Discrete - Time Signals and Systems

Z-Transforms 2

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Z Transform: Review

Given a discrete-time sequence $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$X(z)$ *exists if and only if the infinite series converges*

converging plane is called Region of convergence ROC

$$x[n] \leftrightarrow X(z)$$

$$\delta[n] \leftrightarrow 1 \quad ROC\ all\ Z$$

$$u[n] \leftrightarrow \frac{z}{z-1} \quad ROC\ |Z| > 1$$

$$a^n u[n] \leftrightarrow \frac{z}{z-a} \quad ROC\ |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{z}{z-a} \quad ROC\ |z| < |a|$$

$$b^{|n|} \leftrightarrow \frac{z(1-b^2)}{(1-bz)(z-b)} \quad ROC\ b < |z| < \frac{1}{b}$$

$$Aa^n e^{j\theta n} u[n] \leftrightarrow \frac{Az}{z-ae^{j\theta}} \quad ROC\ |z| > |a|$$

$$a^n u[n] - b^n u[-n-1] \leftrightarrow \frac{z}{z-a} + \frac{z}{z-b} \quad ROC\ \{\ |z| > |a| \cap |z| < |b|\ \}$$

The Inverse Z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \ldots.. + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + \ldots.$$

$Inverse\ Z-transform\ is\ nothing\ but\ obtaining\ the$

$discrete-time\ samples\ from\ the\ polynominal\ X(z)$

$Example1:\ X(z) = 4z^{2} + 2 + 3z^{-1}$ $ROC\ all\ z, except\ z = 0\ \&\ \infty$

Compare with the definition and pick out

the discrete-time samples

$$\{x[-2], x[-1], x[0], x[1]\} = \{4, 0, 2, 3\}$$

$$\{x[-2], x[-1], x[0], x[1]\} = \{4, 0, 2, 3\}$$

$$x[n] = 0 \quad otherwise$$

Writing it as an equation

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

$$\because \delta[n+n_0] \xleftrightarrow{z} z^{n_0}$$

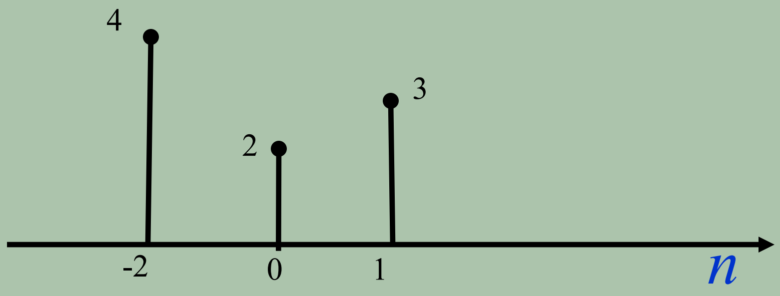


Fig.11.1

Example 2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

Example 3: Division method

$$X(z)=\frac{z^2+1}{z^2+z-2}$$

$$The\ problem\ can\ be\ solved\ through\ expansion,$$

$$though\ this\ can\ be\ done\ in\ two\ ways;$$

$$X(z)=\underbrace{\sum_{n=-\infty}^{-1}x[n]z^{-n}}_{Left.H\ sequence}+\underbrace{\sum_{n=0}^{\infty}x[n]z^{-n}}_{Right.H\ sequence}$$

$$First\ consider\ a\ right\ handed\ sequence:$$

$$X(z)=x[0]+x[1]z^{-1}+x[2]z^{-2}+x[3]z^{-3}+......$$

$$\begin{array}{r|l} & \underline{1-z^{-1}+4z^{-2}-6z^{-3}+.......} \\ z^{2}+z-2 & z^{2}+1 \\ & \underline{z^{2}+z-2} \\ & \qquad -z+3 \\ & \qquad \underline{-z-1+2z^{-1}} \\ & \qquad\qquad 4-2z^{-1} \\ & \qquad\qquad \underline{4+4z^{-1}-8z^{-2}} \\ & \qquad\qquad\qquad -6z^{-1}+8z^{-2} \end{array}$$

$$\therefore X(z)=1-z^{-1}+4z^{-2}-6z^{-3}+.......$$

By inspection,

$$\{x[0],x[1],x[2],x[3],....\}=\{1,-1,4,-6,...\}$$

Now consider a left handed sequence :

$$X(z) = \dots\dots + x[-3]z^3 + x[-2]z^2 + x[-1]z^1$$

$$\begin{array}{r|l}
 & \underline{-(1/2)-(1/4)z-(7/8)z^2+\dots\dots} \\
-2+z+z^2 & 1+0z+z^2 \\
 & \underline{1-(1/2)z-(1/2)z^2} \\
 & \qquad (1/2)z+(3/2)z^2 \\
 & \qquad \underline{(1/2)z-(1/4)z^2-(1/4)z^3} \\
 & \qquad\qquad (7/4)z^2+(1/4)z^3
\end{array}$$

$$\therefore X(z) = -1/2-(1/4)z-(7/8)z^2+\dots\dots$$

By inspection,

$$\{\dots\dots, x[-2], x[-1]\} = \{\dots\dots, -(7/8), -(1/4)\}$$

$$Example4 : X(z) = \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+0.3561z^{-2}}, \textit{ find Right.h sequence } x[n]?$$

$$\underline{1+3z^{-1}+3.6439z^{-2}+2.5756z^{-3}+.......}$$

$$1-z^{-1}+0.3561z^{-2} \;\vdots\; 1+2z^{-1}+z^{-2}$$

$$\underline{1-z^{-1}+0.3561z^{-2}}$$

$$3z^{-1}+0.6439z^{-2}$$

$$\underline{3z^{-1}-3z^{-2}+1.0683z^{-3}}$$

$$3.6439z^{-2}-1.0683z^{-3}$$

$$\underline{3.6439z^{-2}-3.6439z^{-3}+1.297z^{-4}}$$

$$2.575z^{-3}-1.297z^{-4}$$

$$\therefore X(z) = 1+3z^{-1}+3.6439z^{-2}+2.5756z^{-3}+......$$

By inspection, $\{x[0], x[1], x[2], x[3], ...\} = \{1, 3, 3.6439, 2.5756, ...\}$

Example 5: Partial Fraction Method

$$X(z)=\frac{z^2}{(z-1/2)(z-1/3)}$$

$$=\frac{c_1 z}{(z-1/2)}+\frac{c_2 z}{(z-1/3)}$$

$$c_1=X(z)\frac{(z-1/2)}{z}\Bigg|_{z=1/2}=\frac{z}{(z-1/3)}\Bigg|_{z=1/2}=3$$

$$c_2=X(z)\frac{(z-1/3)}{z}\Bigg|_{z=1/3}=\frac{z}{(z-1/2)}\Bigg|_{z=1/3}=-2$$

$$X(z)=\frac{3z}{(z-1/2)}-\frac{2z}{(z-1/3)}$$

Continued Example 5

$$X(z)=\frac{3z}{\left(z-1/2\right)}-\frac{2z}{\left(z-1/3\right)}$$

Depending on ROC multilple results are possible

$if\ \ ROC\ \ is\ \ |z|>1/2$

$$x[n]=3\left(1/2\right)^{n}u[n]-2\left(1/3\right)^{n}u[n]$$

$if\ \ ROC\ \ is\ \ |z|<1/3$

$$x[n]=-3\left(1/2\right)^{n}u[-n-1]+2\left(1/3\right)^{n}u[-n-1]$$

$if\ \ ROC\ \ is\ \ 1/2<|z|<1/3$

$$x[n]=3\left(1/2\right)^{n}u[n]+2\left(1/3\right)^{n}u[-n-1]$$

Example 6: Partial Fraction Method

$$X(z) = \frac{z^{-1}}{1 - 0.25z^{-1} - 0.375z^{-2}}$$

Multiplying the numerator and denominator by z^2,

$$X(z) = \frac{z}{z^2 - 0.25z - 0.375} = \frac{z}{(z-0.75)(z+0.5)}$$

$$X(z) = \frac{z}{(z-0.75)(z+0.5)}$$

$$= \frac{C_1 z}{z-0.75} + \frac{C_2 z}{z+0.5}$$

$$C_1 = \frac{X(z)}{z}(z-0.75)\Big|_{z=0.75} = \frac{(z-0.75)}{(z-0.75)(z+0.5)}\Big|_{z=0.75}$$

$$= \frac{1}{z+0.5}\Big|_{z=0.75} = \frac{4}{5}$$

Continued Example 6

$$\mathrm{C}_2 = \frac{X(z)}{z}(z+0.5)\bigg|_{z=-0.5} = \frac{1}{z-0.75}\bigg|_{z=-0.5} = -\frac{4}{5}$$

$$\therefore \mathrm{X(z)} = \frac{\mathrm{(4/5)z}}{\mathrm{z\text{-}0.75}} - \frac{\mathrm{(4/5)z}}{\mathrm{z+0.5}} = (4/5)\left[\frac{z}{z-0.75} - \frac{z}{z+0.5}\right]$$

$if\ \ ROC\ \ is\ \ |z| > 0.75$

$$x[n] = (4/5)\left(\left(0.75\right)^n u[n] - (-0.5)^n u[n]\right)$$

$if\ \ ROC\ \ is\ \ |z| < 0.5$

$$x[n] = (4/5)\left(-\left(0.75\right)^n u[-n-1] + (-0.5)^n u[-n-1]\right)$$

$if\ \ ROC\ \ is\ \ 0.5 < |z| < 0.75$

$$x(n) = (4/5)\left((0.75)^n u[n] + (-0.5)^n u[-n-1]\right)$$

Example 7: Partial Fraction Method

$$X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})}, \quad ROC \quad |z| > a_1 > 1$$

$$X(z) = \frac{A}{(1 - a_1 z^{-1})} + \frac{B}{(1 - z^{-1})}$$

$$A = X(z)(1 - a_1 z^{-1})\Big|_{z=a_1} = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})}\Bigg|_{z=a_1}$$

$$= \frac{b_0 + b_1 a_1^{-1}}{(1 - a_1^{-1})}$$

$$B = X(z)(1 - z^{-1})\Big|_{z=a_1} = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})}\Bigg|_{z=1}$$

$$= \frac{b_0 + b_1}{(1 - a_1)}$$

Continued Example 7

$$X(z)=\frac{b_0+b_1z^{-1}}{(1-a_1z^{-1})(1-z^{-1})},\qquad ROC\quad |z|>a_1>1$$

$$X(z)=\frac{b_0+b_1a_1^{-1}}{(1-a_1^{-1})}\frac{1}{(1-a_1z^{-1})}+\frac{b_0+b_1}{(1-a_1)}\frac{1}{(1-z^{-1})}$$

$$x[n]=\frac{b_0+b_1a_1^{-1}}{(1-a_1^{-1})}a_1^{\,n}u[n]+\frac{b_0+b_1}{(1-a_1)}u[n]$$

$$x[n]=\left\{\frac{b_0+b_1a_1^{-1}}{(1-a_1^{-1})}a_1^{\,n}+\frac{b_0+b_1}{(1-a_1)}\right\}u[n]$$

Always try to resolve the given equation into library functions so that it is easy to obtain the inverse $Z-$transform

Example 8: Partial Fraction Method

$$X(z) = \frac{1-2.1z^{-1}}{1-0.3z^{-1}-0.4z^{-2}}, \quad ROC \quad |z| > 0.8$$

$$X(z) = \frac{1-2.1z^{-1}}{(1+0.5z^{-1})(1-0.8z^{-1})} = \frac{A}{(1+0.5z^{-1})} + \frac{B}{(1-0.8z^{-1})}$$

$$A = X(z)(1+0.5z^{-1})\Big|_{z=-0.5} = \frac{1-2.1z^{-1}}{(1-0.8z^{-1})}\Bigg|_{z=-0.5} = \frac{1+4.2}{1+1.6} = 2$$

$$B = X(z)(1-0.8z^{-1})\Big|_{z=0.8} = \frac{1-2.1z^{-1}}{(1+0.5z^{-1})}\Bigg|_{z=0.8} = \frac{1-2.1/0.8}{(1+0.5/0.8)} = -1$$

$$X(z) = \frac{2}{(1+0.5z^{-1})} - \frac{1}{(1-0.8z^{-1})}$$

$$x[n] = \left(-(-0.5)^n + (0.8)^n\right)u[n]$$

Z Transform: Properties

Superposition

$Since\ the\ Z-transform\ is\ linear$

$$ax_1[n]+bx_2[n]\overset{z}{\longleftrightarrow}aX_1(z)+bX_2(z)$$

Shifting

$$x[n-n_0]\overset{z}{\longleftrightarrow}z^{-n_0}X(z)$$

$$\therefore\ \ x[n-1]\overset{z}{\longleftrightarrow}z^{-1}X(z)$$

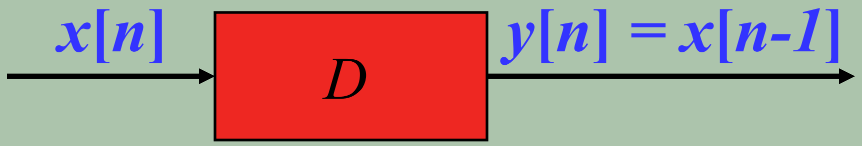
$$x[n+1]\overset{z}{\longleftrightarrow}z^{1}X(z)$$

$$x[n+2]\overset{z}{\longleftrightarrow}z^{2}X(z)$$

$$x[n-2]\overset{z}{\longleftrightarrow}z^{-2}X(z)$$

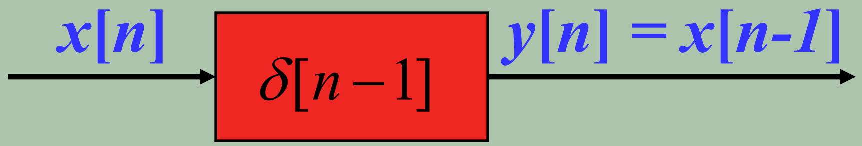
Unit Delay

Symbolic Notation



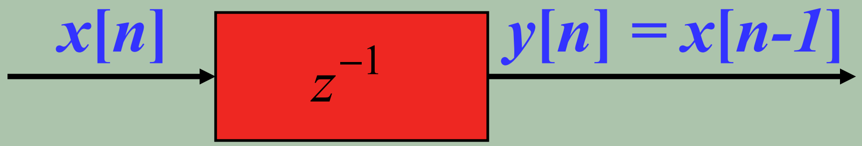
$x[n]$ D $y[n] = x[n-1]$

Mathematical Notation



$x[n]$ $\delta[n-1]$ $y[n] = x[n-1]$

$$\because \quad \delta[n-1] \xleftrightarrow{z} z^{-1}$$



$x[n]$ z^{-1} $y[n] = x[n-1]$

Example 1

Consider the first difference of two successive signal values

$$y[n] = x[n] - x[n-1]$$

Apply the Z – transform on both sides

$$Z\{y[n]\} = Z\{x[n] - x[n-1]\}$$

$$\Rightarrow Y(z) = X(z) - Z^{-1}X(z)$$

$$Y(z) = X(z)(1 - Z^{-1})$$

Computational structure for the above system

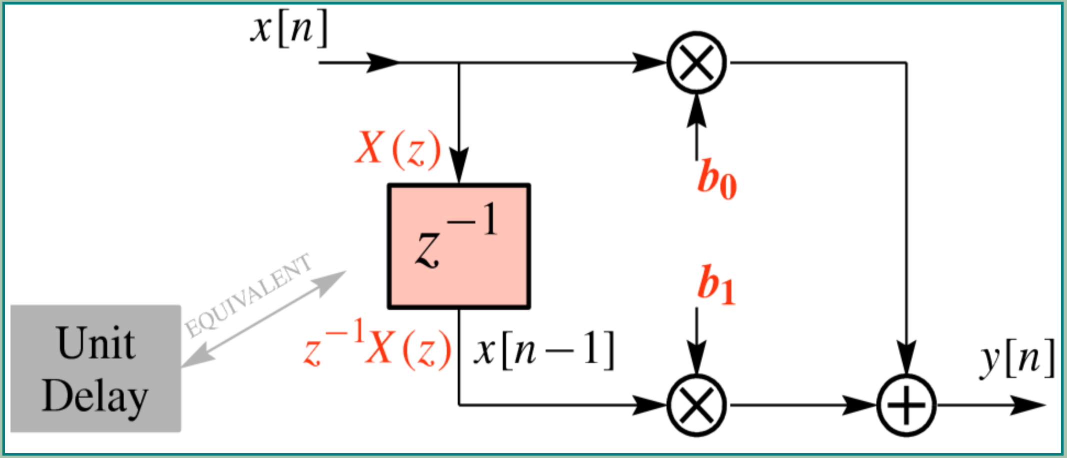


Fig.11.2

Example 2

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

$$Apply\ the\ Z-transform\ on\ both\ sides$$

$$Z\{y[n]\} = Z\left\{\frac{1}{2} y[n-1] + x[n]\right\}$$

$$\Rightarrow Y(z) = \frac{1}{2} Z^{-1} Y(z) + X(z)$$

$$Y(z) - \frac{1}{2} Z^{-1} Y(z) = X(z)$$

$$Y(z)(1 - \frac{1}{2} Z^{-1}) = X(z)$$

$$Y(z) = \frac{X(z)}{(1 - \frac{1}{2} Z^{-1})}$$

Continued Example 2

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

$$Y(z) = \frac{X(z)}{(1 - \frac{1}{2} Z^{-1})}$$

Let the input be $x[n] = \delta[n]$,

the output will be the impulse response $y[n] = h[n]$

$$\because \delta[n] \xleftrightarrow{z} 1, \ X(z) = 1$$

$$Y(z) = H(z)$$

$$H(z) = \frac{1}{(1 - \frac{1}{2} Z^{-1})}$$

$$h[n] = \mathcal{Z}^{-1}\left[H(z)\right]$$

Continued Example 2

$$H(z) = \frac{1}{(1 - \frac{1}{2} Z^{-1})}$$

$$With\ ROC\quad |z| > \frac{1}{2}$$

$$h[n] = \left(1/2\right)^{n} u[n]$$

$$With\ ROC\quad |z| < \frac{1}{2}$$

$$h[n] = -\left(1/2\right)^{n} u[-n-1]$$

A standard notation for Transfer function

$$H(z) = \frac{Y(z)}{X(z)}$$

Example 3

Find the transfer function for the system described by

$$y[n] = -\frac{1}{2}y[n-1] + x[n] + x[n-1]$$

Apply the Z – transform on both sides

$$Z\{y[n]\} = Z\left\{-\frac{1}{2}y[n-1] + x[n] + x[n-1]\right\}$$

$$\Rightarrow Y(z) = -\frac{1}{2}Z^{-1}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z)(1+\frac{1}{2}Z^{-1}) = X(z)(1+z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1+z^{-1})}{(1-\frac{1}{2}Z^{-1})}$$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “7.1-7.4, 8.7--Signal Processing First”, Prentice Hall, 2003