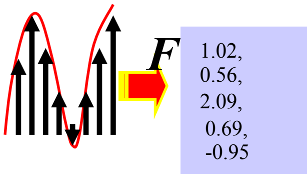

Transform Domain Representation of Discrete Time Signals

The Discrete Fourier Transform

(I)

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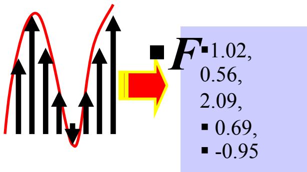


1. Fourier Series ⟶ Periodic waveforms

• Any **periodic waveform**, $f(t)$, can be represented as the sum of an infinite number of sinusoidal and cosinusoidal terms, together with a constant term:

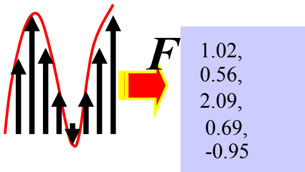
$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \qquad (1)$$

where

$\omega = 2\pi / T_p$ is the first harmonic or fundamental angular frequency

T_p is the repetition period of the waveform

$n\omega$ discrete nth harmonics of ω



$$a_0 = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} f(t)dt$$ is a constant, like a DC voltage level

$$a_n = \frac{2}{T_p} \int_{-T_p/2}^{T_p/2} f(t)\cos(n\omega t)dt$$ and

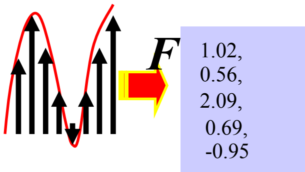
$$b_n = \frac{2}{T_p} \int_{-T_p/2}^{T_p/2} f(t)\sin(n\omega t)dt$$

Eqn (1) can also be represented as

$$f(t) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega t} \qquad (2)$$

where $$d_n = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} f(t) e^{-jn\omega t} dt \qquad (3)$$

is complex and $|d_n|$ has the units of volts



2. Fourier Transform ⟶ Non-periodic waveforms

- Consider a non-periodic waveform, obtained by making the period T_p of the periodic waveform to be infinite, i.e. $T_p \rightarrow \infty$

- As T_p is increased, the spacing between the harmonic components, $1/T_p = \omega / 2\pi$, decreases to $d\omega / 2\pi$, eventually becoming zero.

$$\Downarrow$$

discrete frequency variable $n\omega$ changes to the continuous variable ω

$\Rightarrow$ *Amplitude and phase spectra become continuous*

Thus, $d_n \rightarrow d(\omega)$ as $T_p \rightarrow \infty$

With these changes, eqn (3) becomes

$$d(\omega) = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad (4)$$

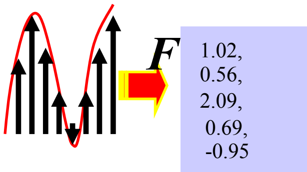
Dividing by $d\omega / 2\pi$,

$$\frac{d(\omega)}{d\omega / 2\pi} = F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad (5)$$

$F(j\omega)$ ⟶ Fourier Transform

$|F(j\omega)|$ ⟶ Amplitude spectral density (continuous with units volts per hertz)

$|F(j\omega)|^2$ ⟶ Energy spectral density (continuous with units joules per hertz)



3. Discrete Fourier Transform

• In practice the Fourier components of data are obtained by digital computation rather than by analog processing.

• So, the analog waveforms are digitized using a sample-and-hold circuit followed by an analog-to-digital converter and under the Nyquist criterion for sampling.

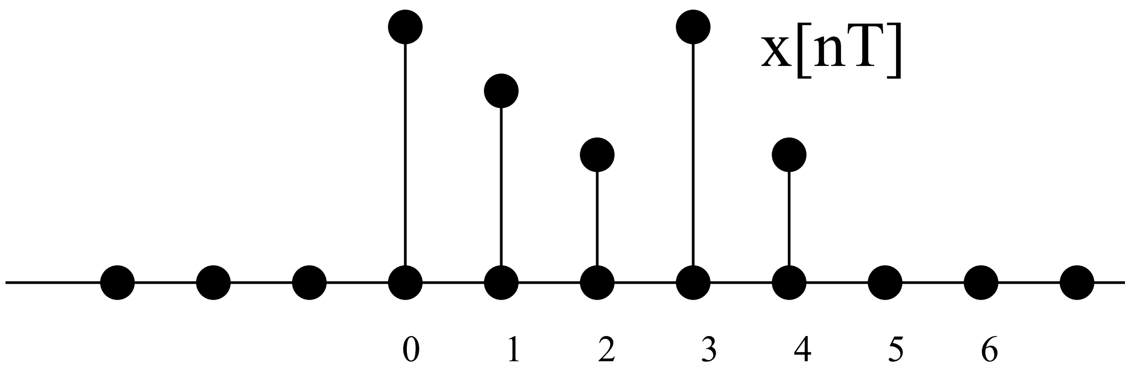
• *Thus the data to be transformed is discrete and probably non-periodic.*

• It is not possible to apply Fourier transform because it is for continuous data.

• Analog transform for use with discrete data

⟶ Discrete Fourier Transform (DFT)

Given: Finite (non-periodic) Duration Discrete - Time Signal



x[nT] = 0   n<0, n>N-1

$\neq 0$ for $0 \leq n \leq N-1$

DFT of x[nT] $\longrightarrow$ X(kΩ) where Ω=2π/NT

$$X(k\Omega) = F_D[x(nT)] = \sum_{n=0}^{N-1} x[nT]e^{-j\Omega nkT}$$

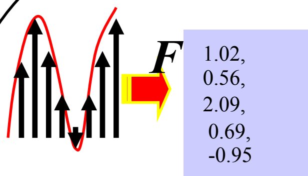
$$= \sum_{n=0}^{N-1} x[nT]e^{-j(\frac{2\pi}{N})nk} \quad _{k=0,......N-1}$$

DFT

$\Updownarrow$

$$x[nT] = \frac{1}{N}\sum_{k=0}^{N-1} X(k\Omega)e^{j(\frac{2\pi}{N})nk} \quad _{n=0,......N-1}$$

IDFT



- **Note**: From now on we assume

X(k) represents X(kΩ), x[n] represents x[nT]

<u>Relation between DFT and Fourier transform</u>

- The DFT equation can be seen analogous to the Fourier transform equation (5) by putting x(nT) = f(t), kΩ = ω, and nT = t.

- Making these substitutions in eqn (5), and putting dt=T and replacing the integral with a summation gives

$$\sum_{n=0}^{N-1} x(nT)e^{-jk\Omega nT}T = F(j\omega) \;\text{ for } 0 \le t \le (N-1)T \qquad (6)$$

- Now comparing eqn (6) with the DFT eqn gives $F(j\omega) = TX(k)$ i.e.

The Fourier transform components may be obtained by multiplying the DFT components by the sampling interval.

Example:

x[n]={1,1,0,0}   n=0,1,2,3

$$X(k) = \sum_{n=0}^{3} x[n] e^{-j(\frac{2\pi}{4})nk} \qquad k=0,1,2,3$$

$X(0)=1+1=2$, $X(1)=1+e^{-j\pi/2}=1-j$

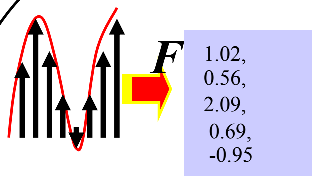
$X(2)=1+e^{-j\pi}=0$, $X(3)=1+e^{j\pi/2}=1+j$

Note X(4)=X(0), X(5)=X(1), X(6)=X(2), and so on: X(k+4)=X(k) k=0,1,2,3

{1,1,0,0} <=>{2,1-j,0,1+j}

Example: Consider an analog signal x(t) sampled with T=0.01 and the sampled values are :

n	0	1	2	3	4	5
x[n]	5.0	-1.5	6.5	-3.0	6.5	-1.5



$$X(k)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})nk} \qquad k=0,1,2,3,4,5$$

$$X(0)=\sum_{n=0}^{5}x[n]=12$$

$$X(1)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})n}=0$$

$$X(2)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})2n}=-3$$

$$X(3)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})3n}=-24$$

$$X(4)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})4n}=-3$$

$$X(5)=\sum_{n=0}^{5}x[n]e^{-j(\frac{2\pi}{6})5n}=0$$

Now taking the IDFT:

$$x[n]=\frac{1}{6}\sum_{k=0}^{5}X(k)e^{j(\frac{2\pi}{6})nk} \qquad n=0,1,2,3,4,5$$

First Write :

$$x[n]=\frac{1}{6}[12-3e^{j(\frac{2\pi}{6})2n}+24e^{j(\frac{2\pi}{6})3n}-3e^{j(\frac{2\pi}{6})4n}$$

Since: $e^{j(\frac{2\pi}{6})4n}=e^{-j(\frac{2\pi}{6})2n}$ Then:

$$x[n]=2-0.5\{e^{j(\frac{2\pi}{6})2n}+e^{-j(\frac{2\pi}{6})2n}\}+4e^{j(\frac{2\pi}{6})3n}$$

$$=2-\cos(\frac{2\pi n}{3})+4\cos(\pi n)$$

$$x[0]=2-\cos(0)+4\cos(0)=5$$

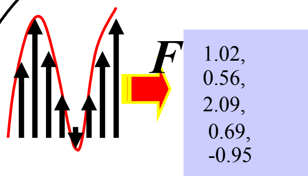
$$x[1]=2-\cos(\frac{2\pi}{3})+4\cos(\pi)=-1.5$$

$$x[2]=2-\cos(\frac{4\pi}{3})+4\cos(2\pi)=6.5$$

$$x[3]=2-\cos(\frac{6\pi}{3})+4\cos(3\pi)=-3$$

$$x[4]=2-\cos(\frac{8\pi}{3})+4\cos(4\pi)=6.5$$

$$x[5]=2-\cos(\frac{10\pi}{3})+4\cos(5\pi)=-1.5$$



$$\{5.0,\ -1.5,\ 6.5,\ -3.0,\ 6.5,\ -1.5\} \underset{\text{IDFT}}{\overset{\text{DFT}}{\Leftrightarrow}} \{12,\ 0,\ -3,\ 24,\ -3,\ 0\}$$

- DFT: N-point transform is unique

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})nk} \quad {}_{k=0,\ldots\ldots N-1} \qquad (7)$$

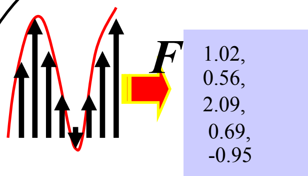
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})nk} \quad {}_{n=0,\ldots\ldots N-1} \qquad (8)$$

$$let..W_n = e^{-j(\frac{2\pi}{N})}$$

$$W_n^{-1} = e^{j(\frac{2\pi}{N})} \ldots Then$$

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn} \quad {}_{k=0,\ldots\ldots N-1} \qquad (9)$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad {}_{n=0,\ldots\ldots N-1} \qquad (10)$$



Suppose we multiply (9) by W_N^{-kp}

where p=0,…..N-1 and sum from k=0 to N-1

$$\sum_{k=0}^{N-1} X(k) W_N^{-kp} = \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x[n] W_N^{k(n-p)}$$

$$\sum_{k=0}^{N-1} x[n] \left[\sum_{n=0}^{N-1} W_N^{k(n-p)} \right]_{p=0,1,\ldots\ldots N-1}$$

$$= \sum_{n=0}^{N-1} W_N^{k(n-p)} = \sum_{n=0}^{N-1} e^{jk(p-n)\frac{2\pi}{N}}$$

Orthogonal Property :

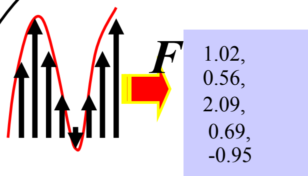
a.) p=n gives

$$\sum_{k=0}^{N-1} 1 = N$$

b.) p≠n rewrite

$$\sum_{k=0}^{N-1} \alpha^k$$

$$\alpha = e^{j(p-n)\frac{2\pi}{N}}$$



$\alpha \neq 1$, p-n is between -(N-1) and +(N+1)

Follows : $\sum_{k=0}^{N-1} \alpha^k = \frac{1-\alpha^N}{1-\alpha} \ldots\ldots \alpha \neq 1$

$$\alpha^N = e^{j(p-n)\frac{2\Pi N}{N}} = 1$$

Implies : $\sum_{k=0}^{N-1} \alpha^k = 0$

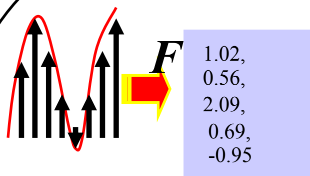
- Proved the orthogonality property of distinct set of complex discrete exponentials :

$$\sum_{k=0}^{N-1} W_N^{k(n-p)} = \begin{cases} N..p=n \\ 0..otherwise \end{cases}$$

- We have now proved:

$$\sum_{k=0}^{N-1} X(k) W_N^{-kp} = \sum_{k=0}^{N-1} x[n] N \delta(n-p)$$

$$= Nx(p)$$



$$X(p) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kp} \qquad {}_{p=0,1,.....N-1}$$

or

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{-j(\frac{2\pi}{N})(kn)}$$

- Conclusion: Given $\{x[n]\}_{n=0,\ 1,\ \ldots\ldots.N-1}$

we obtain a unique set of values :

$\{X(k)\}_{k=0,\ 1,\ \ldots.N-1}$ as a result :

$\{x[n]\} \Leftrightarrow \{X(k)\}$

for n=0,1…N-1 & k=0,1…N-1