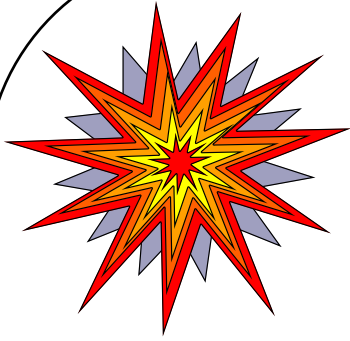# *Transform Domain Representation of Discrete Time Signals*
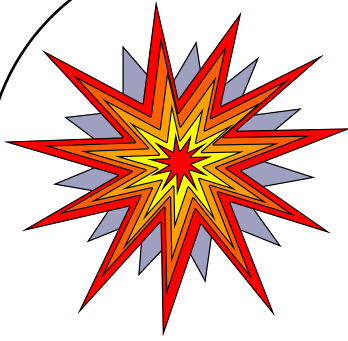
The Z Transform

Yogananda Isukapalli

Introduction

• The z-transform for discrete time signals is the counterpart of the Laplace transform for the continuos-time signals.

• The z-transform is a generalization of the Fourier transform, the principal motivation for introducing the generalization is that the Fourier transform does not converge for all sequences, which limits the class of signals that could be transformed using the Fourier transform.

• The other advantage of the z-transform is that it allows us to bring in the power of complex variable theory to bear on the problems of discrete time signals and systems.
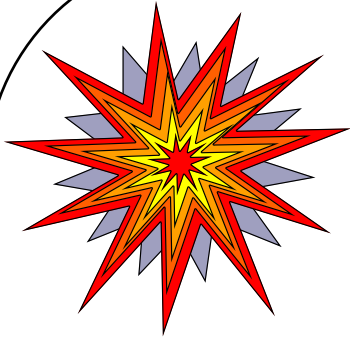
Discrete-time signals and systems

• Given an analog signal x(t), it could be represented as *discrete time signal* by a sequence of weighted & delayed impulses.

 $\{x[t]\}_{t=nT}$ n= …-2, -1, 0, 1, 2, ..

… -2T -T 0   T   2T …..
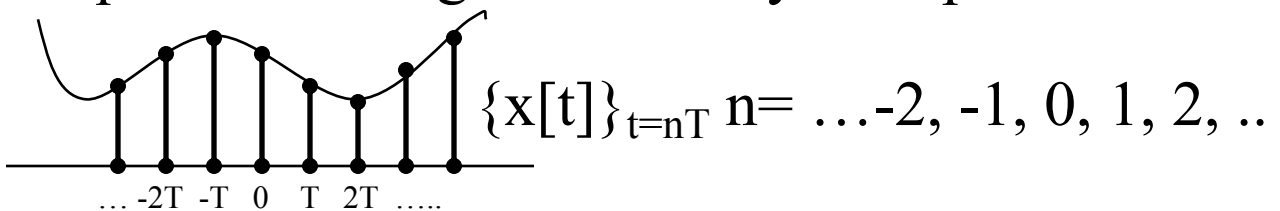
Note: We represent the discrete time signal as x[nT] = x[n] where T is the sampling period and is omitted in the representation.

• A *discrete-time system* is essentially a mathematical algorithm that takes an input sequence, x[n], and produces an output sequence, y[n].

• Linear time-invariant (LTI) systems form an important class of discrete systems used in DSP.

Discrete-time signals and systems contd…

• The input-output relationship of an LTI system is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad (1)$$
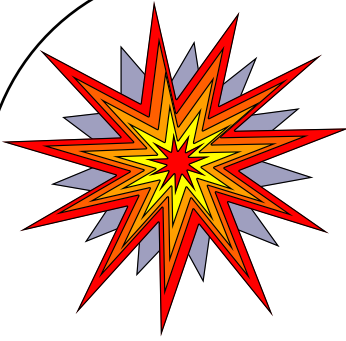
where h(k) is the impulse response of the system.

• The values of h(k) completely define the discrete-time system in the time domain.

• An LTI system is stable if its impulse response satisfies the condition

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \qquad (2)$$

and causal if

$$h(k) = 0, \ k < 0 \qquad (3)$$

Definition of Z-Transform

• For any given sequence x[n], its z-transform is defined as :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad (4)$$

$$= ..... + x[-2]z^2 + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + ...$$
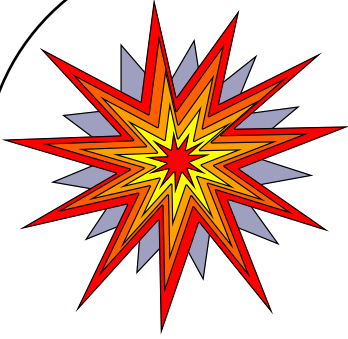
where z is a complex variable.

• If we let $z = re^{jw}$ then the expression reduces to :

$$X(re^{jw}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-jwn} \qquad (5)$$

which can be interpreted as the Fourier transform, for r=1 (i.e.,|z|=1), the z-transform of x[n] reduces to the Fourier transform provided it exists.

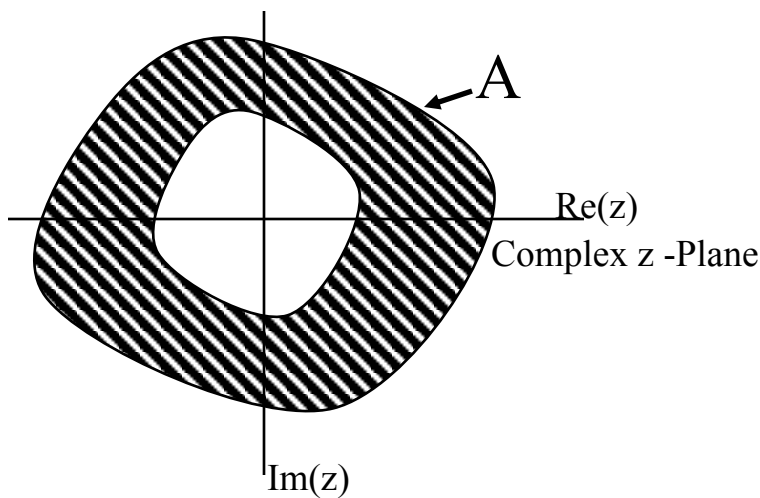• Example : $x[n] = a^n$ n 0
            $= 0$ n< 0 Find the z-transform

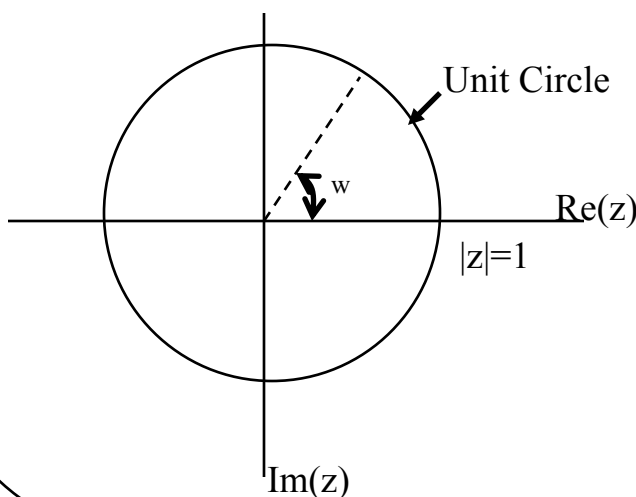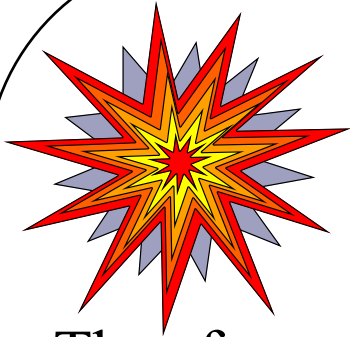Answer: $\dfrac{z}{z-a}$ ⟵⟶ {a^n}, n   0

Region of Convergence

Whenever an infinite series converges, the z-transform $X(z)$ has a finite value in some region A of the complex plane. This region is termed as the *Region of Convergence (ROC)*.

Complex z -Plane

For z=re^{jw} & r=1 thus making |z|=1, this contour in the z-plane is a circle of unity radius and is termed as the *unit circle*
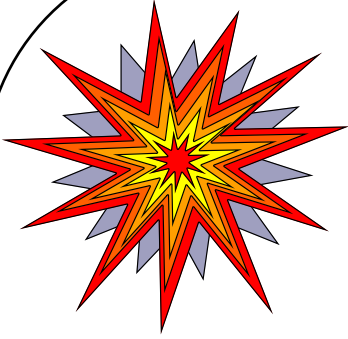
Region of Convergence contd...

Thus for any given sequence, the set of values of z for which the z-transform converges is called the region of convergence, which we abbreviate as ROC.

Convergence of the Fourier transform requires the sequence to be absolutely summable but in the case of the z-transform applying the above condition we get : $\sum\limits_{n=-\infty}^{\infty}|x[n]r^{-n}| < \infty$ for absolute convergence.

It is clear from the above equation that because of the multiplication of the sequence by the real exponential r^{-n}, it is possible for the z-transform to converge even if the Fourier transform does not. Thus for the Fourier transform to converge it must include the unit circle in its ROC.
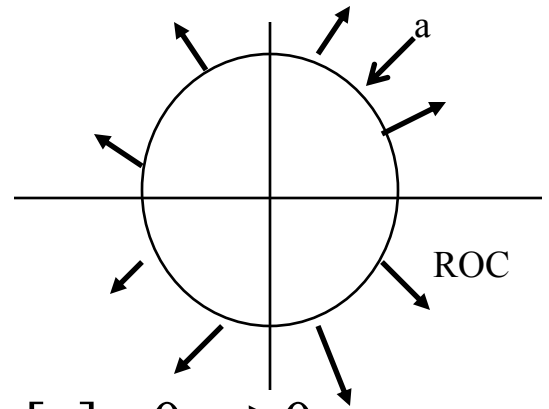
Types of Sequences

Right-handed sequences : x[n]≠0 n≥0

$$=0 \quad n<0$$

Example x[n]=$Aa^n e^{j\theta n}$ n≥0, =0   n<0

Answer: X(z)= $\dfrac{\mathbf{A.z}}{\mathbf{z-a.e^{j\theta}}}$ **..ROC:|z|>|a|**

Left-handed sequences : x[n] =0 n≥0

$$\neq 0 \quad n<0$$

Example x[n]=$-a^n u[-n-1]$

Answer: X(z)= $\dfrac{\mathbf{z}}{\mathbf{z-a}}$ **..ROC:|z|<|a|**
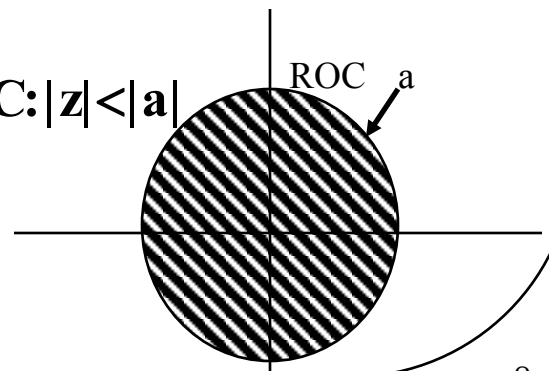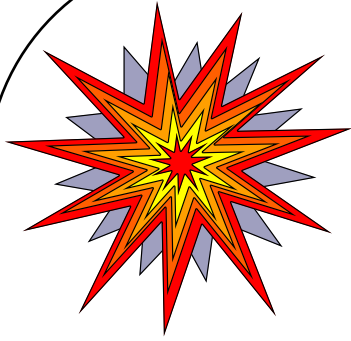
Types of Sequences contd..

Mixed sequences : $x[n] \neq 0 \quad n \geq 0$

$\neq 0 \quad n < 0$

Example $x[n] = a^n u[n] - b^n u[-n-1]$

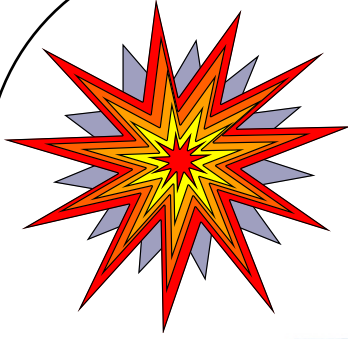Answer: $X(z) = \dfrac{z}{z-a} + \dfrac{z}{z-b}$ **..ROC:$|z| > |a|$ & $|z| < |b|$**

If $|b| < |a|$ ROC : empty set

If $|b| > |a|$ ROC : exists

ROC: $a \cap b$
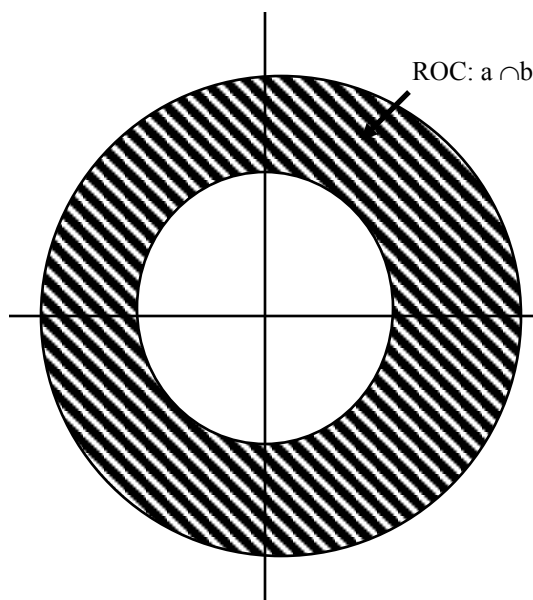
$|a| < |z| < |b|$

Z-transforms of some common sequences

| Entry number	Discrete-time sequence x(n), n ≥ 0	z-transform X(z)	Region of convergence of X(z)
1	$k\delta(n)$	k	Everywhere
2	k	$\dfrac{kz}{z-1}$	$\|z\|>1$
3	kn	$\dfrac{kz}{(z-1)^2}$	$\|z\|>1$
4	kn^2	$\dfrac{kz(z+1)}{(z-1)^3}$	$\|z\|>1$
5	$ke^{-\alpha n}$	$\dfrac{kz}{z-e^{-\alpha}}$	$\|z\|>e^{-\alpha}$
6	$kne^{-\alpha n}$	$\dfrac{kze^{-\alpha}}{(z-e^{-\alpha})^2}$	$\|z\|>e^{-\alpha}$
7	$1-e^{-\alpha n}$	$\dfrac{z(1-e^{-\alpha})}{z^2-z(1+e^{-\alpha})+e^{-\alpha}}$	$\|z\|>e^{-\alpha}$
8	$\cos(\alpha n)$	$\dfrac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}$	$\|z\|>1$
9	$\sin(\alpha n)$	$\dfrac{z\sin\alpha}{z^2-2z\cos\alpha+1}$	$\|z\|>1$
10	$e^{-\alpha n}\sin(\alpha n)$	$\dfrac{ze^{-\alpha}\sin\alpha}{z^2-2e^{-\alpha}z\cos\alpha+e^{-2\alpha}}$	$\|z\|>e^{-\alpha}$
11	$e^{-\alpha n}\cos(\alpha n)$	$\dfrac{ze^{-\alpha}(ze^{\alpha}-\cos\alpha)}{z^2-2ze^{-\alpha}\cos\alpha+e^{-2\alpha}}$	$\|z\|>e^{-\alpha}$
12	$\cosh(\alpha n)$	$\dfrac{z^2-z\cosh\alpha}{z^2-2z\cosh\alpha+1}$	$\|z\|>\cosh\alpha$
13	$\sinh(\alpha n)$	$\dfrac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}$	$\|z\|>\sinh\alpha$
14	$k\alpha^n$	$\dfrac{kz}{z-\alpha}$	$\|z\|>\alpha$
15	$kn\alpha^n$	$\dfrac{k\alpha z}{(z-\alpha)^2}$	$\|z\|>\alpha$
16	$2\|c\|\|p\|^n\cos(n\angle p+\angle c)$	$\dfrac{cz}{z-p}+\dfrac{c^*z}{z-p^*}$	

k and α are constants; c is a complex number.
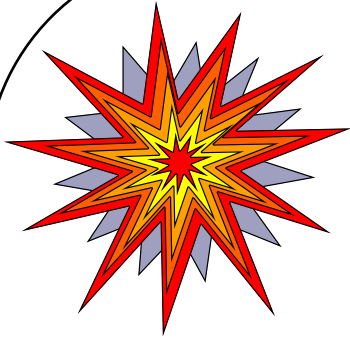
Examples

1. $x[n] = b^{|n|}$ for all 'n'

$$X(z) = \sum_{n=-\infty}^{-1} b^{-n} z^{-n} + \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=1}^{\infty} b^n z^n + \sum_{n=0}^{\infty} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (bz)^n - 1 + \sum_{n=0}^{\infty} (bz^{-1})^n$$

$$= \frac{1}{1-bz} - 1 + \frac{1}{1-bz^{-1}}$$

$$= \frac{bz}{1-bz} + \frac{z}{z-b}$$

$$= \frac{z(1-b^2)}{(1-bz)(z-b)}$$

ROC: $b \cap 1/b$
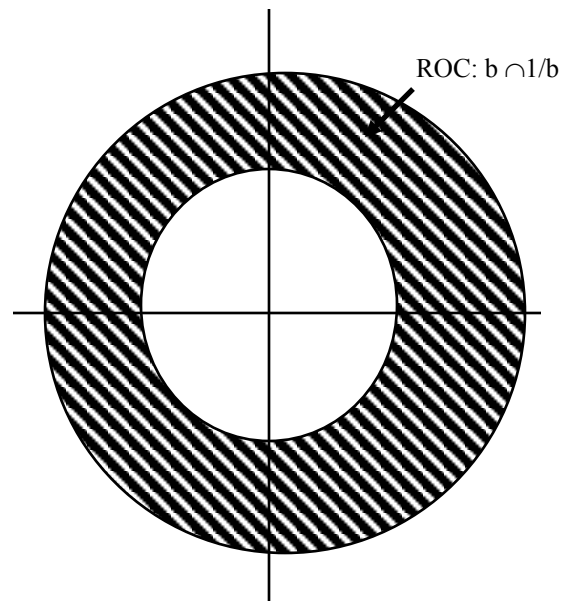
$|b| < |z| < |1/b|$

The ROC is $(1 - bz) > 0$ and $(z - b) > 0$, i.e.

$|b| < |z| < 1/|b|$

2.(a)(*i*) Given $x_1[n] = (0.3)^n \mu[n]$

Determine the ROC.

The Z-transform of $x_1[n]$ is given by

$$X_1[z] = \sum_{n=-\infty}^{n=\infty} x_1[n]z^{-n}$$

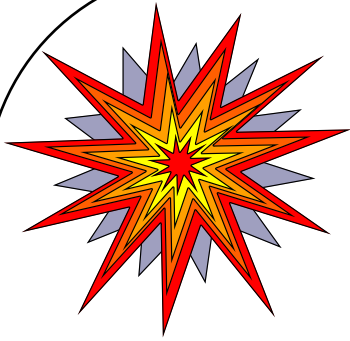$$= \sum_{n=-\infty}^{n=\infty} (0.3)^n \mu[n]z^{-n}$$

Considering that $\mu[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

$$X_1[z] = \sum_{n=0}^{n=\infty} (0.3)^n z^{-n}$$

This power series converges to

$$X_1[z] = \frac{1}{1 - 0.3z^{-1}}$$

The ROC is given by $\Re_1 : |z| > 0.3$

Examples contd…

(*ii*) For $x_2[n] = (-0.5)^n \mu[n]$

Similarly, $X_2[n] = \dfrac{1}{1 + 0.5z^{-1}}$ and ROC $\mathfrak{R}_2 : |z| > 0.5$
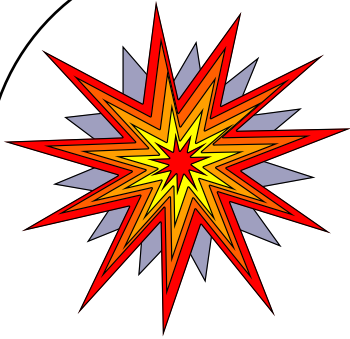
b (*i*)

To Determine the ROC of the sequence

$y_1[n] = x_1[n] + x_2[n]$

Now ROC of $X_1[z]$ is given by $\mathfrak{R}_1 : |z| > 0.3$

ROC of $X_2[z]$ is given by $\mathfrak{R}_2 : |z| > 0.5$

$\therefore$ ROC of $Y_1[z]$ is given by $\mathfrak{R}_1 \cap \mathfrak{R}_2 = \mathfrak{R}_2 : |z| > 0.5$

Properties of ROC

1. The ROC is a ring or disc in the z-plane centered at the origin i.e.,$0 \ \ r_R < |z| < r_L \ \ \infty$
2. The Fourier Transform of x[n] converges absolutely if and only if the ROC of the z-transform includes the unit circle

3. ROC cannot contain any poles

4. If x[n] is a finite duration sequence, then the ROC is the entire z-plane except possibly z=0 or $z=\infty$

5. If x[n] is a right sided sequence, the ROC extends outward from the outermost finite pole in X(z) to $z=\infty$

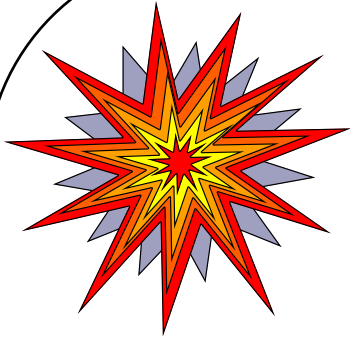6. If x[n] is a left sided sequence, the ROC extends inward from the innermost finite pole in X(z) to z= 0
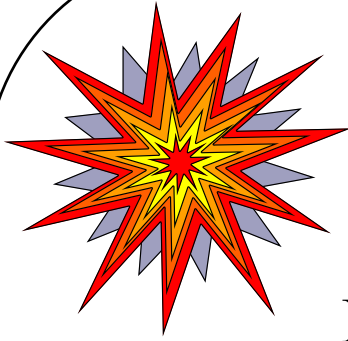
Properties of the Z-Transform

Let : $x_1[n] \xleftrightarrow{z} X_1(z)$, ROC = R_{x1}

$x_2[n] \xleftrightarrow{z} X_2(z)$, ROC = R_{x2}

1. Linearity: $ax_1[n]+bx_2[n] \xleftrightarrow{z} aX_1(z)+bX_2(z)$
   ROC contains R_{x1} R_{x2}

2. Time Shifting: $x[n-k] \xleftrightarrow{z} z^{-k}X(z)$
   ROC = R_{x1}(except for the possible addition or deletion of z=0 or z=∞)

3. Multiplication by an exponential Sequence:
   $z_0^n x[n] \xleftrightarrow{z} X(z/z_0)$ ROC = $|z_0|R_{x1}$

4. Differentiation of X(z) : $nx[n] \leftrightarrow -z\dfrac{dX(z)}{dz}$
   ROC = R_{x1}(except for the possible addition or deletion of z=0 or z=∞)

Properties of the Z-Transform contd…

5. Conjugation of a complex Sequence:

$$x^*[n] \xleftarrow{\ z\ } X^*(z^*), \qquad ROC = |z_0| R_{x1}$$

6. Time Reversal : $x[-n] \xleftarrow{\ z\ } X(1/z), ROC = 1/R_{x1}$

7. Convolution of Sequences :

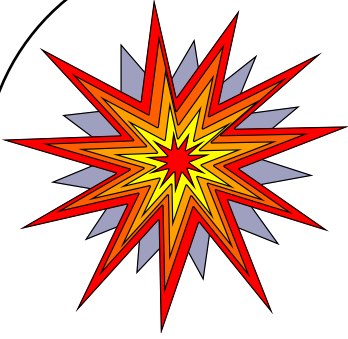$$x_1[n]\ x_2[n] \xleftarrow{\ z\ } X_1(z).X_2(z), ROC = R_{x1} \cap R_{x2}$$

8. Relationship with the Laplace transform:

Let $z = e^{sT}$, where s is the complex variable given by $s = d + j\omega$, then $z = e^{(d+j\omega)T} = e^{dT}e^{j\omega T}$
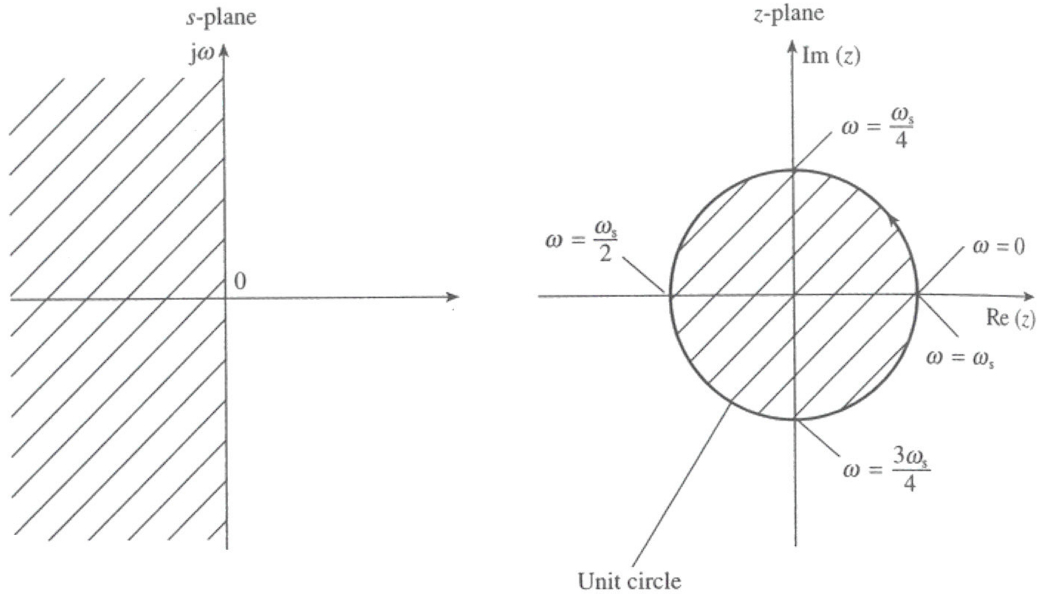
Thus $|z| = e^{dT}$ and $\angle z = \omega T = 2\pi f / F_s = 2\pi\omega / \omega_s$ where ω_s (rad s^{-1}) is the sampling frequency.

As ω varies from $-\infty$ to ∞ the s-plane is mapped to the z-plane as shown in the following figure.

Properties of the Z-Transform contd…

s-plane: ω (rad s^{-1})	z-plane: ωT (rad)
0	0
$\omega_s/4$	$\pi/2$
$\omega_s/2$	π
$3\omega_s/4$	1.25π
ω_s	2π
$1.25\omega_s$	$\pi/2$
$1.5\omega_s$	π
$1.75\omega_s$	1.25π
$2\omega_s$	2π

Figure: Mapping of frequencies from the s-plane to the z-plane (ωT is also referred to as θ)
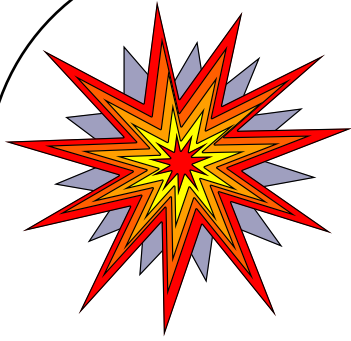
Example

Solution for Differential Equations:

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{L} b_k x[n-k$$

example)
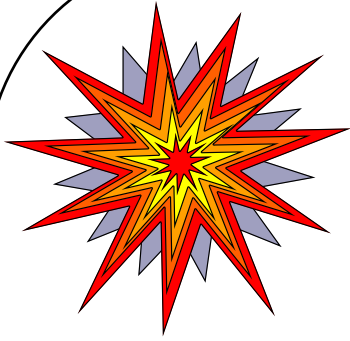
$$y[n] = -\frac{1}{2} y[n-1] + x[n] + x[n-1]$$

$$Y(z) = -\frac{1}{2} z^{-1} Y(z) + X(z) + z^{-1} X(z)$$

$$[1 + \frac{1}{2} z^{-1}] Y(z) = [1 + z^{-1}] X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{[1 + z^{-1}]}{[1 + \frac{1}{2} z^{-1}]}$$

Thus here the transfer function H(z) is specified by the above equation.

Inverse Z - Transform

• The inverse z-transform (IZT) allows us to recover the discrete-time sequence x(n), given its z-transform.

• In practice, X(z) is often expressed as a ratio of two polynomials in z^{-1} or equivalently in z:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_M z^{-M}} \qquad (6)$$

In this form, the inverse z-transform, x(n), may be obtained using one of several methods including the following three:

1. Power series expansion method

2. Partial fraction expansion method

3. Residue method
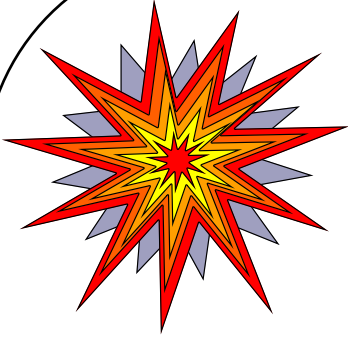
1. *Power series method*

• Given the z-transform, X(z), of a causal sequence as in eqn (6), it can be expanded into an infinite series in z⁻¹ or z by *long division* or *synthetic division*:

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_M z^{-M}}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + ...$$

• The coefficients of z^{-n} are the values of x(n).

• The numerator and denominator are first expressed in either descending powers of z or ascending powers of z^{-1} and the quotient is then obtained by long division.
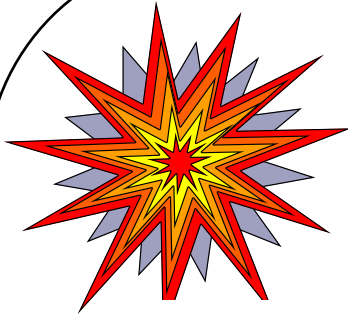
Power series method examples

1. Given the following z-transform of a causal LTI system, obtain its IZT by expanding it into a power series using long division:
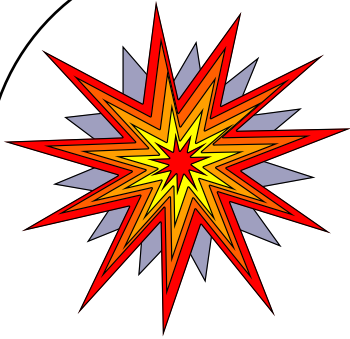
$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561z^{-2}}$$

Sol) First, we expand $X(z)$ into a power series with the numerator and denominator polynomials in ascending powers of z^{-1} and then perform the usual long division.

$$
\begin{array}{r}
1 + 3z^{-1} + 3.6439z^{-2} + 2.5756z^{-3} + \ldots \\
1 - z^{-1} + 0.3561z^{-2} \overline{\big)\ 1 + 2z^{-1} + z^{-2}} \\
\underline{1 - \ z^{-1} + 0.3561z^{-2}} \\
3z^{-1} + 0.6439z^{-2} \\
\underline{3z^{-1} - 3z^{-2} + 1.0683z^{-3}} \\
3.6439z^{-2} - 1.0683z^{-3} \\
\underline{3.6439z^{-2} - 3.6439z^{-3} + 1.297\ 592\ 7z^{-4}} \\
2.5756z^{-3} - 1.297\ 592\ 7z^{-4}
\end{array}
$$

Alternatively, we may express the numerator and denominator in positive powers of z, in descending order, and then perform the long division:

$$\frac{z^2 + 2z + 1}{z^2 - z + 0.3561}$$

$$
\begin{array}{r}
1 + 3z^{-1} + 3.6439z^{-2} + 2.5756z^{-3} + \ldots \\
z^2 - z + 0.3561 \overline{\big)\ z^2 + 2z + 1} \\
\underline{z^2 - \ z + 0.3561} \\
3z + 0.6439 \\
\underline{3z - 3 + 1.0683z^{-1}} \\
3.6439 - 3.643\ 91z^{-1} + 1.297\ 592\ 7z^{-2} \\
2.5756z^{-1} - 1.297\ 592\ 7z^{-2}
\end{array}
$$

Power series method examples contd…

Either way, the z-transform is now expanded into the familiar power series, that is

$$X(z) = 1 + 3z^{-1} + 3.6439z^{-2} + 2.5756z^{-3} + ...$$

The inverse z-transform can now be written directly:

$$x(0) = 1; x(1) = 3; x(2) = 3.6439; x(3) = 2.5756;...$$

- *The values of x(n) are also obtained recursively*:

$$x(0) = b_0 / a_0$$
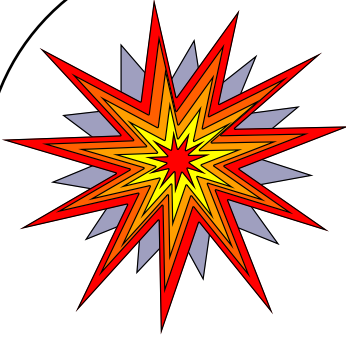$$x(1) = [b_1 - x(0)a_1] / a_0$$
$$x(2) = [b_2 - x(1)a_1 - x(0)a_2] / a_0$$
$$\vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \quad (7)$$

$$x(n) = \left[b_n - \sum_{i=1}^{n} x(n-i)a_i \right] \Big/ a_0, \qquad n = 1,2,....$$

where

$$x(0) = b_0 / a_0$$

2. We now redo the previous example using the recursive approach.

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.3561z^{-2}}$$

Comparing the coefficients of X(z) with those of the general transform in eqn (6),

$$b_0 = 1, b_1 = 2, b_2 = 1, a_0 = 1, a_1 = -1, a_2 = 0.3561; N = M = 2,$$

From eqn (7), we have

$$x(0) = b_0/a_0 = 1$$

$$x(1) = [b_1 - x(0)a_1]/a_0 = [2 - 1 \times (-1)] = 3$$

$$x(2) = [b_2 - x(1)a_1 - x(0)a_2] = 1 - 3 \times (-1) - 1 \times 0.3561 = 3.6439$$
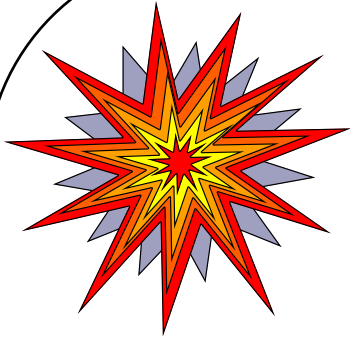
$$x(3) = [b_3 - x(2)a_1 - x(1)a_2 + x(0)a_3]$$

$$= 0 - x(2)a_1 - x(1)a_2 = 0 - 3.6439 \times (-1) - 3 \times 0.3561 = 2.5756$$

Thus the first four values of the inverse z-transform are

$$x(0) = 1, \ x(1) = 3, \ x(2) = 3.6439, \ x(3) = 2.5756$$

It is seen that both the recursive and direct, long division methods lead to identical solutions.

2. *Partial fraction expansion method*

- Consider eqn (6) again

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_M z^{-M}}$$

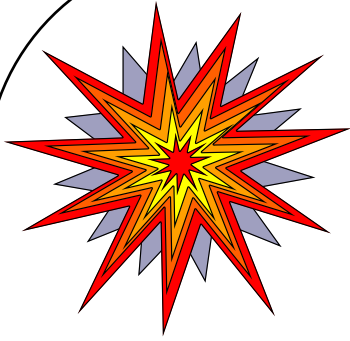If the poles of X(z) are first order and N=M, then X(z) can be expanded as:

$$X(z) = B_0 + \frac{C_1}{1 - p_1 z^{-1}} + \frac{C_2}{1 - p_2 z^{-1}} + ... + \frac{C_M}{1 - p_M z^{-1}}$$

$$= B_0 + \frac{C_1 z}{z - p_1} + \frac{C_2 z}{z - p_2} + ... + \frac{C_M z}{z - p_M}$$

$$= B_0 + \sum_{k=1}^{M} \frac{C_k z}{z - p_k} \qquad (8)$$

where $\quad B_0 = b_N / a_N$

$\quad\quad p_k$ are the poles of *X(z)* (assumed distinct)

$\quad\quad C_k$ are the partial fraction coefficients, also known as *residues* of X(z).
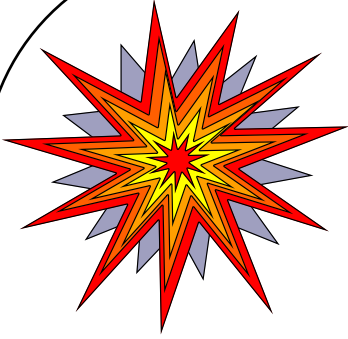
Partial fraction expansion method…

• If $N<M$, $B_0=0$

• If $N>M$, X(z) must be first reduced, to make N≤M, by long division and the remainder can be expressed as in eqn (8).

• The coefficient C_k associated with the pole p_k can be obtained as:

$$C_k = \frac{X(z)}{z}(z - p_k)\Big|_{z=p_k} \qquad (9)$$

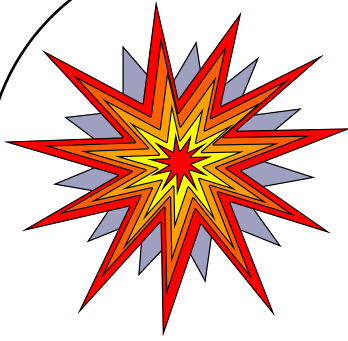• The inverse z-transform of each partial fraction is then obtained from z-transform tables.

Partial fraction expansion method…

• *Multiple-order poles*: If X(z) contains an *m*th-order pole, the partial fraction expansion must include terms of the form:

$$\sum_{i=1}^{m} \frac{D_i}{(z - p_k)^i} \qquad (10)$$

The coefficients, D_i, may be obtained from the relation

$$D_i = \frac{1}{(m-i)!} \frac{d^{m-i}}{dz^{m-i}} \left[(z - p_k)^m \frac{X(z)}{z} \right]_{z = p_k} \qquad (11)$$

Partial fraction expansion method examples

1. *X(z) contains simple, first order poles*:

$$X(z) = \frac{z^{-1}}{1 - 0.25z^{-1} - 0.375z^{-2}}$$

Find the inverse z-transform.

Soln) Multiplying the numerator and denominator by z^2 :

$$X(z) = \frac{z}{z^2 - 0.25z - 0.375} = \frac{z}{(z - 0.75)(z + 0.5)}$$
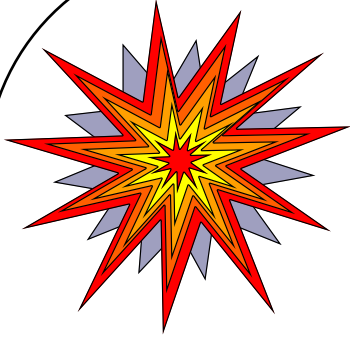
Since N < M,

$$X(z) = \frac{z}{(z - 0.75)(z + 0.5)} = \frac{C_1 z}{z - 0.75} + \frac{C_2 z}{z + 0.5}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{1}{(z - 0.75)(z + 0.5)} = \frac{C_1}{z - 0.75} + \frac{C_2}{z + 0.5}$$

We have poles $p_1 = 0.75$ and $p_2 = -0.5$

From eqn (9),

$$C_1 = \frac{X(z)}{z}(z - p_1)\bigg|_{z = p_1} = \frac{X(z)}{z}(z - 0.75)\bigg|_{z = 0.75}$$

$$= \frac{(z - 0.75)}{(z - 0.75)(z + 0.5)}\bigg|_{z = 0.75}$$

$$= \frac{1}{z + 0.5}\bigg|_{z = 0.75}$$

$$= \frac{4}{5}$$

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Similarly,

$$C_2 = \frac{X(z)}{z}(z - p_2)\Big|_{z=p_2} = \frac{X(z)}{z}(z + 0.5)\Big|_{z=-0.5}$$

$$= \frac{1}{z - 0.75}\Big|_{z=-0.5}$$

$$= -\frac{4}{5}$$

Therefore,

$$X(z) = \frac{(4/5)z}{z - 0.75} - \frac{(4/5)z}{z + 0.5} = (4/5)\left[\frac{z}{z - 0.75} - \frac{z}{z + 0.5}\right]$$

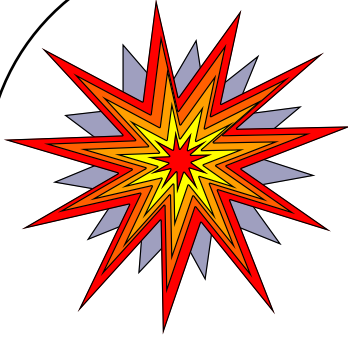*So,*

$$x(n) = \frac{4}{5}\left[(0.75)^n - (-0.5)^n\right], \ n > 0$$

2. *X(z) contains a second order pole*:

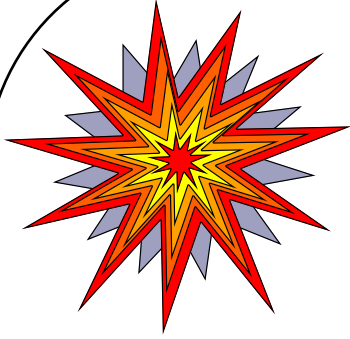$$X(z) = \frac{z^2}{(z - 0.5)(z - 1)^2}$$

Find the inverse z-transform.

Partial fraction method examples….

Soln)
$$X(z) = \frac{z^2}{(z-0.5)(z-1)^2} = \frac{C}{z-0.5} + \frac{D_1}{z-1} + \frac{D_2}{(z-1)^2}$$

$$C = \left. \frac{X(z)}{z}(z-0.5) \right|_{z=0.5}$$

$$= \left. \frac{z^2(z-0.5)}{z(z-0.5)(z-1)^2} \right|_{z=0.5}$$

$$= \left. \frac{z}{(z-1)^2} \right|_{z=0.5}$$

$$= 2$$

To obtain D_1 we use eqn (11), with *i*=1, *m*=2.

$$D_1 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z} \right]_{z=1} = \frac{d}{dz}\left[\frac{(z-1)^2 z^2}{z(z-0.5)(z-1)^2} \right]_{z=1}$$

$$= \frac{d}{dz}\left(\frac{z}{z-0.5} \right)_{z=1}$$

$$= \left. \frac{z-0.5-z}{(z-0.5)^2} \right|_{z=1}$$

$$= -2$$

Partial fraction method examples....
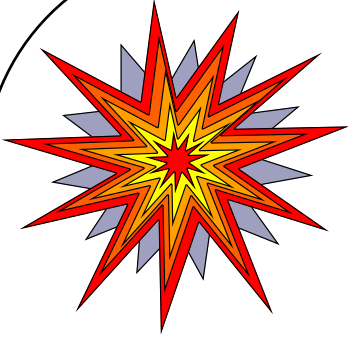
Similarly, for D_2 we use eqn (11), with $i=2$, $m=2$.

$$D_2 = \left.\frac{(z-1)^2 X(z)}{z}\right|_{z=1} = \left.\frac{(z-1)^2 z^2}{z(z-0.5)(z-1)^2}\right|_{z=1}$$

$$= \frac{1}{1-0.5}$$

$$= 2$$

Combining the results, X(z) becomes

$$X(z) = \frac{2z}{z-0.5} - \frac{2z}{z-1} + \frac{2z}{(z-1)^2}$$

Therefore,

$$x(n) = 2(0.5)^n - 2 + 2n$$

$$= 2\big[(n-1) + (0.5)^n\big], \quad n \geq 0$$

3. Residue method

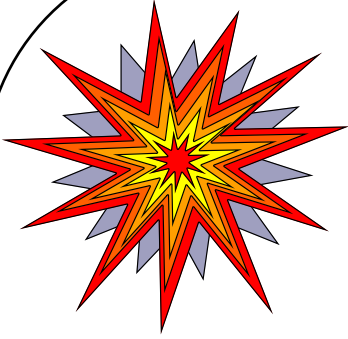• The inverse z-transform is obtained by evaluating the contour integral

$$x(n) = \frac{1}{2\pi j} \oint_c z^{n-1} X(z) dz \qquad (12)$$

where C is the path of integration enclosing all the poles of z.

• Eqn (12) is evaluated using Cauchy's residue theorem:

$$x(n) = \frac{1}{2\pi j} \oint_c z^{n-1} X(z) dz$$

= sum of the residues of $z^{n-1} X(z)$ at all the poles inside C

Residue method…

- The residue of $z^{n-1}X(z)$ at the pole p_k is given by:

$$\operatorname{Re} s[F(z), p_k] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - p_k)F(z)\right]_{z=p_k} \quad (13)$$
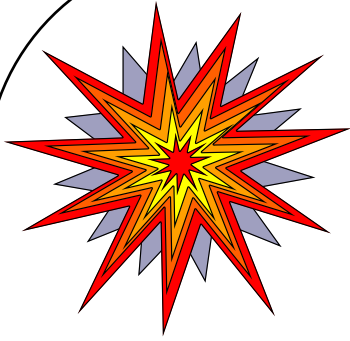
where $F(z) = z^{n-1}X(z)$,

m is the order of the pole at p_k,

Res$[F(z), p_k]$ is the residue of $F(z)$ at z= p_k.

- For a simple (distinct) pole, eqn (13) reduces to

$$\operatorname{Re} s[F(z), p_k] = (z - p_k)F(z) = (z - p_k)z^{n-1}X(z)\Big|_{z=p_k} (14)$$

Residue method examples

1.
$$X(z) = \frac{z}{(z-0.75)(z+0.5)}$$

Find x(n). Assume C is the circle |z|=1.

Soln) This problem is the same as example 1 under z-transform examples:

If we let *F(z)= z^{n-1}X(z)* then

$$F(z) = \frac{z^{n-1}z}{(z-0.75)(z+0.5)} = \frac{z^n}{(z-0.75)(z+0.5)}$$
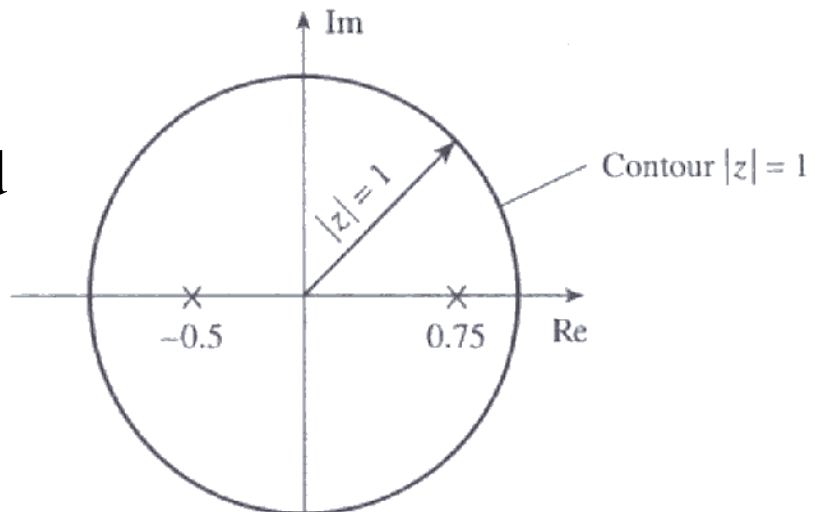
F(z) has poles at z = 0.75 and z = -0.5.
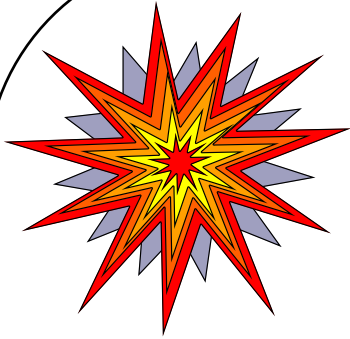

Fig: A sketch of the contour of integration with the position of poles

Residue method examples…
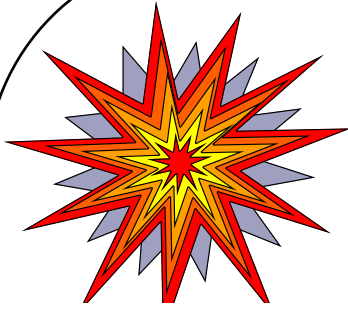
$x(n) = \text{Res}[F(z), 0.75] + \text{Res}[F(z), -0.5]$

Since the poles are first order, eqn (14) will be used. Thus,

$$\text{Re}\,s[F(z), 0.75] = (z - 0.75)F(z)\big|_{z=0.75}$$

$$= \frac{(z - 0.75)z^n}{(z - 0.75)(z + 0.5)}\bigg|_{z=0.75}$$

$$= \frac{(0.75)^n}{0.75 + 0.5}$$

$$= \frac{4}{5}(0.75)^n$$

$$\text{Re}\,s[F(z), -0.5] = (z + 0.5)F(z)\big|_{z=-0.5}$$

$$= \frac{(z + 0.5)z^n}{(z - 0.75)(z + 0.5)}\bigg|_{z=-0.5}$$

$$= -\frac{4}{5}(-0.5)^n$$

Thus, $x(n) = (4/5)[(0.75)^n - (-0.5)^n]$

which is identical to the result obtained by the partial fraction expansion method.

Residue method examples…

2. *X(z) contains a second-order pole* Find the discrete-time sequence, $x(n)$, with the following z-transform:

$$X(z) = \frac{z^2}{(z - 0.5)(z - 1)^2}$$

This example is the same as Example ■ under partial fraction expansion. According to the residue method the discrete-time sequence is given by

$$x(n) = \sum_{k=1}^{M} \text{Res}[F(z), p_k]$$

where

$$F(z) = z^{n-1}X(z) = \frac{z^{n+1}}{(z - 0.5)(z - 1)^2}$$

$F(z)$ has a simple pole at $z = 0.5$ and a second-order pole at $z = 1$; thus $x(n)$ is given by

$$x(n) = \text{Res}[F(z), p_1] + \text{Res}[F(z), p_2]$$

$$\text{Res}[F(z), 0.5] = \frac{(z - 0.5)z^{n+1}}{(z - 0.5)(z - 1)^2} = \left.\frac{z^{n+1}}{(z - 1)^2}\right|_{z=0.5}$$

$$= 0.5(0.5)^n/(0.5)^2 = 2(0.5)^n$$

$$\text{Res}[F(z), 1] = \frac{d}{dz}\left[\frac{(z - 1)^2 z^{n+1}}{(z - 0.5)(z - 1)^2}\right]$$

$$= \left.\frac{(z - 0.5)(n + 1)z^n - z^{n+1}}{(z - 0.5)^2}\right|_{z=1}$$

$$= [(0.5)(n + 1) - 1]/(0.5)^2 = 2(n - 1)$$

Combining the results, we have

$$x(n) = 2[(n - 1) + (0.5)^n]$$

which is the same result as for the partial fraction expansion method.