function, $H(s)$, are found.
- (3) Denormalize the prototype analog LP filter, $H(s)$, by frequency transformation and scaling to obtain a new transfer function, $H'(s)$, as described previously.
- (4) Apply the BZT to obtain the desired digital filter transfer function, $H(z)$, by replacing s in the frequency scaled transfer function, $H'(s)$, as described previously.

Design Example

Lowpass filter A lowpass digital filter meeting the following specifications is required:

passband	0–500 Hz
stopband	2–4 kHz
passband ripple	3 dB
stopband attenuation	20 dB
sampling frequency	8 kHz

Determine the following:

- (1) pass- and stopband edge frequencies for a suitable analog prototype lowpass filter;
- (2) order, N , of the prototype lowpass filter;
- (3) coefficients, and hence the transfer function, of the discrete-time filter using the bilinear z -transform.

Assume a Butterworth characteristic for the filter.

Solution (1) From the specifications, the prewarped frequencies are

$$\omega_p' = \tan\left(\frac{2\pi \times 500}{2 \times 8000}\right) = 0.198\,912$$

$$\omega_s' = \tan\left(\frac{2\pi \times 2000}{2 \times 8000}\right) = 1$$

$$\omega_s^p = \frac{\omega_s'}{\omega_p'} = 1/0.198\,912 = 5.0273$$

Thus, the prewarped pass- and stopband edge frequencies for the prototype LP filter are: 0, 1, 5.0273.

- (2) Next, we use Equation 8.25 and the values of the parameters above to determine the order of the filter.

Now

$$10^{4/10} - 1 = 10^{20/10} - 1 = 99; \quad 10^{4s/10} - 1 = 10^{1/10} - 1 = 0.995\,262\,3;$$

$$\log\left(\frac{99}{0.995\,262\,3}\right) = 1.997\,697$$

For the prototype LPF

$$\omega_p^p = 1; \quad \omega_s^p = 5.0273; \quad \log\left(\frac{\omega_s^p}{\omega_p^p}\right) = 2 \log(5.0273) = 1.402\,66.$$

$$N \geq \frac{1.997\,697}{1.402\,66} = 1.424. \text{ Let } N = 2$$

(3) The poles of the prototype filter are (from Equation 8.26)

$$s_{p,1} = \cos\left[\frac{(2+2-1)\pi}{4}\right] + j \sin\left[\frac{(2+2-1)\pi}{4}\right] = -\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$$

$$s_{p,2} = -\frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2}$$

The s -plane transfer function, $H(s)$, is

$$H(s) = \frac{1}{(s - s_{p,1})(s - s_{p,2})} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The frequency scaled s -plane transfer function is

$$H'(s) = H(s) \Big|_{\frac{s}{\omega_p}} = \frac{1}{\left(\frac{s}{\omega_p}\right)^2 + \sqrt{2}\frac{s}{\omega_p} + 1}$$

$$= \frac{\omega_p'^2}{s^2 + \sqrt{2}s\omega_p' + \omega_p'^2}$$

Applying the BZT:

$$H(z) = H'(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{\omega_p'^2}{\left(\frac{z-1}{z+1}\right)^2 + \sqrt{2}\omega_p'\left(\frac{z-1}{z+1}\right) + \omega_p'^2}$$

$$= \frac{\omega_p'^2(z+1)^2}{(z-1)^2 + \sqrt{2}\omega_p'(z-1)(z+1) + \omega_p'^2(z+1)^2}$$

After simplification and dividing top and bottom by z^2 , we have

$$H(z) = \frac{\omega_p'^2}{1 + \sqrt{2}\omega_p' + \omega_p'^2} \times \frac{1 + 2z^{-1} + z^{-2}}{1 + \frac{2(\omega_p'^2 - 1)z^{-1}}{1 + \sqrt{2}\omega_p' + \omega_p'^2} + \frac{(1 - \sqrt{2}\omega_p' + \omega_p'^2)z^{-2}}{1 + \sqrt{2}\omega_p' + \omega_p'^2}}$$

Using the values of the parameters

$$1 + \sqrt{2}\omega_p' + \omega_p'^2 = 1.320\ 87; \quad \omega_p'^2 - 1 = -0.960\ 43$$

$$1 - \sqrt{2}\omega_p' + \omega_p'^2 = 0.758\ 285\ 8; \quad \omega_p'^2 = 0.039\ 565\ 9$$

and substituting in the equation above and simplifying, we have

$$H(z) = \frac{0.029\ 95(1 + 2z^{-1} + z^{-2})}{1 - 1.4542z^{-1} + 0.574\ 08z^{-2}}$$

References

1. “Digital Signal Processing – A Practical Approach” -
Emmanuel C. Ifeakor and Barrie W. Jervis
Second Edition