Discrete - Time Signals and Systems

Z-Transforms 1

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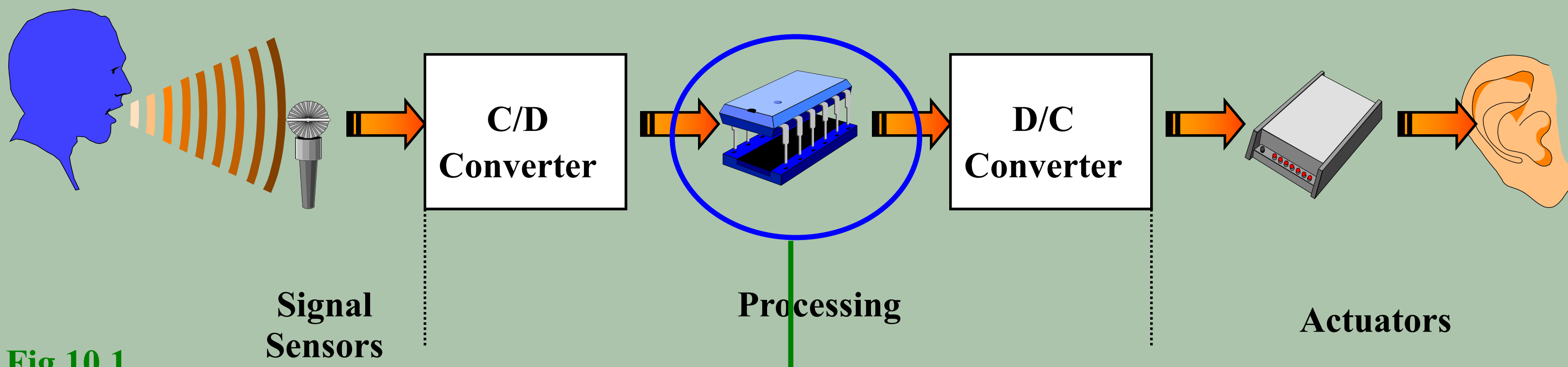


Fig.10.1

Discrete-Time System

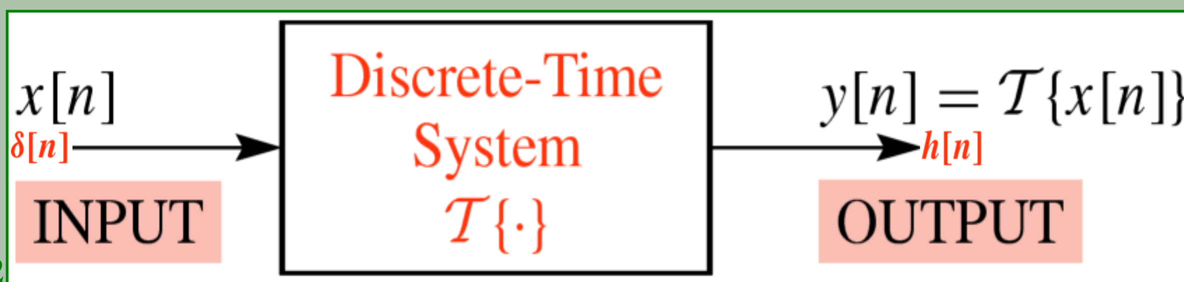


Fig.10.2

There are many mathematical descriptions for the general discrete-time system shown above

a). Difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

b). Convolution sums for FIR systems

$$y[n] = \sum_{k} x[k]h[n-k] = \sum_{k} h[k]x[n-k]$$

c). System functions

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Z Transform: Definition

Given a discrete-time sequence

$$x[n] = \{\ldots.., \; x[-2], \; x[-1], \; x[0], \; x[1], \; x[2], \ldots..\}$$

$$X(z), \;\; Z-transform \;\; of \;\; x[n] \;\; is \;\; defined \;\; as;$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots.. + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + \ldots..$$

X(z) **is a complex valued function evaluated in a complex plane**

Whenever an infinite series converges, the z-transform X(z) has a finite value in some region ***A*** of the complex plane. This region is termed as the *Region of Convergence* ***(ROC)***

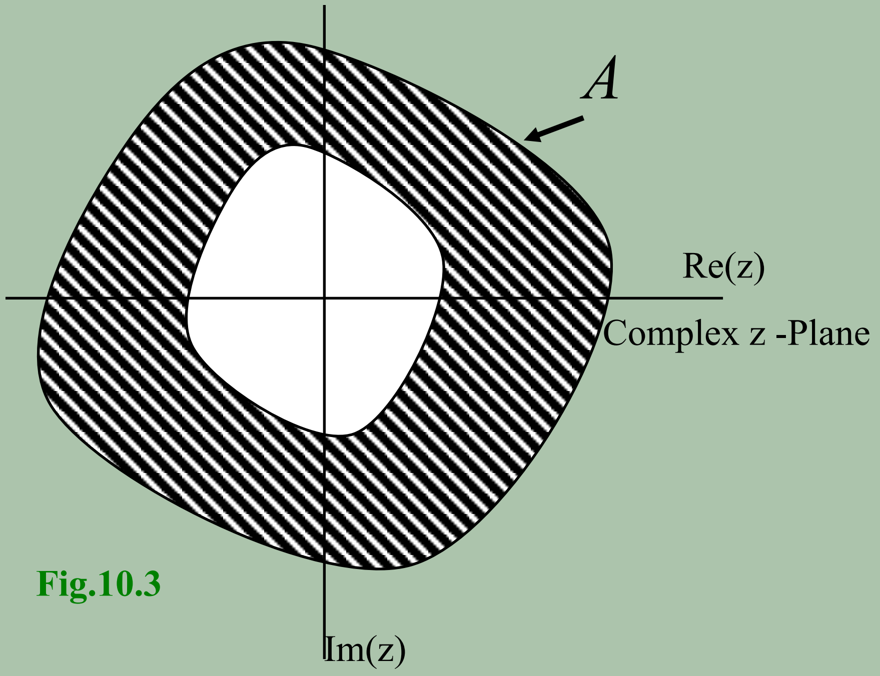


Fig.10.3

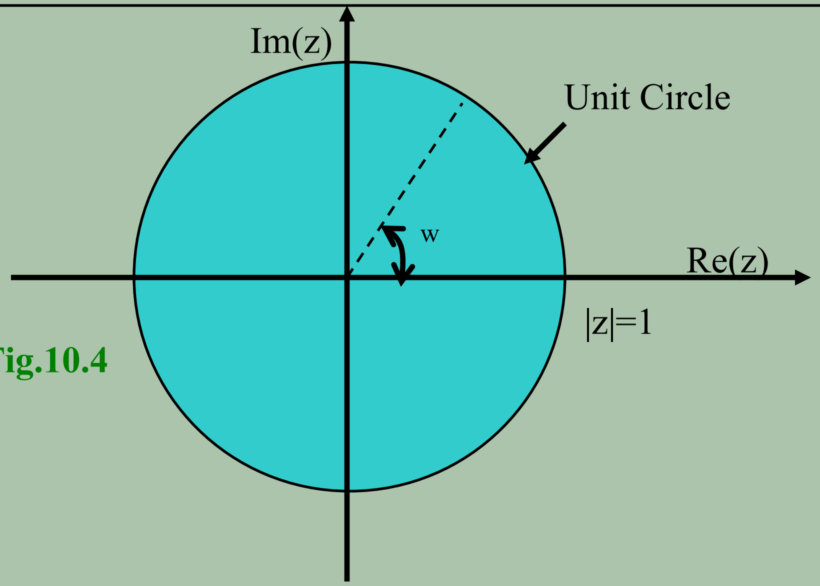


Fig.10.4

For $z=re^{jw}$ & $r=1$ thus making $|z|=1$, this contour in the z-plane is a circle of unity radius and is termed as the *unit circle*

$$x[n] \leftrightarrow X(z)$$

Why Z-transform?

The Z-transform of a finite discrete-time signal results in a polynomial

The mathematics of polynomials is a well developed concept. The analysis part is simplified in Z-domain

Z-Transform can be applied to FIR Filter

$$h[n] \leftrightarrow H(z)$$

Convolution will become simple multiplication

$$Y(z) = X(z)H(z)$$

Example 1

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

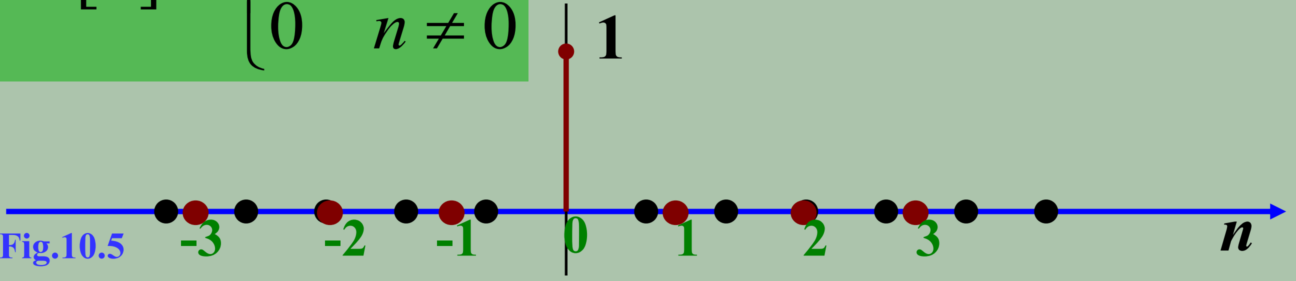


Fig.10.5

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots.. + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots..$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \ldots. + 0 + 0\ldots. + 1z^{0} + 0 + 0\ldots\ldots$$

$$= 1 \quad \textit{ROC all Z}$$

Example 2

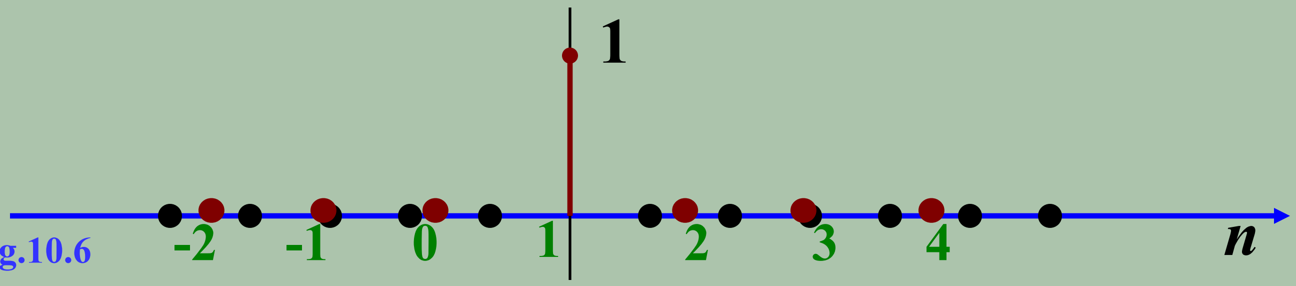


Fig.10.6

$$x[n] = \delta[n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = ...0 + 0.... + 1z^{-1} + 0..... = z^{-1}$$

$\delta[n-1] \leftrightarrow z^{-1}$ $ROC \ all \ Z, \ except \ Z = 0$

Example 3

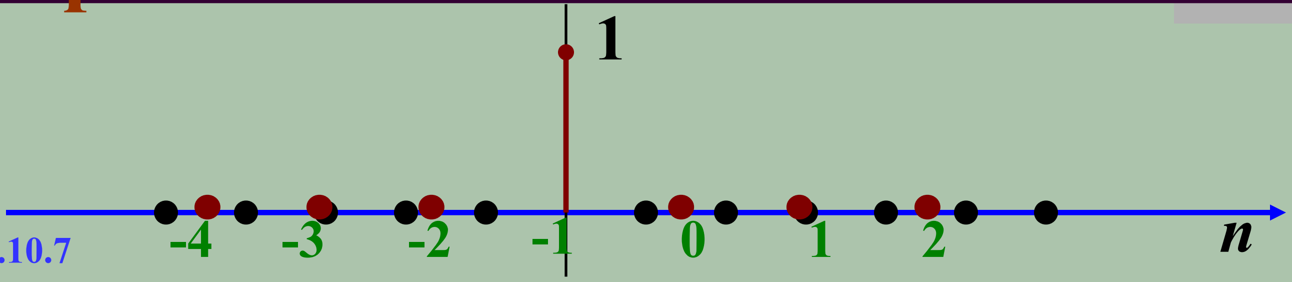


Fig.10.7

$$x[n] = \delta[n+1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = \ldots.0 + 0\ldots. + 1z^{1} + 0\ldots.$$

$$= z^{1}$$

$\delta[n+1] \leftrightarrow z$ $\quad ROC \ \ all \ \ Z, \ \ except \ \ Z = \infty$

Example 4

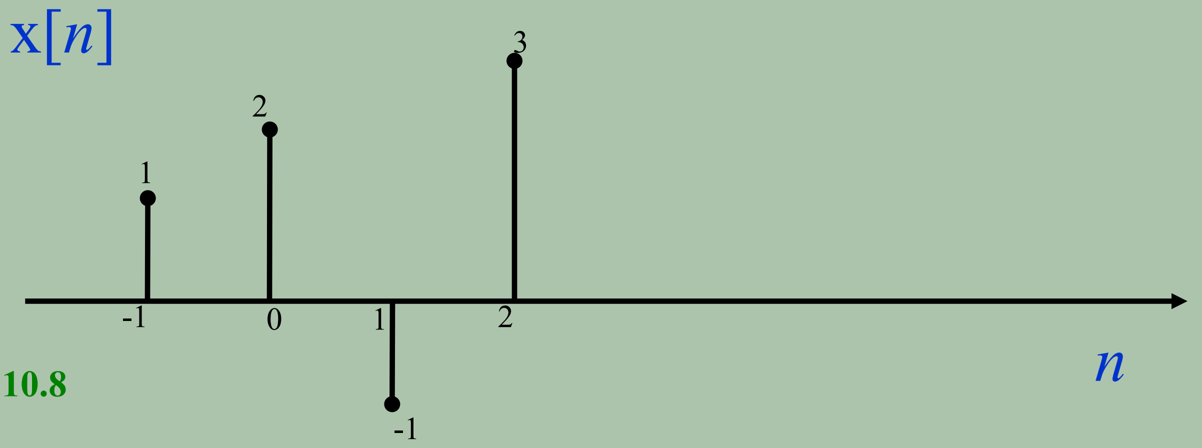


Fig.10.8

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots\ldots + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots\ldots$$

$$= \ldots + 0 + z^{1} + 2z^{0} - 1z^{-1} + 3z^{-2} + 0 + \ldots\ldots$$

$$= z^{1} + 2z^{0} - 1z^{-1} + 3z^{-2}$$

Example 5: Infinite Unit Sequence

$$x[n] = u[n] = 1 \quad for\ all\ n, \qquad n \geq 0$$

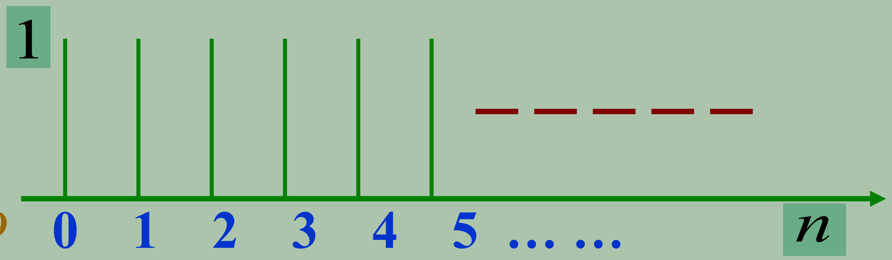


Fig.10.9

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots\ldots + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots\ldots$$

$$= \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$

$$= \frac{1}{1 - z^{-1}} (using\ sum\ of\ infinite\ series) = \frac{z}{z-1} \quad ROC\ |z| > 1$$

$$\sum_{n=0}^{\infty} \alpha^{n} = 1 + \alpha + \alpha^{2} + + \alpha^{3} + \ldots = \frac{1}{1-\alpha} \quad only \quad if \quad |\alpha| < 1;$$

Example 6: Finite Unit Sequence

$$x[n] = u[n] = 1 \quad for \quad 0 \le n \le N-1$$

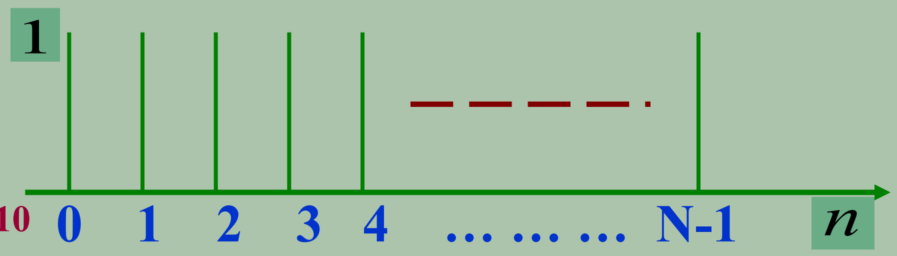


Fig.10.10

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=0}^{N-1} u[n]z^{-n} = \sum_{n=0}^{N-1} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \ldots + z^{-(N-1)}$$

$$= \frac{1 - z^{-N}}{1 - z^{-1}} \; (using\ sum\ of\ finite\ series) \quad ROC\ all\ z,\ except\ z = 0$$

$$\sum_{n=0}^{N-1} \alpha^n = 1 + \alpha + \alpha^2 + \ldots + \alpha^{N-1}$$

$$= \frac{1 - \alpha^N}{1 - \alpha}$$

Example 7: Infinite Right handed sequence

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \ldots\ldots + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots\ldots$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \ldots$$

$$= \frac{1}{1 - az^{-1}} (using\ sum\ of\ infinite\ series) = \frac{z}{z - a}$$

Example 7 continued...

The issue of convergence

Consider the sum of infinite geometric series

$$\sum_{n=0}^{\infty}(\alpha)^{n} = 1 + \alpha + \alpha^{2} + +\alpha^{3} + \ldots$$

$$= \frac{1}{1-\alpha} \quad only \quad if \ |\alpha| < 1;$$

higher order terms will become zero when $|\alpha| < 1$

$|az^{-1}| < 1$ *or* $|z| > |a|$, *for the problem at hand*

$|z| > |a|$ *is known as Region of Convergence* (ROC)

Example 7 continued...

$$x[n] = a^n u[n], \qquad X(z) = \frac{z}{z-a}$$

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \qquad ROC\ |z| > |a|$$

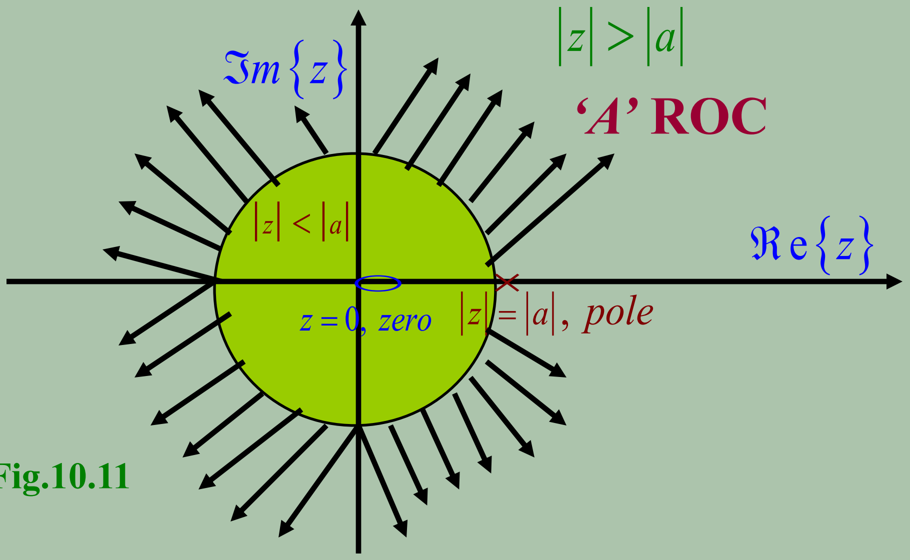


Fig.10.11

Example 8: Finite Right handed sequence

$$x[n] = a^n u[n] \quad for \quad 0 \le n \le N-1$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{N-1} a^n u[n] z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n}$$

$$= \sum_{n=0}^{N-1} \left(az^{-1} \right)^n = 1 + az^{-1} + a^2 z^{-2} + \ldots.. + a^{(N-1)} z^{-(N-1)}$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} (using\ sum\ of\ finite\ series) \quad ROC\ all\ Z$$

Example 9: Left handed sequence

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} \qquad \because (-n-1) \geq 0, \quad -\infty < n \leq -1$$

The sign of 'n' can be changed, n = -n

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

Example 9 continued...

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} + 1 - 1$$

$$= 1 - \left(\sum_{n=1}^{\infty} a^{-n} z^{n} + 1 \right)$$

$$= 1 - \left(\sum_{n=1}^{\infty} a^{-n} z^{n} + a^{-0} z^{0} \right) = 1 - \sum_{n=0}^{\infty} a^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} \left(a^{-1} z \right)^{n}$$

$$= 1 - \frac{1}{1 - a^{-1} z} \quad (\textit{using sum of infinite series})$$

$$= \frac{1 - a^{-1}z - 1}{1 - a^{-1}z} = \frac{z}{z - a}$$

Region of convergence

In the infinite sum,

$$\left| a^{-1}z \right| < 1 \quad or \quad |z| < |a|$$

Notice the interesting result,

$$-a^{n}u[-n-1] \quad \leftrightarrow \quad \frac{z}{z-a} \qquad ROC \ |z| < |a|$$

$$a^{n}u[n] \quad \leftrightarrow \quad \frac{z}{z-a} \qquad ROC \ |z| > |a|$$

Same Z – transform for two different signals with different ROC's

Example 9 continued...

$$x[n] = -a^n u[-n-1], \quad X(z) = \frac{z}{z-a}$$

$$-a^n u[-n-1] \leftrightarrow \frac{z}{z-a} \quad ROC \; |z| < |a|$$

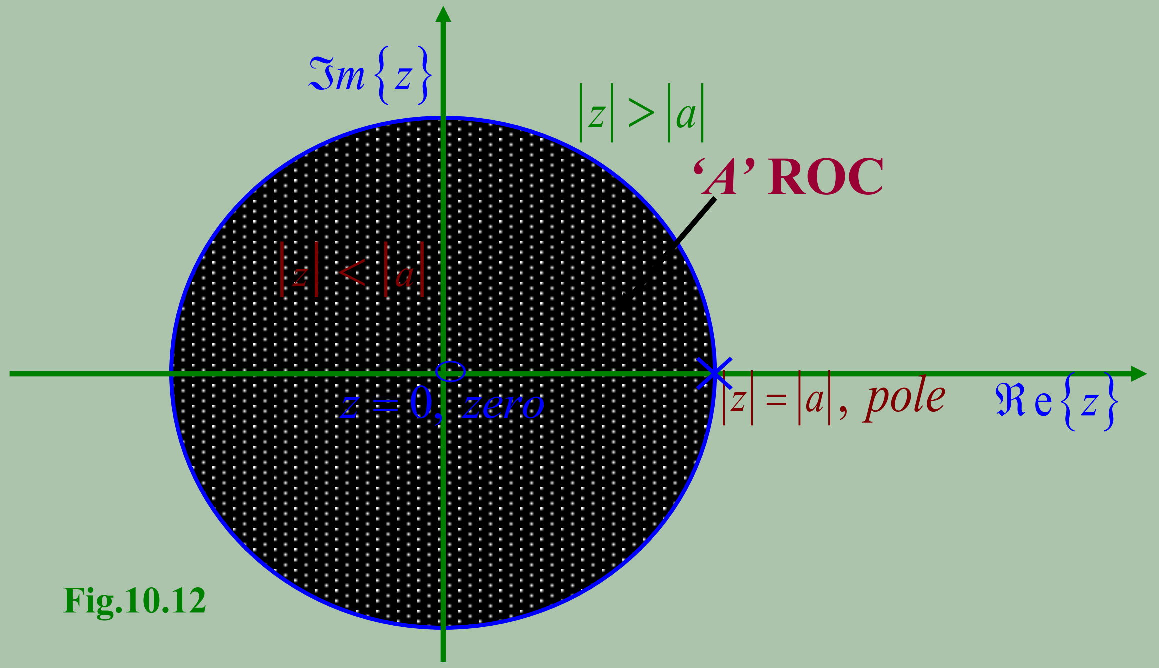


Fig.10.12

Example 10: Mixed Sequences

$$x[n] = a^n u[n] - b^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a^n u[n] - b^n u[-n-1]\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a^n u[n] z^{-n} - b^n u[-n-1] z^{-n}\right)$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-b^n u[-n-1] z^{-n}\right)$$

Example 10 continued...

From the previous two examples

$$= \underbrace{\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}}_{Right.H\ \ Sequence} + \underbrace{\sum_{n=-\infty}^{\infty} \left(-b^n u[-n-1] z^{-n}\right)}_{Left.Handed\ \ Sequence}$$

$$X(z) = \underbrace{\frac{z}{z-a}}_{ROC\ \ |z|>|a|} + \underbrace{\frac{z}{z-b}}_{ROC\ \ |z|<|b|}$$

ROC has to be the overlapping area of the two regions, $\{\ |z| > |a| \cap |z| < |b|\ \}$

Example 10 continued ...

If $|b| < |a|$, *there is no overlapping region hence* $X(z)$ *doesn't exist*

If $|b| > |a|$

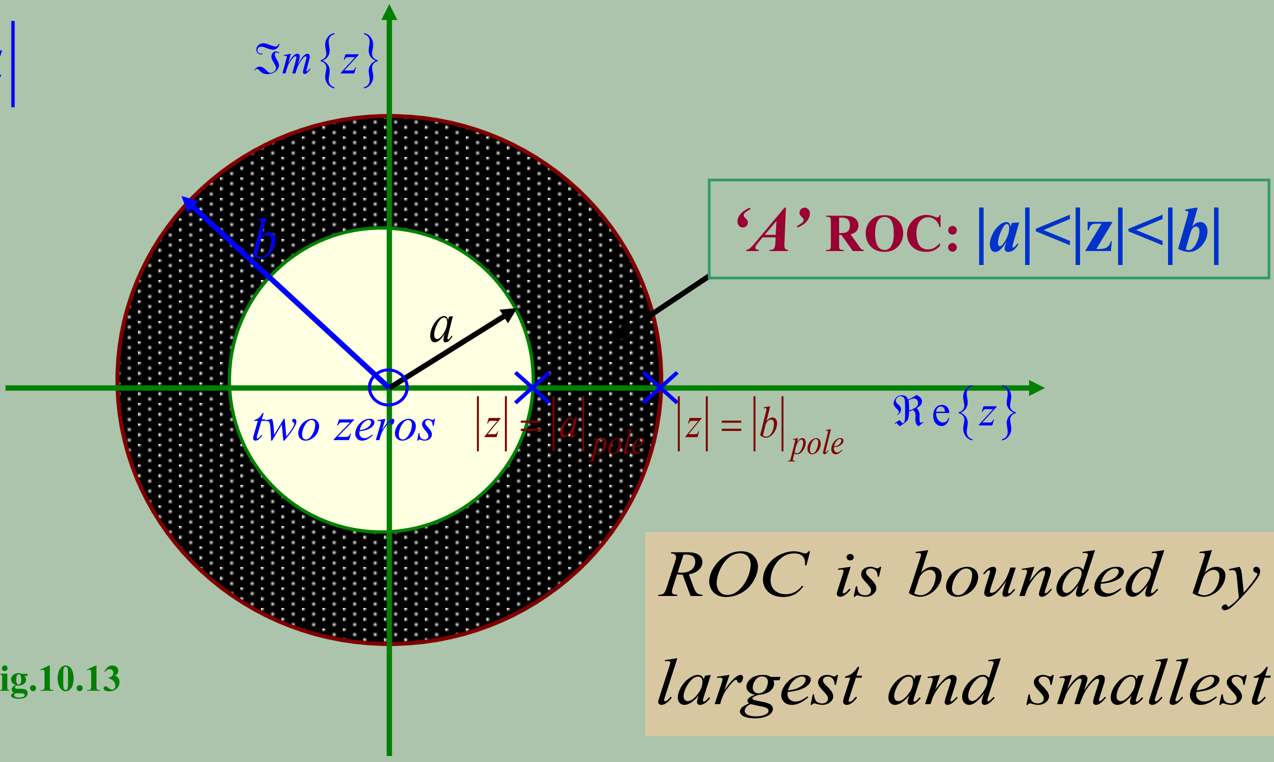


Fig.10.13

ROC is bounded by the largest and smallest poles

Example 11: Mixed Sequence

$$x[n] = b^{|n|}, \quad for\ all\ 'n'$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$x[n]$ *can be split into two sequences;*

$$x[n] = \underbrace{b^{-n} u[-n-1]}_{Left.H\ sequence} + \underbrace{b^{n} u[n]}_{R.Handed}$$

$$X(z) = \sum_{n=-\infty}^{-1} b^{-n} z^{-n} + \sum_{n=0}^{\infty} b^{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} b^{n} z^{n} + \sum_{n=0}^{\infty} b^{n} z^{-n}$$

Example 11 continued...

$$= \sum_{n=0}^{\infty} (bz)^n - 1 + \sum_{n=0}^{\infty} (bz^{-1})^n$$

$$= \frac{1}{1-bz} - 1 + \frac{1}{1-bz^{-1}}$$

$$= \underbrace{\frac{bz}{1-bz}}_{ROC\ |z|<\frac{1}{b}} + \underbrace{\frac{z}{z-b}}_{ROC\ |z|>b}$$

$$= \frac{z(1-b^2)}{(1-bz)(z-b)}$$

$$ROC \quad b < |z| < \frac{1}{b}$$

Example 11 continued...

If $|b| > 1$, *ROC doesn't exist*

If $0 < |b| < 1$

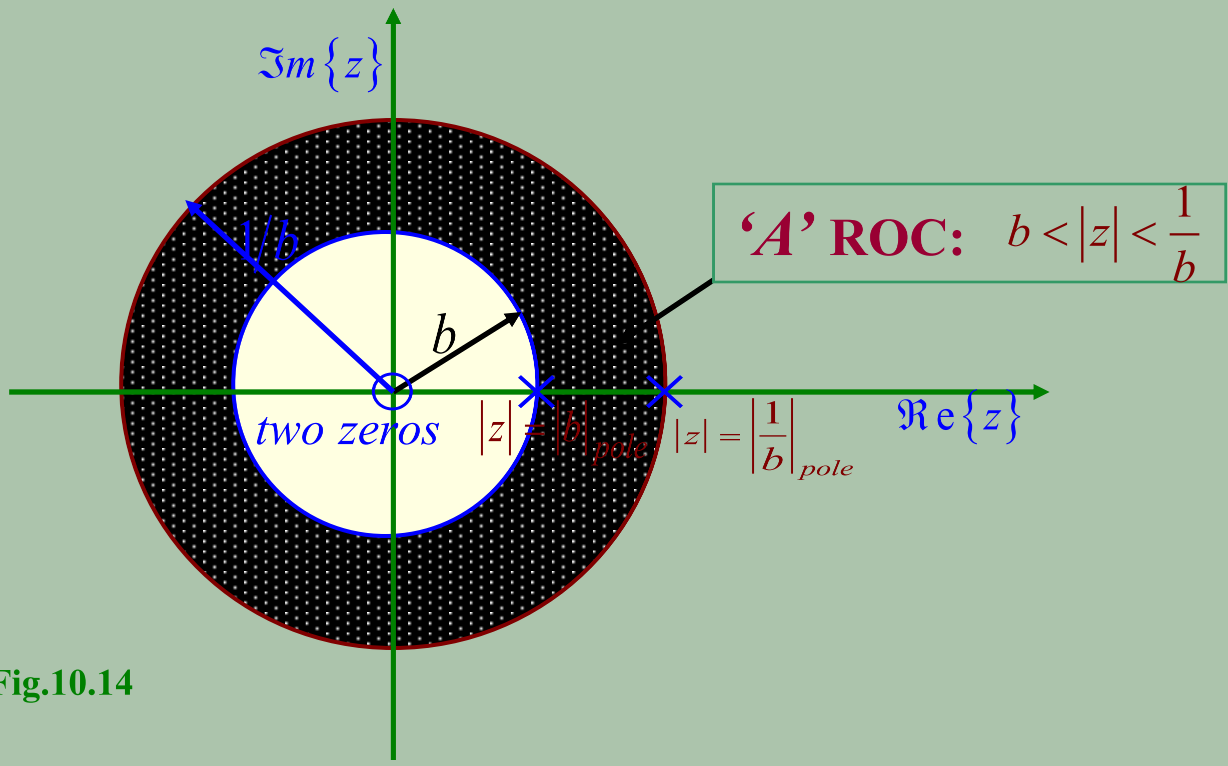


Fig.10.14

Example 12: Two right handed signals

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since both are right handed signals,

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad ROC \ |z| > |a|$$

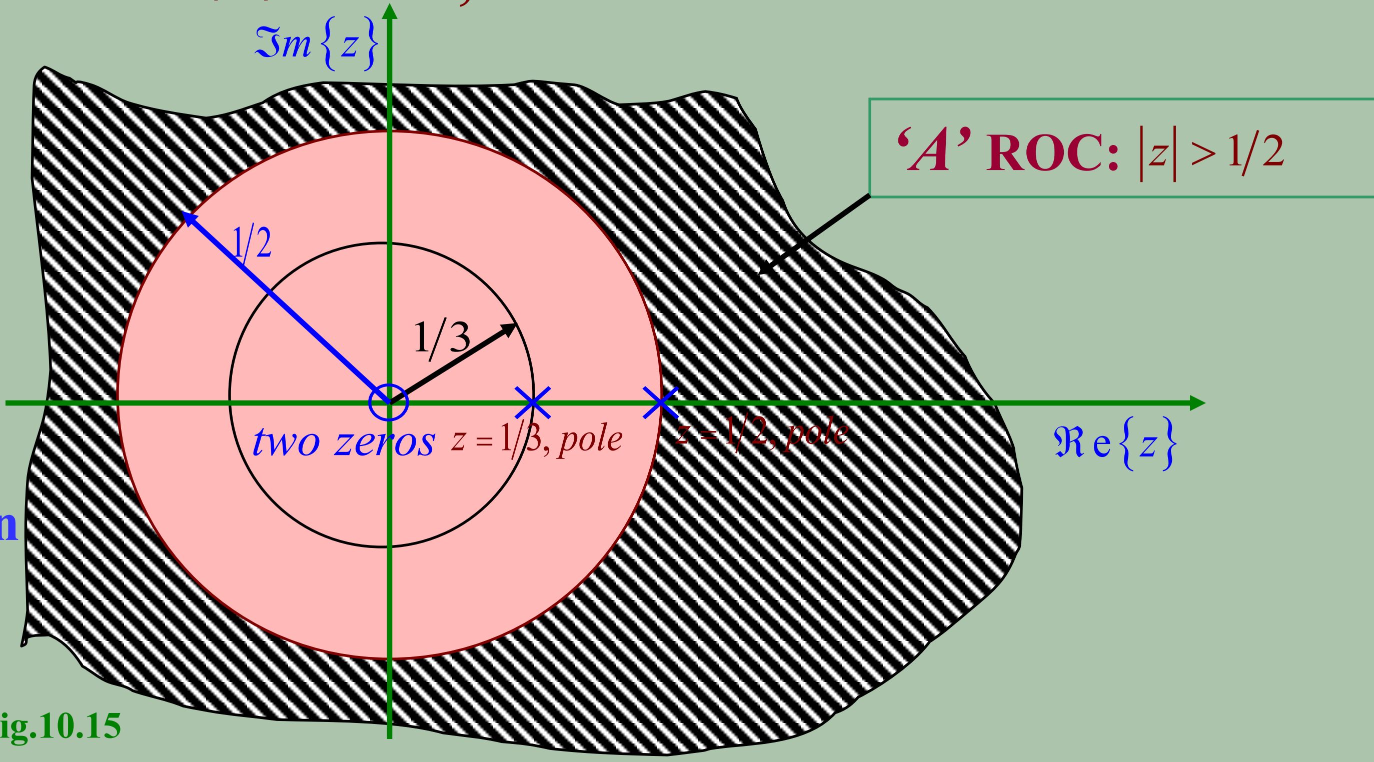
$$\left(\frac{1}{3}\right)^n u[n] \leftrightarrow \frac{z}{z-1/3} \quad ROC \ |z| > 1/3$$

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{z}{z-1/2} \quad ROC \ |z| > 1/2$$

Example 12 continued...

$$X(z)=\frac{7z}{z-1/3}-\frac{6z}{z-1/2}$$

$$ROC,\ \left\{\ |z|>1/3\cap|z|>1/2\ \right\}\Rightarrow|z|>1/2$$



The shape is drawn arbitrarily

Fig.10.15

Example 13

$$x[n] = Aa^n e^{j\theta n} u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad ROC \ |z| > |a|$$

$$x[n] = A\left(ae^{j\theta}\right)^n u[n]$$

$$Aa^n e^{j\theta n} u[n] \leftrightarrow \frac{z}{z-ae^{j\theta}} \quad ROC \ |z| > \left|ae^{j\theta}\right|$$

$$\therefore X(z) = \frac{z}{z-ae^{j\theta}} \quad ROC \ |z| > |a| \quad \left(\because \left|e^{j\theta}\right| = 1\right)$$

Example 14

$$x[n] = \cos[\alpha n]u[n]$$

$$\cos[\alpha n] = \frac{e^{j\alpha n} + e^{-j\alpha n}}{2}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{e^{j\alpha n} + e^{-j\alpha n}}{2} \right) z^{-n} u[n]$$

$$= \frac{1}{2} \left(\sum_{n=0}^{\infty} e^{j\alpha n} z^{-n} + \sum_{n=0}^{\infty} e^{-j\alpha n} z^{-n} \right)$$

$$= \underbrace{\frac{z/2}{z - e^{j\alpha}}}_{ROC|z|>1} + \underbrace{\frac{z/2}{z - e^{-j\alpha}}}_{ROC|z|>1}$$

$$= \frac{1}{2} \left(\frac{z(2z - 2\cos\alpha)}{z^2 - 2z\cos\alpha + 1} \right) \quad ROC\,|z| > 1$$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “7.1-7.2,--Signal Processing First”, Prentice Hall, 2003