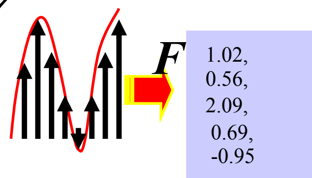
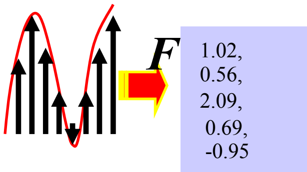


Transform Domain Representation of Discrete Time Signals

The Discrete Fourier Transform

(II)

Yogananda Isukapalli



Review of DFT (N-Point Transform)

DFT

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})nk} \qquad k=0,......N-1 \qquad (1)$$

IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})nk} \qquad n=0,......N-1 \qquad (2)$$

$$let..W_N = e^{-j(\frac{2\pi}{N})}$$

$$W_N^{-1} = e^{j(\frac{2\pi}{N})} ...Then$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k=0,......N-1 \qquad (3)$$

IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n=0,......N-1 \qquad (4)$$

Family of Mutually Orthogonal Complex Exponentials

We Have proved :

$$\sum_{k=0}^{N-1} W_N^{\,k(n-p)} = \begin{cases} N..p=n \\ 0..otherwise \end{cases}$$

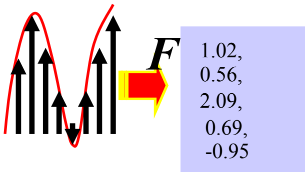
and they are periodic in *k* and in n.

- DFT inherits the periodicity from the periodicity of complex digital exponentials:

From Equation (1)

$$X(k+N) = \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})n(k+N)}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})nk} e^{-j(\frac{2\pi}{N})nN}$$



$$= \sum_{n=0}^{N-1} x[n] e^{-j(\frac{2\pi}{N})nk} = X(k)$$

$$e^{-j(\frac{2\pi}{N})nN} = 1$$

From Equation (2)

$$x[n+N] = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})k(n+N)}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})kn} e^{j(\frac{2\pi}{N})kN}$$

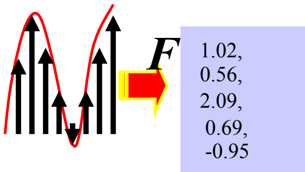
$$= \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j(\frac{2\pi}{N})kn} = x[n]$$

$$e^{j(\frac{2\pi}{N})kN} = 1$$

Periodicity

X(k+N) = X(k) k = 0, 1, …..N-1

x[n+N] = x[n] n = 0, 1, …..N-1



DFT Properties :

1) DFT is Linear :

$$DFT_N\ (\ ax_1[n] + bx_2[n]\ )$$

$$=DFT_N\ (\ ax_1[n]\ ) + DFT_N\ (\ bx_2[n]\ )$$

$$=aX_1(k) + bX_2(k)$$

2) IDFT is Linear :

$$IDFT_n(aX_1(k) + bX_2(k))$$

$$= IDFT_n(aX_1(k)) + IDFT_n(bX_2(k))$$

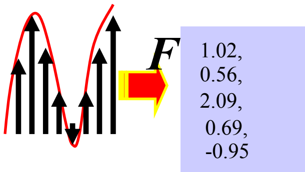
$$= ax_1[n] + bx_2[n]$$

3) DFT is periodic

$$X(k+N) = X(k)$$

4) IDFT is periodic

$$x[n] = x[n+N]$$



DFT representations :

The period : X(0)…….X(N-1) standard frequencies span from:

$$0 - 2\pi(N-1)/N \text{ rad/sample}$$

The period centered around 0 :

{X(-N/2)…..,X(0),……..X(N/2-1)}

{Normalized frequencies - π to π-2π/N rad/sample

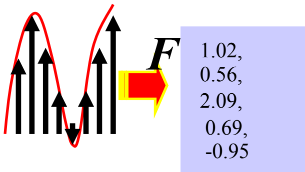
Also Note :

$$X_{-\frac{N}{2}} = X_{\frac{N}{2}}$$

$$X_{-\frac{N}{2}+1} = X_{\frac{N}{2}+1}$$

$$\ldots\ldots$$

$$X_{-1} = X_{N-1}$$



5) DFT: Time Shift

$$DFT\left[x[n-n_0]\right] = \sum_{n=0}^{N-1} x[n-n_0] e^{-j(\frac{2\pi}{N})nk}$$

$$= \sum_{<N>} x[m] e^{-j(\frac{2\pi}{N})k(m+n_0)}$$

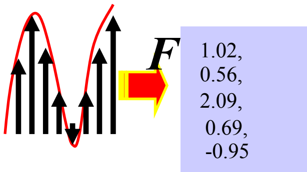
$$= e^{-j(\frac{2\pi}{N})kn_0} DFT\left(x[m]\right)$$

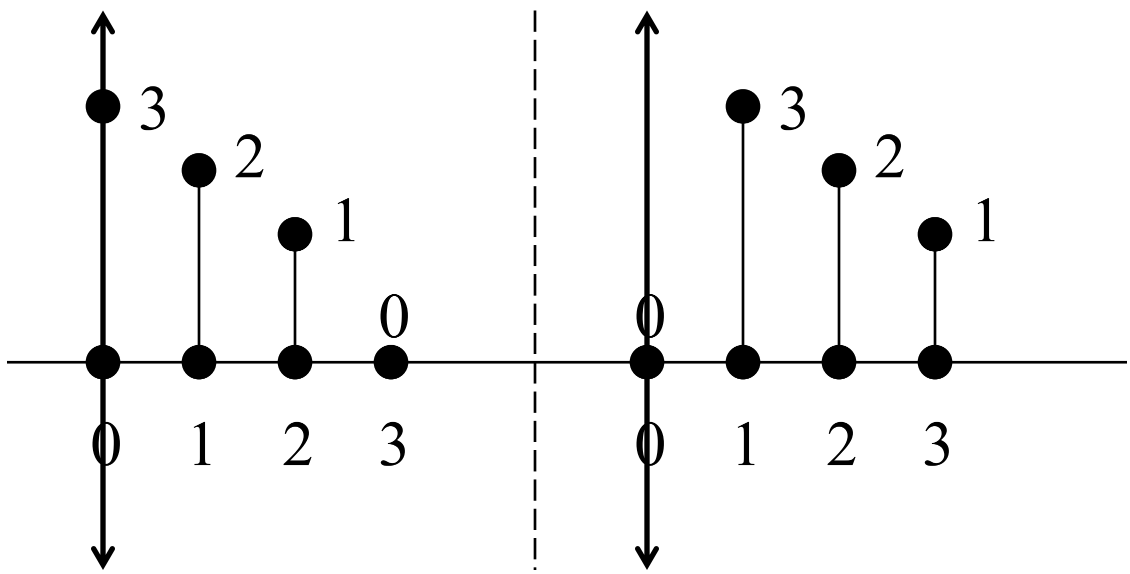
$$= e^{-j(\frac{2\pi}{N})kn_0} X(k)$$

DFT: If $X_2(k) = X_1(k) e^{-j(\frac{2\pi}{N})km}$

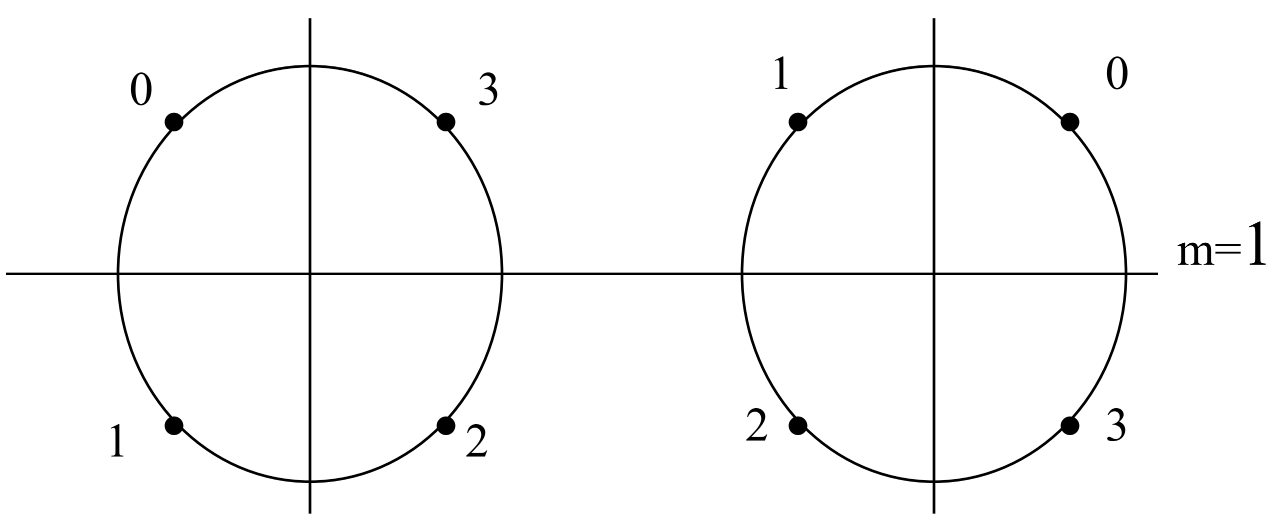
Then:

$$x_2[n] = x_1[n \oplus m] \leftarrow circular..shift$$





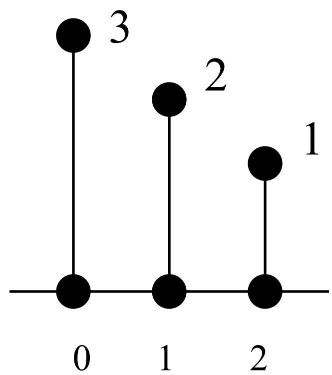
Circular shift for m=1

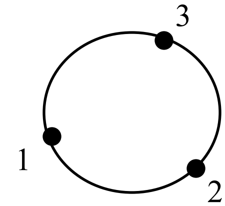


$$x_2[n] = x_1[n \oplus m] \leftarrow circular..shift$$

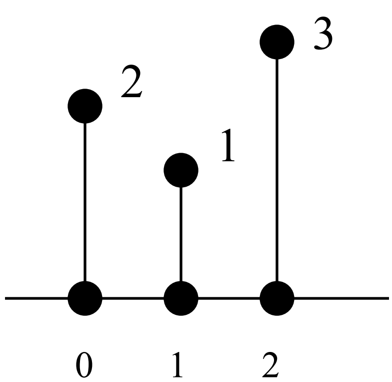
Example:

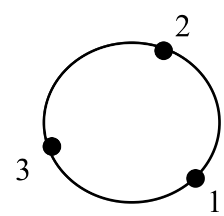
Let:





If $x_2[n] = x_1[n+1]$ m = -1





Which Implies : $X_2(k) = X_1(k)e^{j(\frac{2\pi}{3})k}$

6) Evaluation of IDFT from DFT(x[n])

$$x[n] = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k) e^{-j(\frac{2\pi}{N})nk}\right]^*$$

$$= \frac{1}{N}\left[DFT[X^*(k)\right]^*$$

Thus we see that IDFT is the complex conjugate of the DFT of $X^*(k)$ multiplied by 1/N

7) DFT: Symmetry Properties

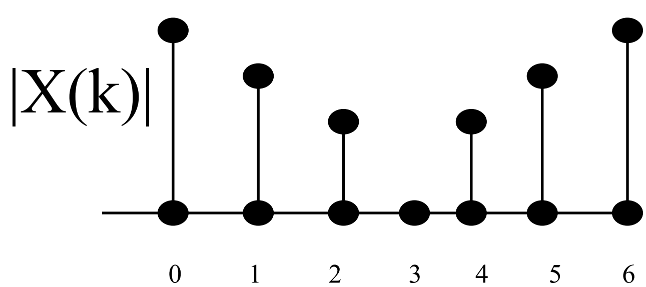
x[n] is real

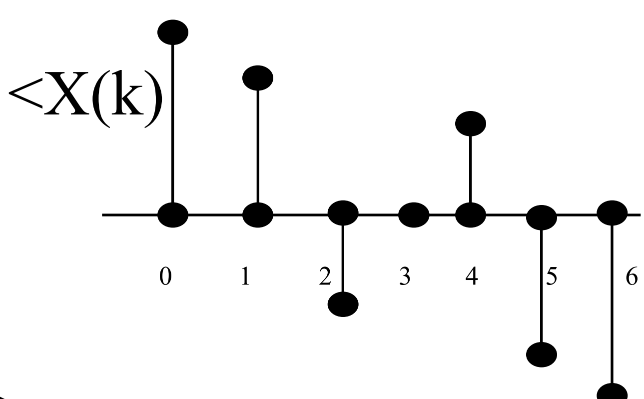
(1) Re(X(k)) = Re(X(N-k)) $_{k = 1, 2, \ldots N/2-1}$ N even

(2) Im(X(k)) = Im(X(N-k)) $_{k = 1, 2, \ldots N-1/2}$ N odd

(3) |X(k)| = |X(N-k)| $_{k = 1, 2, \ldots N/2-1}$ N even

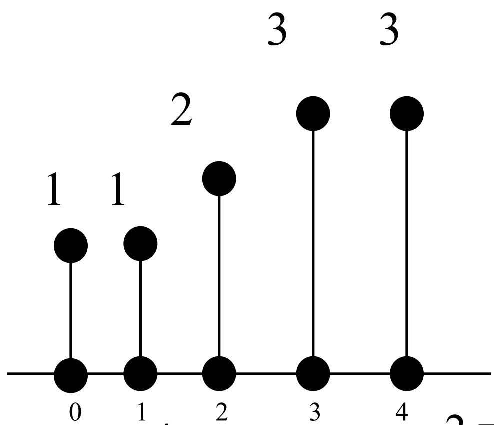
(4) <X(k) = -<X(N-k) $_{k = 1, 2, \ldots N-1/2}$ N odd





Example:

x[n]



$$X(k)=\sum_{n=0}^{4} x[n]e^{-j(\frac{2\pi}{5})kn}$$

Verify :

$$X(0)=10$$

$$X(1)=1+1e^{-j(\frac{2\pi}{5})}+2e^{-j(\frac{4\pi}{5})}+3e^{-j(\frac{6\pi}{5})}+3e^{-j(\frac{8\pi}{5})}$$

$$X(1)=3.08e^{j0.7\pi}$$

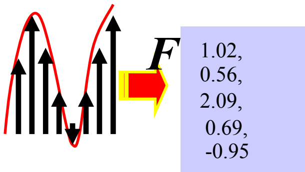
$$X(2)=1+1e^{-j(\frac{4\pi}{5})}+2e^{-j(\frac{8\pi}{5})}+3e^{-j(\frac{12\pi}{5})}+3e^{-j(\frac{16\pi}{5})}$$

$$X(2)=0.73e^{j0.9\pi}$$

Follows :

$$X(3)=X^{*}(2)=0.73e^{-j0.9\pi}$$

$$X(4)=X^{*}(1)=3.08e^{-j0.7\pi}$$



8) Even functions: If x(n) is an even function $x_e(n)$, i.e. $x_e(n) = x_e(-n)$, then

$$F_D[x_e(n)] = X_e(k) = \sum_{n=0}^{N-1} x_e(n)\cos(k\Omega nT)$$

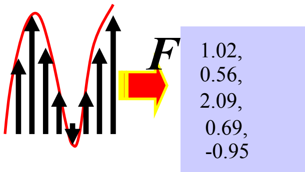
9) Odd functions: If x(n) is an odd function $x_o(n)$, i.e. $x_o(n) = -x_o(-n)$, then

$$F_D[x_o(n)] = X_o(k) = -j\sum_{n=0}^{N-1} x_o(n)\sin(k\Omega nT)$$

10) Parseval's theorem: The normalized energy in the signal is given by either of the expressions

$$\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

11) *Delta function*: $F_D[\delta(nT)] = 1$



12) The linear and circular cross-correlations of two data sequences may be computed using DFTs.

For example, the circular correlation of two finite length periodic sequences $x_{1p}(n),\ x_{2p}(n)$ can be calculated as

$$r_{cx_1x_2}(j) = \frac{1}{N}\sum_{n=0}^{N-1} x_{1p}(n)x_{2p}(n+j), \qquad j = 0,\ldots..,N-1$$

$$= F_D^{-1}[X_1^*(k)X_2(k)]$$

13) DFTs may also be used in the computation of circular and linear convolutions, for example

$$x_{3p}(n) = x_{1p}(n) \otimes x_{2p}(n)$$

$$= F_D^{-1}[X_1(k)X_2(k)]$$

where $\otimes$ denotes circular convolution, and

$x_{1p}(n), x_{2p}(n)$, and $x_{3p}(n)$ are finite periodic

sequences of equal length