Discrete - Time Signals and Systems

Z-Transform-FIR filters

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FIR filters-Review

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \ldots \textit{general difference equation for FIR filters}$$

Filter Order = M: ***No. of memory blocks required in the filter implementation***

Filter Length, L = M+1: ***Total No. of samples required in calculating the output, M from memory (past) and one present sample***

Filter coefficients $\{b_k\}$: ***Completely defines an FIR filter. All the properties of the filter can be understood through the coefficients***

$Z-transform\ of\ impulse\ response\ h[n]\ results\ in\ 'transfer$
$function',\ it\ is\ also\ known\ as\ 'system\ function'$

$$H(z)=\sum_n h[n]z^{-n}$$

$From\ the\ previous\ lecture\ recall\ that\ Transfer\ function$

$$H(z)=\frac{Y(z)}{X(z)}$$

$$\Rightarrow Y(z)=H(z)X(z)$$

$Notice\ the\ mathematical\ simplicity\ of\ the\ above\ result$

$Convolution\ becomes\ a\ simple\ multiplication$

$$h[n]*x[n]\leftrightarrow H(z)X(z)$$

Calculating the output of a FIR filter using Z-transforms

Steps involved

1) *Find the Z-transform of input signal* $x[n]$

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

2) *Find the Z-transform of impulse response* $h[n]$

$$H(z)=\sum_{n=0}^{M} h[n]z^{-n}$$

3) *Multiply* $X(z)$ *and* $H(z)$ *to get* $Y(z)$

4) *Obtain output* $y[n]$ *by applying inverse Z-transform to* $Y(z)$

$$y[n]\xleftrightarrow{\mathbb{Z}^{-1}} Y(z)$$

Why to operate in transforms?

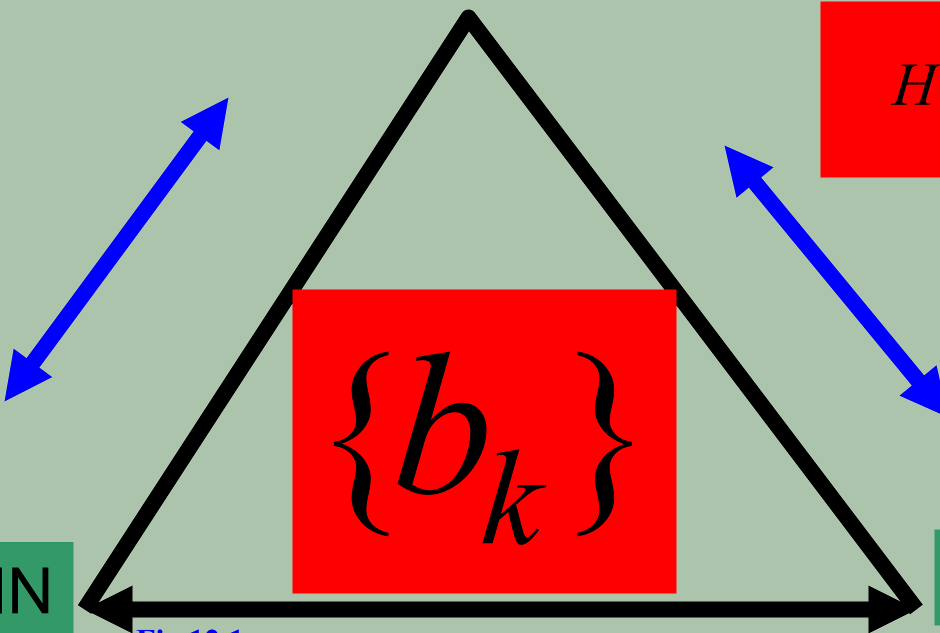


Fig 12.1

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

$\therefore z = e^{j\hat{\omega}}$, *makes both domains equal*

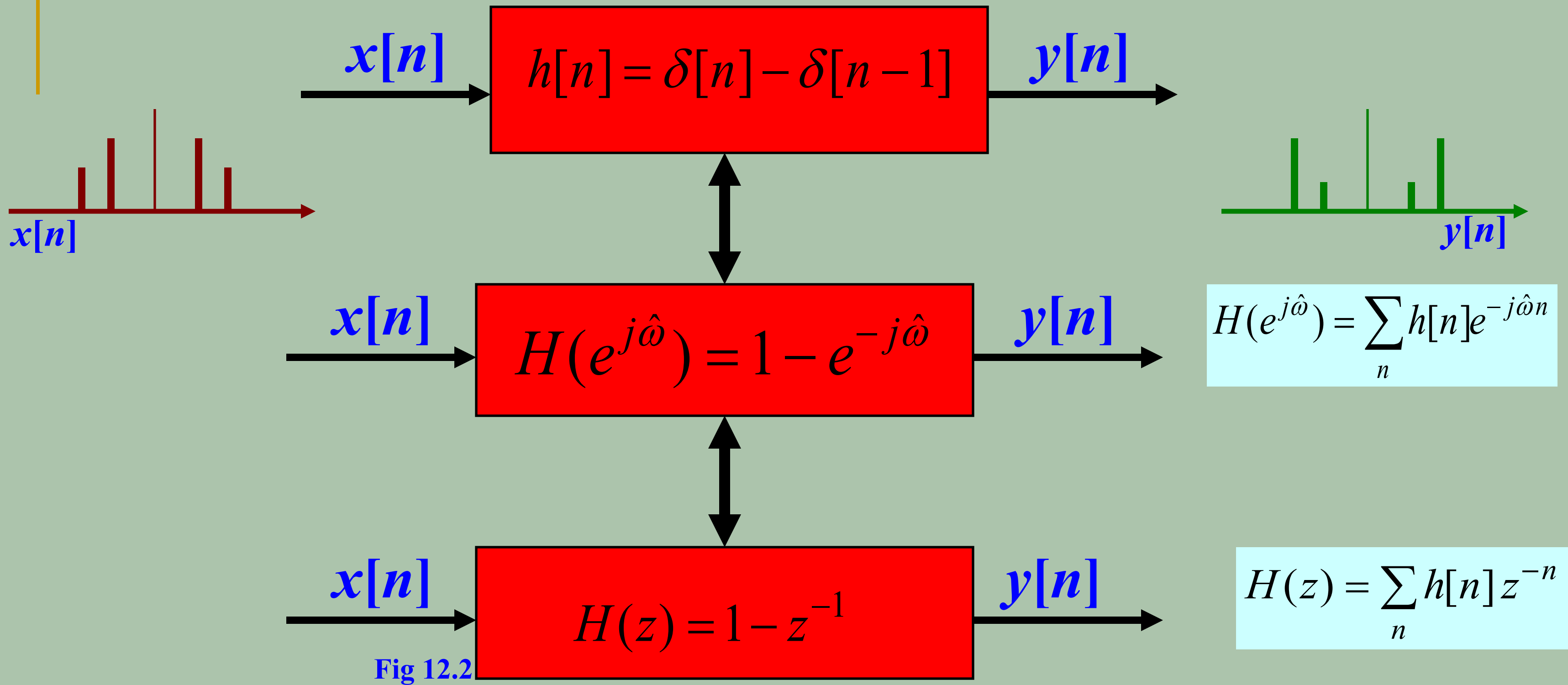


Fig 12.2

Same output from all three domains

Computational complexity determines the domain

Some of the filter properties are better understood in frequency or Z-domains

Z and frequency transforms are related

Example 1

Calculating the transfer function of the FIR filter with impulse response co-efficients given by,

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = \sum_{n=0}^{4} h[n] z^{-n}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}$$

$$= 2 + 0z^{-1} - 3z^{-2} - 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

Example 2

Calculating the transfer function of the FIR filter described by the difference equation

$$y[n] = x[n] + 2x[n-1] - 3x[n-2] - 4x[n-3]$$

$$h[n] = b_k$$

$$h[n] = \{1, 2, -3, -4\}$$

$$H(z) = \sum_{n=0}^{3} h[n] z^{-n}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$

$$= 1 + 2z^{-1} - 3z^{-2} - 4z^{-3}$$

Example 3

Calculating the output of the FIR filter

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$\because \delta[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}$$

$$X(z) = z^{-1} - z^{-2} + z^{-3} - z^{-4}$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

$$\because Y(z) = X(z)H(z)$$

$$= \left(z^{-1} - z^{-2} + z^{-3} - z^{-4}\right)\left(1 + 2z^{-1} + 3z^{-2} + 4z^{-3}\right)$$

$$=z^{-1}+(-1+2)z^{-2}+(1-2+3)z^{-3}+(-1+2-3+4)z^{-4}$$

$$+(-2+3-4)z^{-5}+(-3+4)z^{-6}+(-4)z^{-7}$$

$$=z^{-1}+z^{-2}+2z^{-3}+2z^{-4}-3z^{-5}+z^{-6}-4z^{-7}$$

Apply the inverse Z − transform

$$\because\ z^{-n_0} \xleftrightarrow{Z^{-1}} \delta[n-n_0]$$

$$Y(z)=z^{-1}+z^{-2}+2z^{-3}+2z^{-4}-3z^{-5}+z^{-6}-4z^{-7}$$

$$y[n]=\delta[n-1]+\delta[n-2]+2\delta[n-3]+2\delta[n-4]-3\delta[n-5]$$

$$+\delta[n-6]-4\delta[n-7]$$

Example 4

Find the impulse response of the FIR filter

$$x[n] = \delta[n-2]$$

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$\because \delta[n-n_0] \xleftrightarrow{z} z^{-n_0}$$

$$X(z) = z^{-2}$$

$$Y(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

$$\because H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 + 2z^{-1} + 3z^{-2} + 4z^{-3}}{z^{-2}} = z^2 + 2z + 3 + 4z^{-1}$$

$$h[n] = \delta[n+2] + 2\delta[n+1] + 3\delta[n] + 4\delta[n-1] \ldots \textit{non causal}$$

Example 5

Calculating the output of the FIR filter

$$x[n] = \begin{cases} A & 0 \le n \le 4 \\ 0 & elsewhere \end{cases}$$

$$h[n] = \begin{cases} (1/2)^{n} & 0 \le n \le 3 \\ 0 & elsewhere \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{4} A z^{-n} = A(1 + z^{-1} + + z^{-2} + z^{-3} + z^{-4})$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$=\sum_{n=0}^{3}\left(1/2\right)^{n} z^{-n} =$$

$$=(1+\left(1/2\right)z^{-1}+\left(1/4\right)z^{-2}+\left(1/8\right)z^{-3})$$

$$\because Y(z)=H(z)X(z)$$

$$=A(1+z^{-1}++z^{-2}+z^{-3}+z^{-4})(1+\left(1/2\right)z^{-1}+\left(1/4\right)z^{-2}+\left(1/8\right)z^{-3})$$

$$=A(1+\left(3/2\right)z^{-1}+\left(7/4\right)z^{-2}+\left(15/8\right)z^{-3}+\left(15/8\right)z^{-4}$$

$$+\left(7/8\right)z^{-5}+\left(3/8\right)z^{-6}+\left(1/8\right)z^{-7})$$

$$y[n]=A(\delta[n]+\left(3/2\right)\delta[n-1]+\left(7/4\right)\delta[n-2]+\left(15/8\right)\delta[n-3]$$

$$+\left(15/8\right)\delta[n-4]+\left(7/8\right)\delta[n-5]+\left(3/8\right)\delta[n-6]$$

$$+\left(1/8\right)\delta[n-7])$$

Cascading Systems

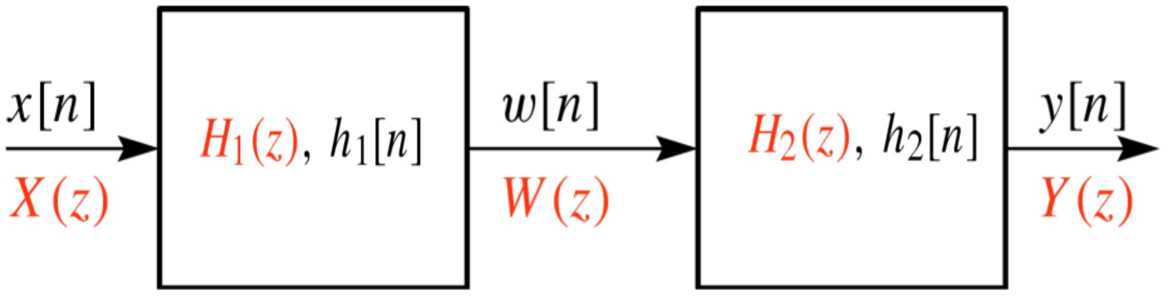


Fig 12.3

$x[n]$...*input signal to the* 1^{st} *FIRfilter*

$h[n]$...*impulse response of the* 1^{st} *FIRfilter*

$w[n]$...*output of the* 1^{st} *FIR filter and input to the* 2^{nd} *FIR filter*

$y[n]$...*output of the* 2^{nd} *filter, also this is the overall output*

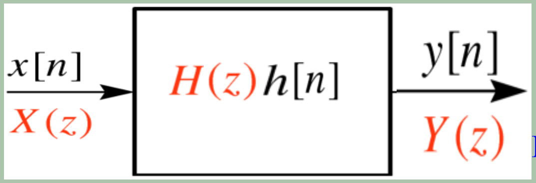


Fig 12.4

$$h[n] = h_1[n] * h_2[n] \overset{z}{\longleftrightarrow} H(z) = H_1(z)H_2(z)$$

Cascaded system can be replaced by a single filter with system function $H(z)$

Example 1

The impulse responses in a cascaded system are

$$h_1[n] = \delta[n] - \delta[n-1]$$

$$h_2[n] = \delta[n] + \delta[n-1]$$

Find the impulse response of an effective system that can replace the cascading arrangement

$$\because \delta[n - n_0] \xleftrightarrow{z} z^{-n_0}$$

$$h_1[n] \xleftrightarrow{z} H_1(z), \; h_2[n] \xleftrightarrow{z} H_2(z)$$

$$H_1(z) = 1 - z^{-1}$$

$$H_2(z) = 1 + z^{-1}$$

$$H(z) = H_1(z)H_2(z) = \left(1 - z^{-1}\right)\left(1 + z^{-1}\right) = 1 - z^{-2}$$

$$h[n] = \delta[n] - \delta[n-2]$$

Example 2

Consider a system described by the difference equations

$$w[n] = 3x[n] - x[n-1]$$

$$y[n] = 2w[n] - w[n-1]$$

Find the impulse response of an effective system that can replace the cascading arrangement

$$h_1[n] = b_1(k) = [3, -1]$$

$$h_1[n] = 3\delta[n] - \delta[n-1]$$

$$h_2[n] = b_2(k) = [2, -1]$$

$$h_2[n] = 2\delta[n] - \delta[n-1]$$

$$\because\ \delta[n - n_0] \xleftrightarrow{z} z^{-n_0}$$

$$h_1[n] \xleftrightarrow{z} H_1(z),\ h_2[n] \xleftrightarrow{z} H_2(z)$$

$$H_1(z) = 3 - z^{-1}$$

$$H_2(z) = 2 - z^{-1}$$

$$H(z) = H_1(z)H_2(z) = \left(3 - z^{-1}\right)\left(2 - z^{-1}\right)$$

$$= 6 - 5z^{-1} + z^{-2}$$

$$h[n] = 6\delta[n] - 5\delta[n-1] + \delta[n-2]$$

The system can now be expressed as one difference equation

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

Example 3

The impulse response of an effective system,

$$h[n] = \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3]$$

Split the above filter into two cascaded filters such that the 1st system is described by,

$$w[n] = x[n] - x[n-1]$$

$$\because \delta[n - n_0] \xleftrightarrow{z} z^{-n_0}$$

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$h_1[n] = \delta[n] - \delta[n-1]$$

$$H_1(z) = 1 - z^{-1}$$

$$\because H(z) = H_1(z)H_2(z)$$

$$H_2(z) = \frac{H(z)}{H_1(z)}$$

$$= \frac{1-2z^{-1}+2z^{-2}-z^{-3}}{1-z^{-1}}$$

$$= 1 - z^{-1} + z^{-2}$$

$$h_2[n] = \delta[n] - \delta[n-1] + \delta[n-2]$$

$$\left.\begin{aligned} y[n] &= w[n] - w[n-1] + w[n-2] \\ w[n] &= x[n] - x[n-1] \end{aligned}\right\} \textit{Complete system}$$

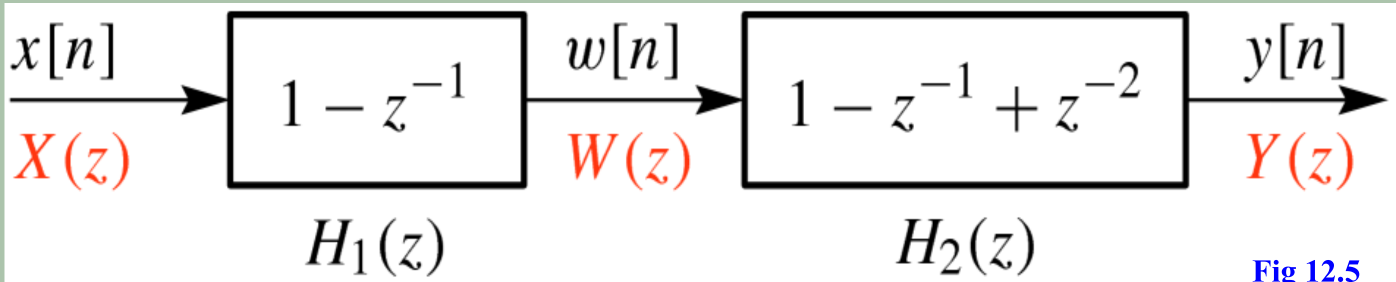


Fig 12.5

Example 4: Deconvolution

Cascading of filters has an important practical application

Undoing the effects of the first filter

Example : Communication channel

Undoing the effects of channel on signal is called equalization

$$Y(z) = H_1(z)H_2(z)X(z)$$

$$if \;\; Y(z) = X(z)$$

$$\Rightarrow H_1(z)H_2(z) = 1$$

$$Assume \;\; H_1(z) = 1 - z^{-1}$$

$$H_2(z) = \frac{1}{1 - z^{-1}}$$

Z-Transform & Unit circle

$The\ frequency\ or\ \hat{\omega}-domain\ is\ a\ subset\ of\ z-domain$

$The\ general\ expression\ for\ Z\ is,$

$z = re^{j\hat{\omega}},$

$where,\ 'r'\ is\ the\ radius\ of\ the\ circle$

$z = e^{j\hat{\omega}},\ makes\ both\ domains\ equal$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$\because |z| = \left|e^{j\hat{\omega}}\right| = 1,\ the\ unit\ circle\ has\ a\ unique\ significance$

$in\ the\ z-domain$

Gray dots, $z = \{-j, 1, j\}$

$$\hat{\omega} = \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

General Z − plane

$z = e^{j\hat{\omega}}$*, special case*

$z - domain = \hat{\omega} - domain$

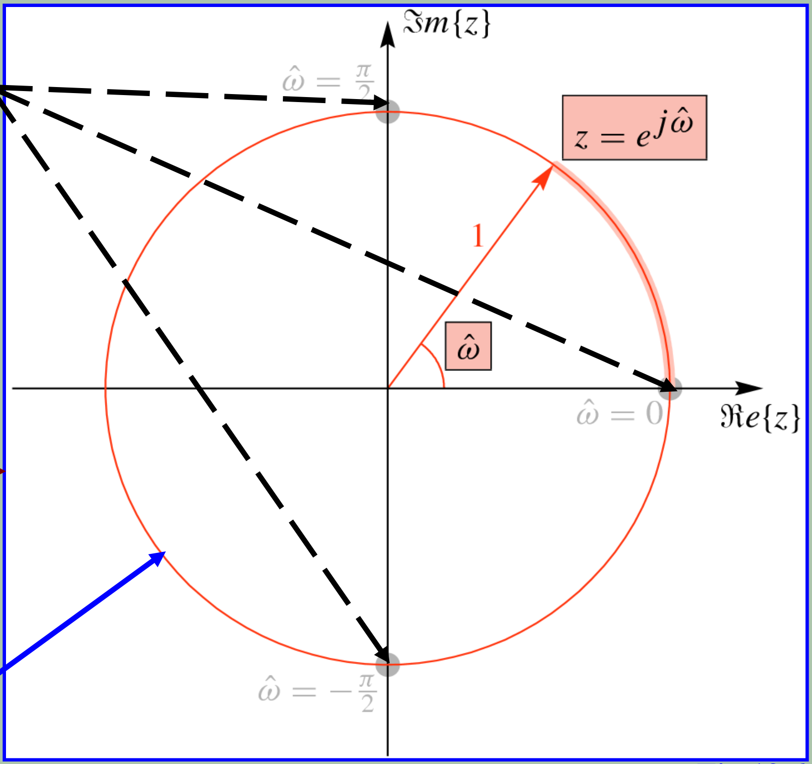


Fig 12.6

Note that '$\hat{\omega}$*' is the discrete − time frequency discussed in previous lectures*

$$-\pi \le \hat{\omega} \le \pi \longleftrightarrow -1 \le z \le 1$$

Periodicity in $\hat{\omega}$ *− domain − −2π radians, one cycle in z − domain*

Zeros & Poles

Consider the transfer function of an FIR filter

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

Convert the above function as a polynomial in 'z'

$$H(z) = \frac{\left(1 - 2z^{-1} + 2z^{-2} - z^{-3}\right)z^{3}}{z^{3}}$$

$$= \frac{z^{3} - 2z^{2} + 2z - 1}{z^{3}}$$

write the numerator in factored form,

$$z^{3} - 2z^{2} + 2z - 1 = (z-1)(z - e^{j\pi/3})(z - e^{-j\pi/3})$$

$$H(z) = \frac{(z-1)(z-e^{j\pi/3})(z-e^{-j\pi/3})}{z^3}$$

Zeros....The values of z for which $H(z) = 0$

In this example,

$$H(z) = 0 \;\; for \;\; z = \left\{1, e^{j\pi/3}, e^{-j\pi/3}\right\}$$

Poles....The values of z for which $H(z) \to \infty$

$$H(z) \to \infty \;\; for \;\; z = \{0, 0, 0\}$$

note that $z^3 = 0$ *results in 3 roots*

It is also known as a 3^{rd} *order pole at* $z = 0$

Significance of zeros in an FIR system:

Except for a constant FIR system is completely characterized by the zeros

The difference equation which desribes the relation between $x[n]$ and $y[n]$, can be found with the knowledge of zero locations

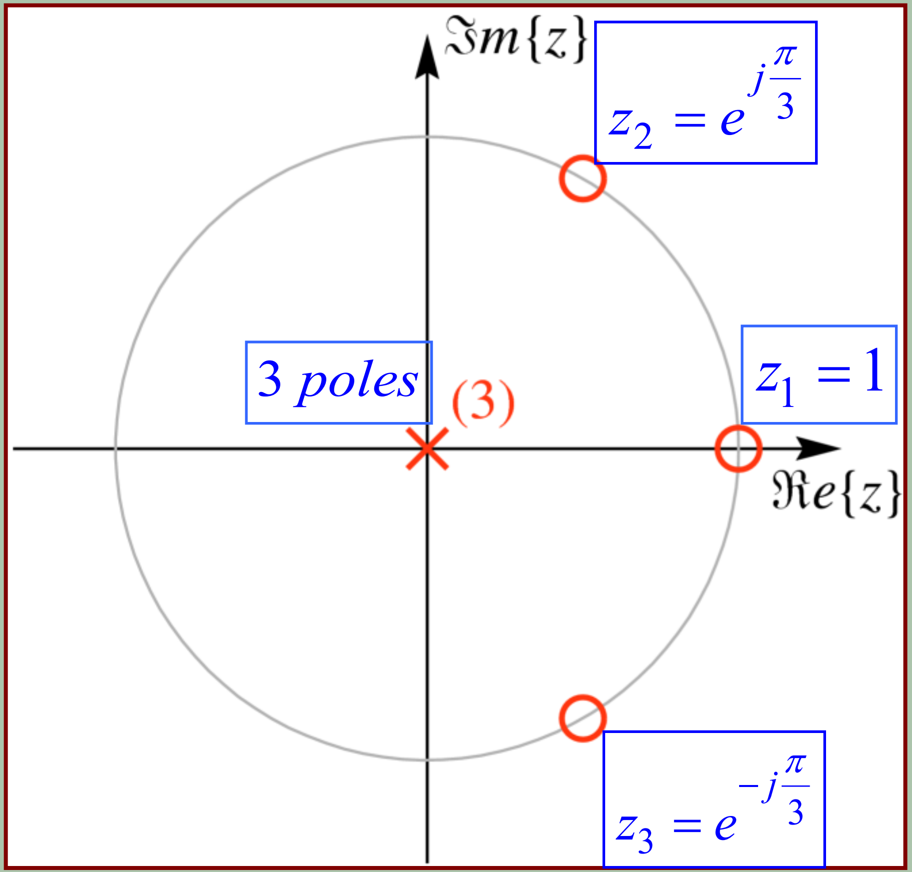


Fig 12.7

Example 1

$Find\ the\ poles\ and\ zeros\ for\ the\ transfer\ function?$

$H(z) = 1 - 3z^{-1} + 2z^{-2}$

$Converting\ into\ a\ polynomial\ in\ 'z'$

$$H(z) = \frac{z^2 - 3z + 2}{z^2}$$

$write\ the\ numerator\ in\ factored\ form,$

$z^2 - 3z + 2 = (z-2)(z-1)$

$zeros = [z_1, z_2] = [2, 1]$

$poles = [p_1, p_2] = [0, 0] = 2^{nd}\ order\ pole\ at\ z = 0$

Example 1 continued…

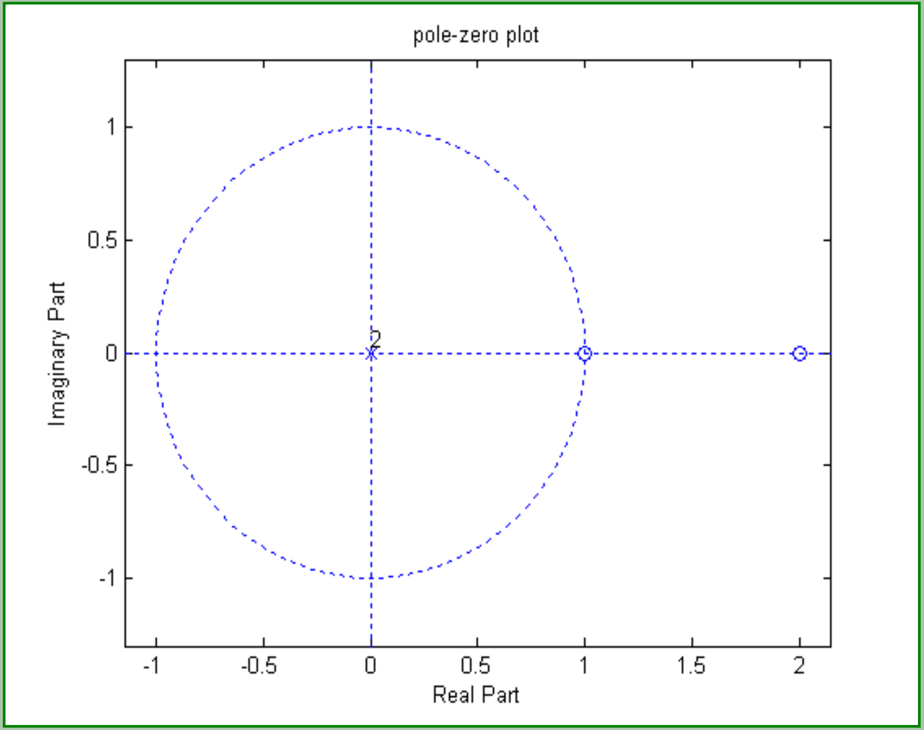


Fig 12.8

Notice that both the zeros are on real axis, also notice one pole is not included in the unit circle. There is a double pole at z=0

Example 2

Find the poles and zeros of an FIR system described by the difference equation,

$$y[n] = x[n] - x[n-1] + x[n-2]$$

Theimpulseresponsefunctionisgivenby

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2]$$

$$\because \delta[n - n_0] \xleftrightarrow{z} z^{-n_0}$$

$$H(z) = 1 - z^{-1} + z^{-2}$$

Converting into a polynomial in 'z'

$$H(z) = \frac{z^2 - z + 1}{z^2}$$

$$\textit{write the numerator in factored form,}$$

$$z^2 - z + 1 = (z - e^{j\pi/3})(z - e^{-j\pi/3})$$

$$zeros = [e^{j\pi/3}, e^{-j\pi/3}], \textit{ both on unit circle}$$

$$poles = [0,0] = 2^{nd} \textit{ order pole at } z = 0$$

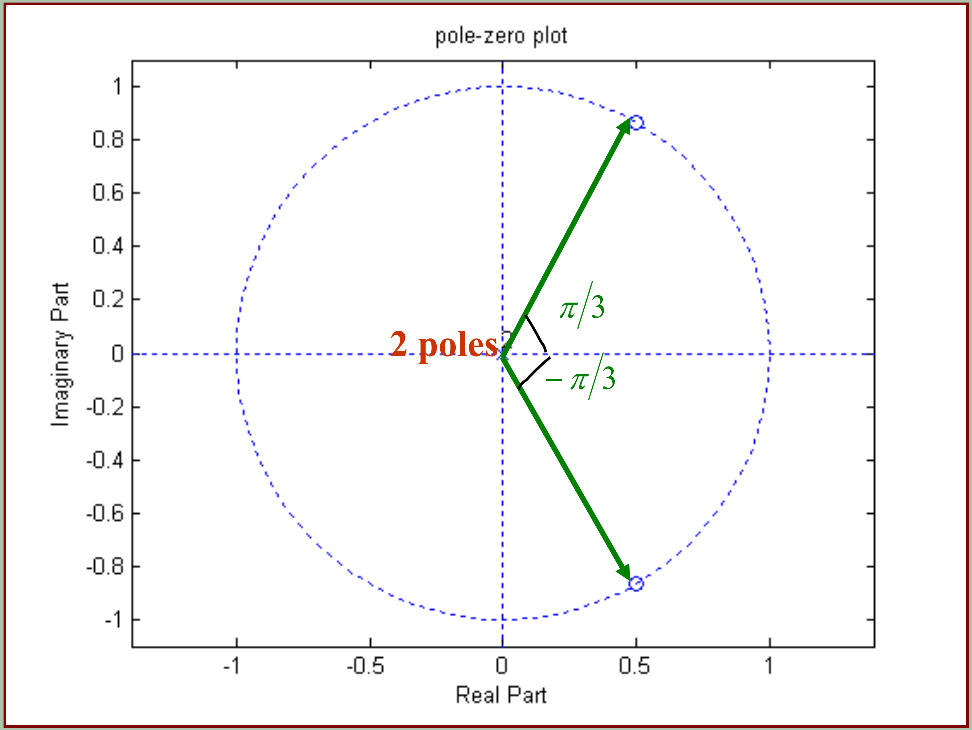


Fig 12.9

Example 3

The zeros and poles of an FIR filter are given,

$$zeros = [e^{j\pi/4}, e^{-j\pi/4}]$$

$$poles = [0, 0]$$

Find the difference equation of the filter

The numerator is obtained through multiplication of factors

$$(z - e^{j\pi/4})(z - e^{-j\pi/4}) = z^2 - \sqrt{2}z + 1$$

denominator through poles, z^2

$$H(z) = \frac{z^2 - \sqrt{2}z + 1}{z^2}$$

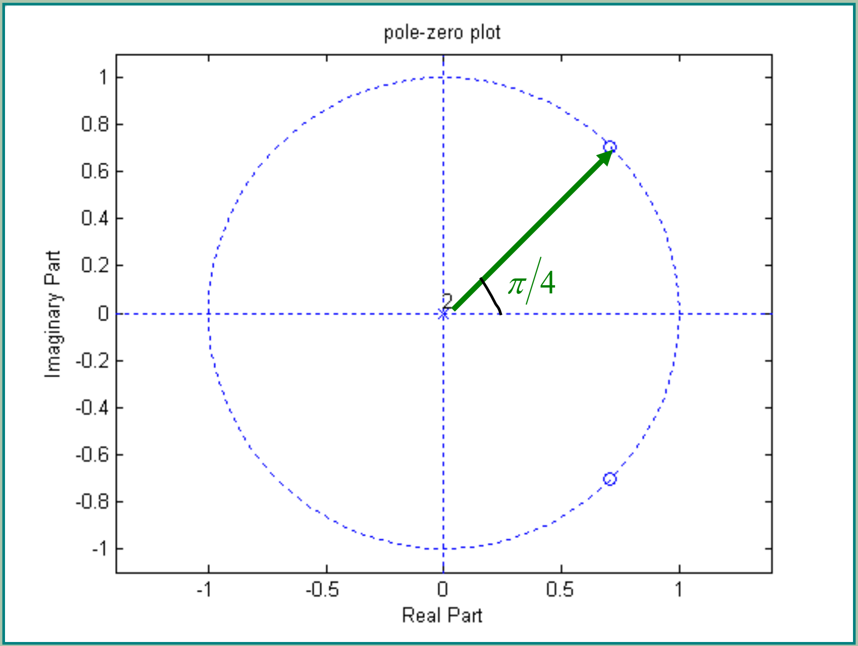
$$= 1 - \sqrt{2}z^{-1} + z^{-2}$$

Apply inverse Z − transform,

$$h[n] = \delta[n] - \sqrt{2}\delta[n-1] + \delta[n-2]$$

The difference equation

$$y[n] = x[n] - \sqrt{2}x[n-1] + x[n-2]$$



Notice that, because of conjugate zeros a sinusoid signal with frequency $\pi/4$ will be nulled out by this FIR filter

Fig 12.10

Example 4, Application: Nulling filters

The Graphical Design of a Comb filter

- In medical applications, the 60Hz frequency of the power supply is often "picked up" by the test equipment (EKG recorder)

- Also harmonically related frequencies such as $f_2 = 2x60 = 120Hz$, and $f_3 = 3x60 = 180$ Hz are generated because of non-linear phenomena.

- The object of a digital filter design is to eliminate or suppress these unwanted frequencies which distort or mask up the signals of interest

Thus the desired response of the filter would be,

Assume a sampling frequency of 360Hz

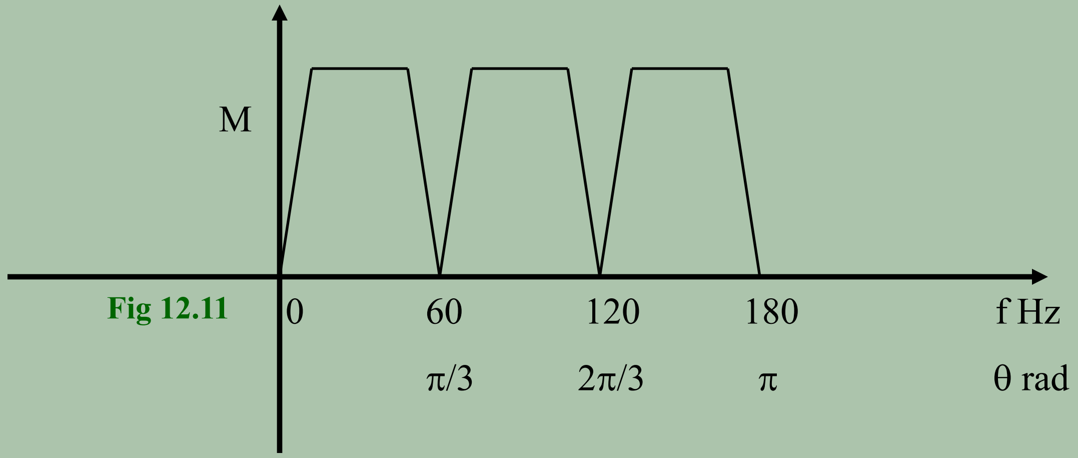


Fig 12.11

Thus we have :

$\theta_1 = \omega_1 T = 2\pi(60)/360 = \pi/3$

$\theta_2 = \omega_2 T = 2\pi(120)/360 = 2\pi/3$

$\theta_3 = \omega_3 T = 2\pi(180)/360 = \pi$

1) Complex zeros must occur in conjugate pairs

2) θ = 0 is added to eliminate any DC component in the signal

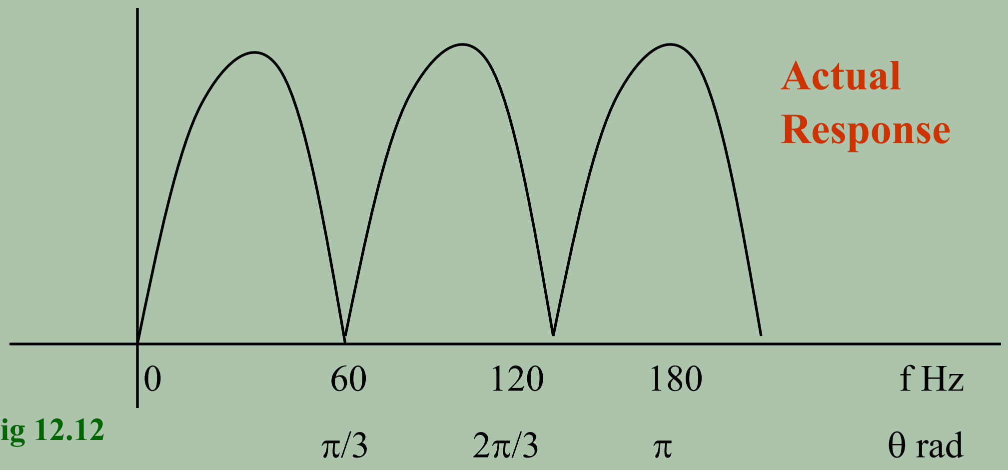


Fig 12.12

$$H(z) = (z-1)(z-e^{j\frac{\pi}{3}})(z-e^{j\frac{2\pi}{3}})(z-e^{j\pi})(z-e^{j\frac{4\pi}{3}})(z-e^{j\frac{5\pi}{3}})$$

$$H(z) = z^6 - 1$$

Implies : $y[n] = x[n+6] - x[n]$

But the above obtained filter is non-causal !! To make it causal filter we place six poles at z = 0.

$$H(z) = \frac{z^6 - 1}{z^6} \qquad \therefore \Rightarrow H(z) = (1 - z^{-6})$$

Thus the required causal FIR comb filter is:

$$y[n] = x[n] - x[n-6]$$

Filter properties & the location zeros

There is an extremely close relationship between the frequency response of the filter and the location of zeros on the unit circle

One can design a filter with required frequency by placing zeros in the appropriate place. The difference equation of the filter can be obtained by multiplying the factors associated with zeros

Example: Running average filter

$$y[n] = \sum_{k=0}^{L-1} x[n-k]$$

The system function

$$H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1-z^{-L}}{1-z^{-1}}$$

$$= \frac{z^L - 1}{z^{L-1}\left(z-1\right)}$$

Zeros..from numerator,

$$z^L - 1 = 0$$

$$z = e^{j2\pi k/L}, \qquad k = 0,1,2 \ldots L-1$$

Zeros are equally spaced around the circle

Poles..from numerator,

$$z^{L-1}(z-1)=0$$

$z^{L-1}=0,\ \ L-1^{th}$ *order pole at* 0

And another pole at $z=1$

Note that this pole and zero at $z=1$ *get cancelled*

The primary reason for the filter to become a lowpass filter is this cancellation of the zero at $z=1$ *due to the pole present in the same location*

Example 1

$L = 11$

Running average filter

Equi spaced zeros

More gap bewteen zeros

close to $z = 1$

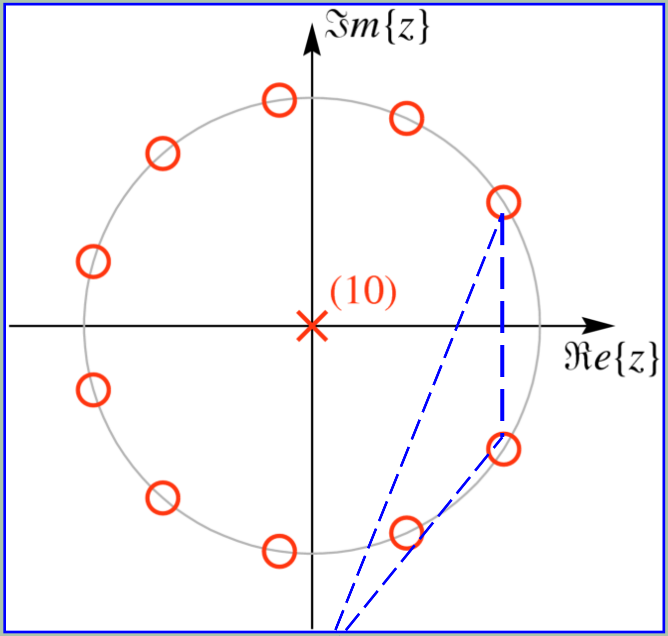


Fig 12.13

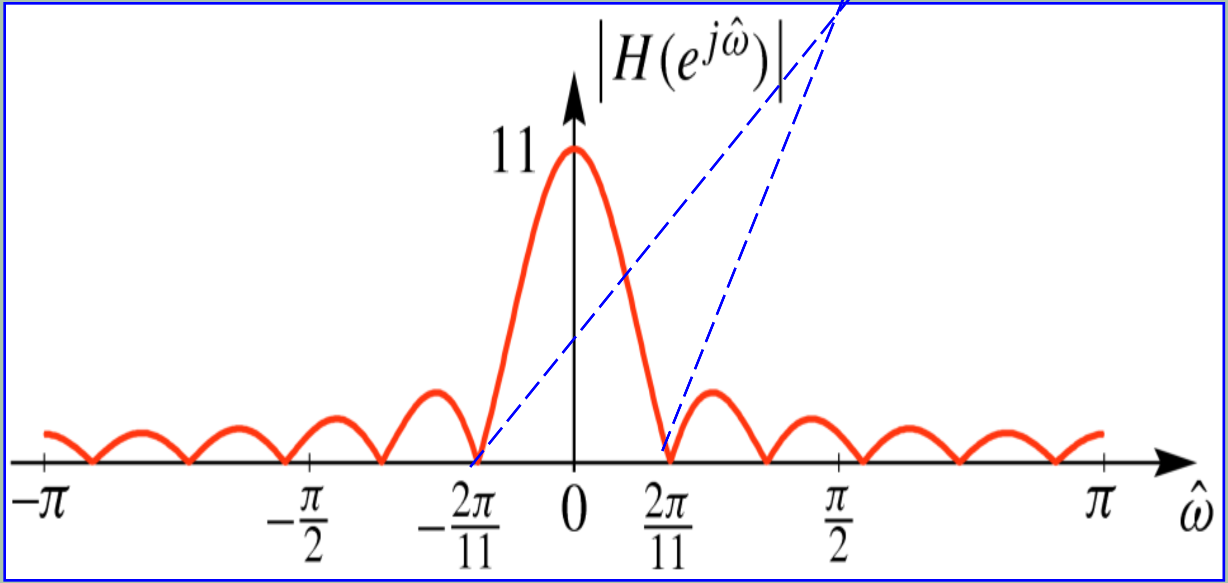


Fig 12.14

3 dimensional view of the frequency response

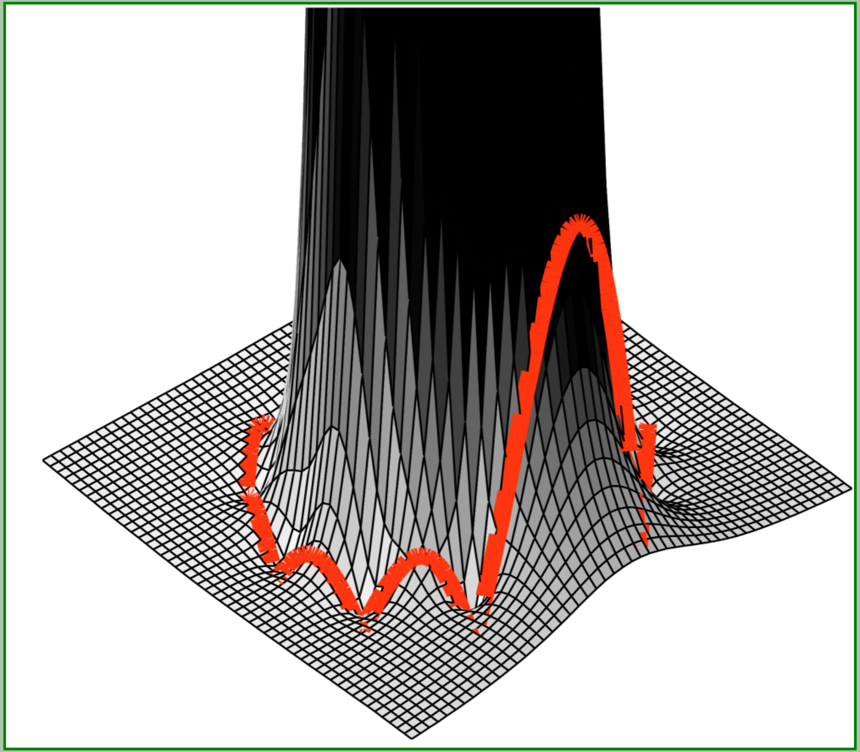


Fig 12.15

If the zeros locations are known, the transfer function of the filter can be obtained through, $H(z) = \prod_{k=1}^{L-1}\left(1 - e^{j2\pi k/L} z^{-1}\right)$

Example 2

$L = 10$

Running average filter

9^{th} *order pole at* $z = 0$

More gap bewteen zeros

close to $z = 1$

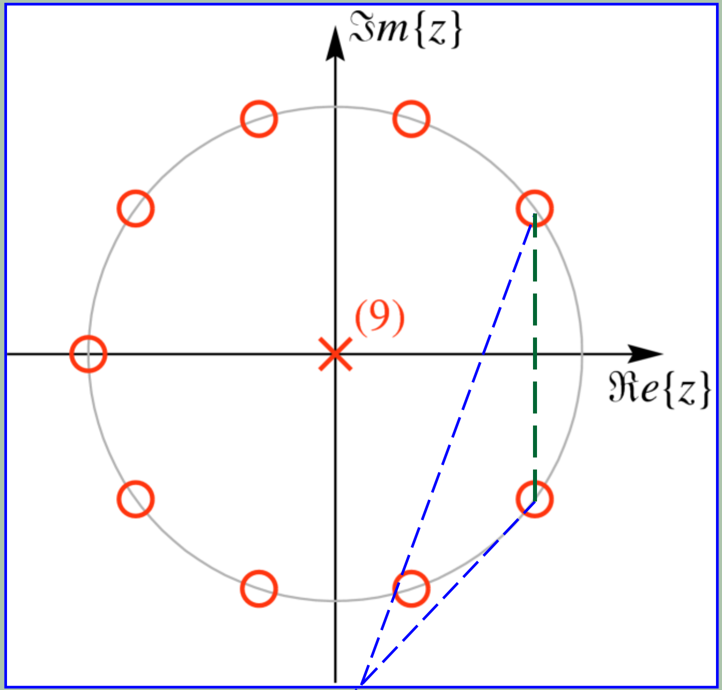


Fig 12.16

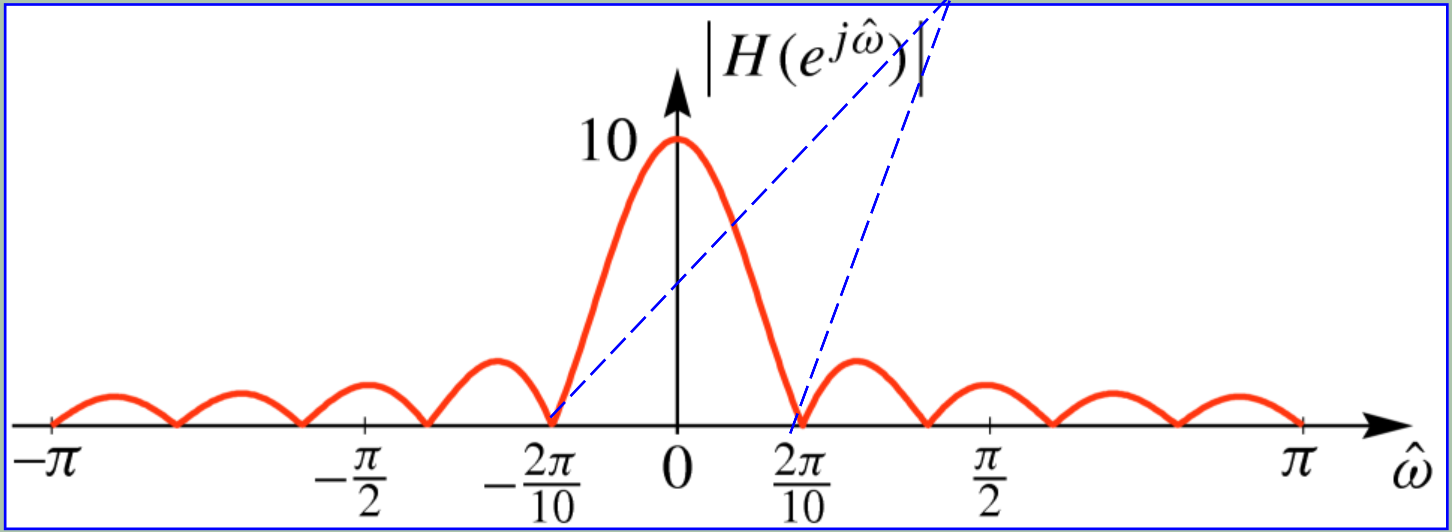


Fig 12.17

$Example\ 3$

$L = 10$

$Band\ pass\ filter$

$A\ simple\ trick: cancel\ the\ zero\ in$

$a\ different\ location$

$Transfer\ function\ shifts;$

$$H(z) = \sum_{k=0}^{L-1} z^{-k} e^{j2\pi k_0 k/L}$$

$k_0 \ldots determines\ the$

$shift$

$Since\ there\ is$

$no\ conjugate$

$zero.\ this\ results$

$in\ a\ complex\ filter$

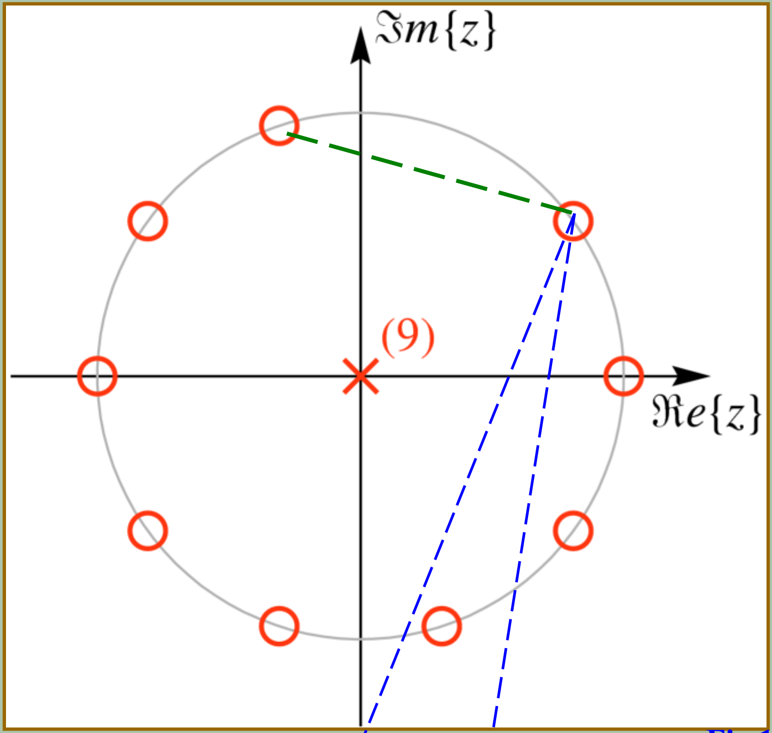


Fig 12.18

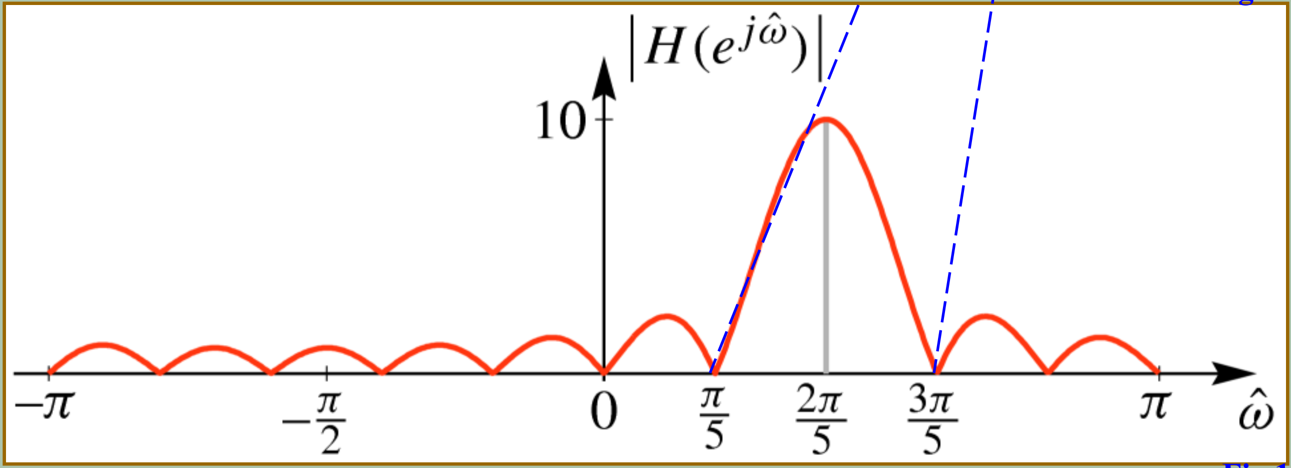


Fig 12.19

same example

due to conjugate symmetry this results in a filter with real coefficients

$$H(z) = \frac{1}{2}\sum_{k=0}^{L-1} z^{-k} e^{j2\pi k_0 k/L} + \frac{1}{2}\sum_{k=0}^{L-1} z^{-k} e^{-j2\pi k_0 k/L}$$

$$H(z) = H_1(z) + H_2(z)$$

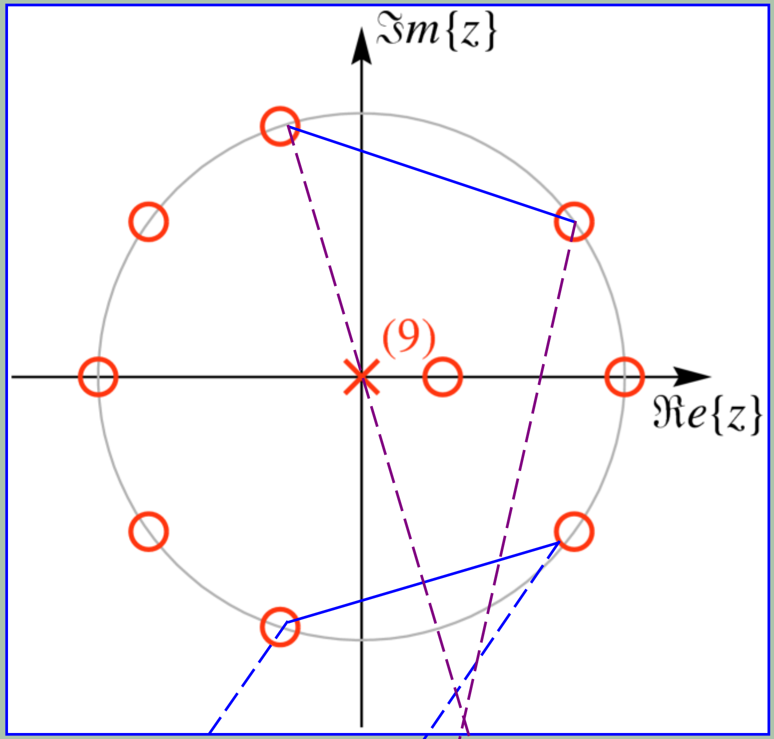


Fig 12.20

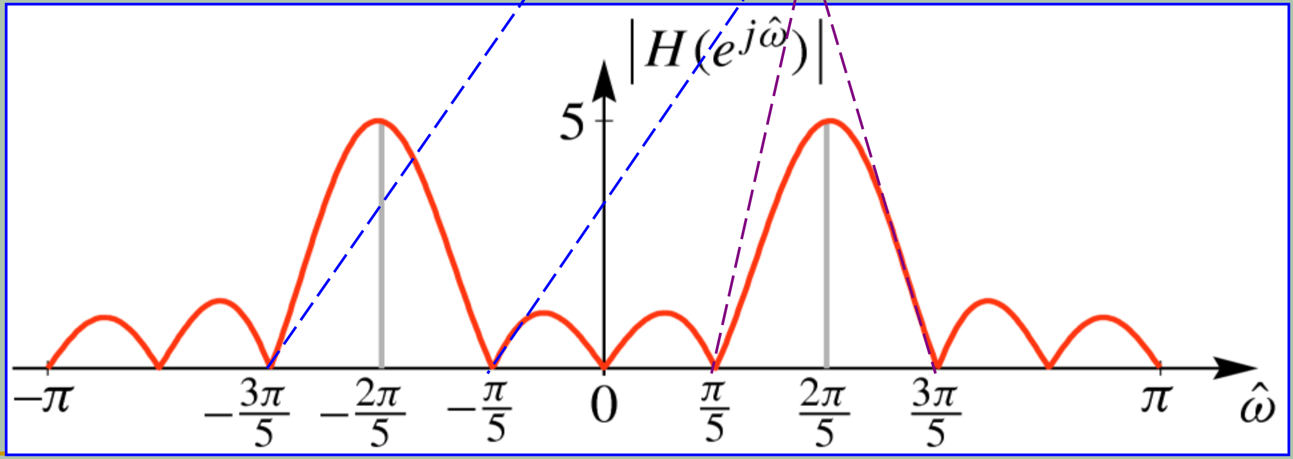


Fig 12.21

FIR filter with two zeros

Notice changes in the impulse response h[n] and the frequency response as the complex zero pair is moved around the unit circle (changing the angle of the zeros)

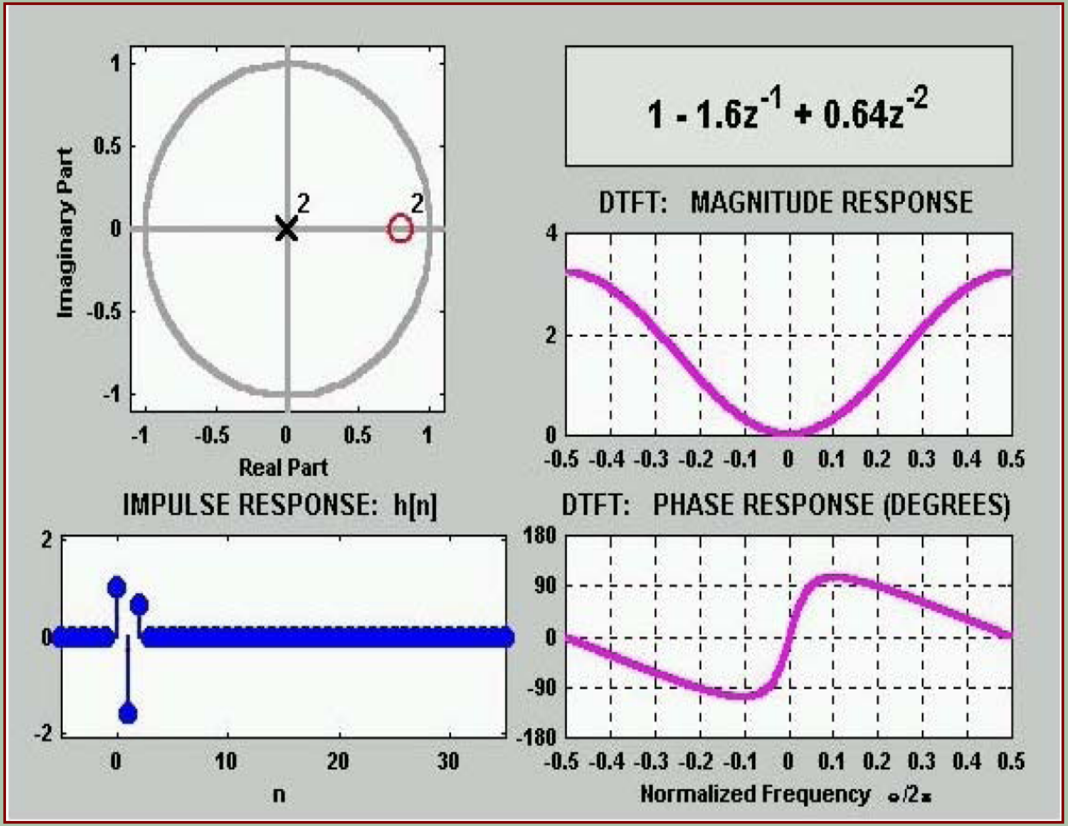


Fig 12.22

$$h[n] = \delta[n] - 2\cos(\hat{\omega})\delta[n-1] + \delta[n-2]$$

FIR filter with three zeros; one is held fixed at z = -1

Notice changes in the impulse response h[n] and the frequency response as the complex zero pair is moved around the unit circle (changing angle)

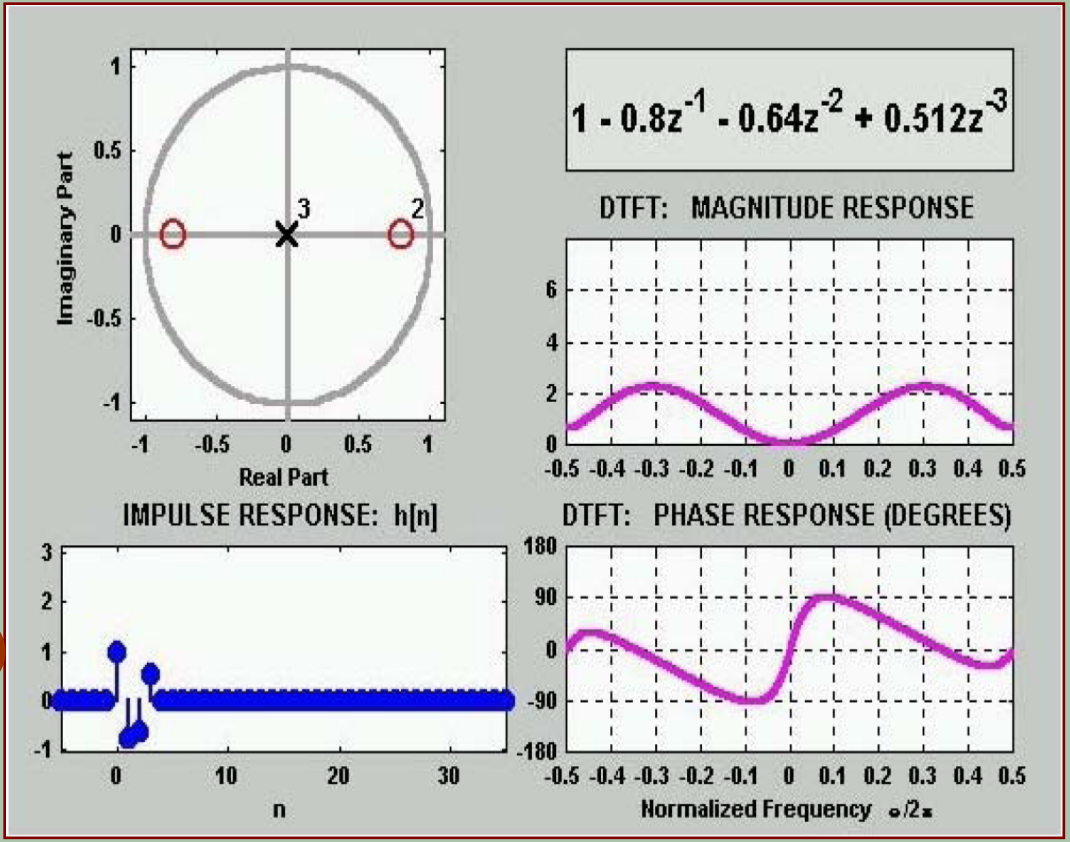


Fig 12.23

FIR filter with ten zeros equally spaced around the unit circle

Notice changes in the impulse response h[n] and the frequency response as the zero at z = 1, is moved radially

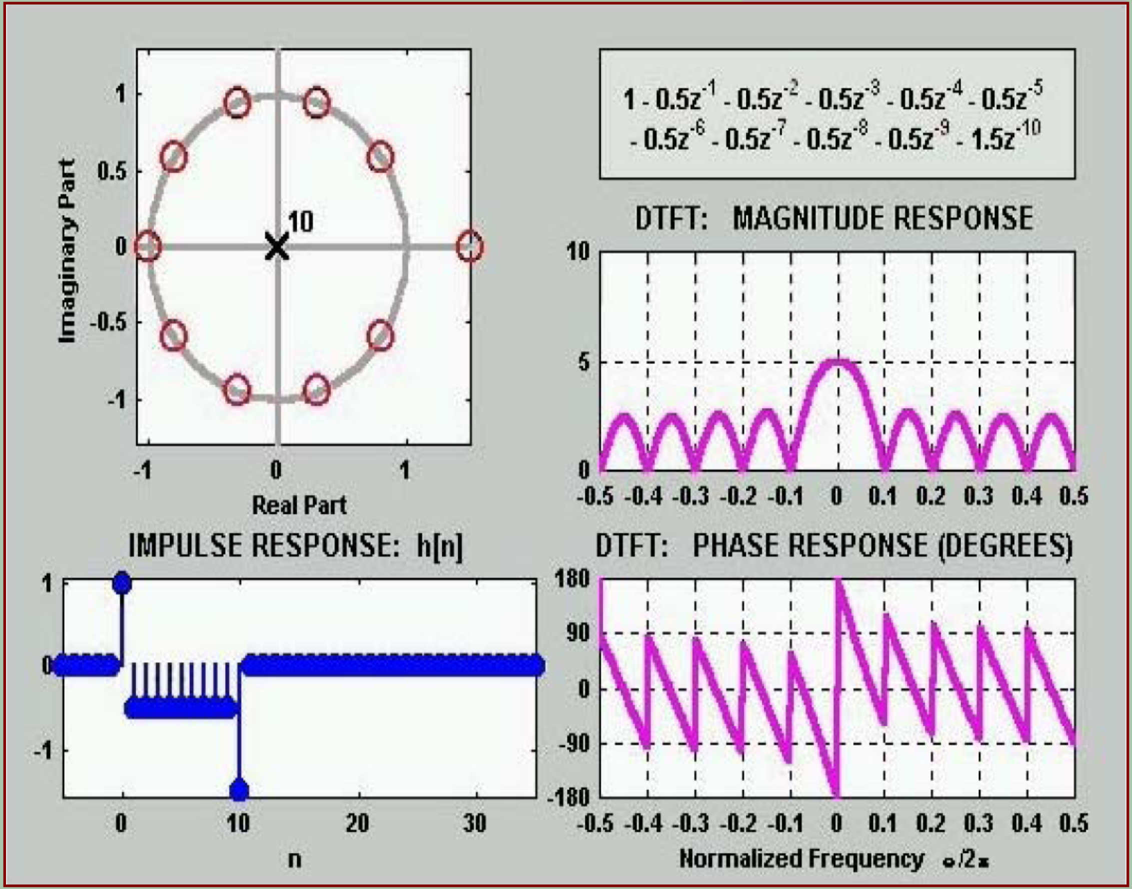


Fig 12.24

FIR filter with ten zeros equally spaced around the unit circle

Notice changes in the impulse response h[n] and the frequency response as the zero pair at 72 degrees is moved radially.

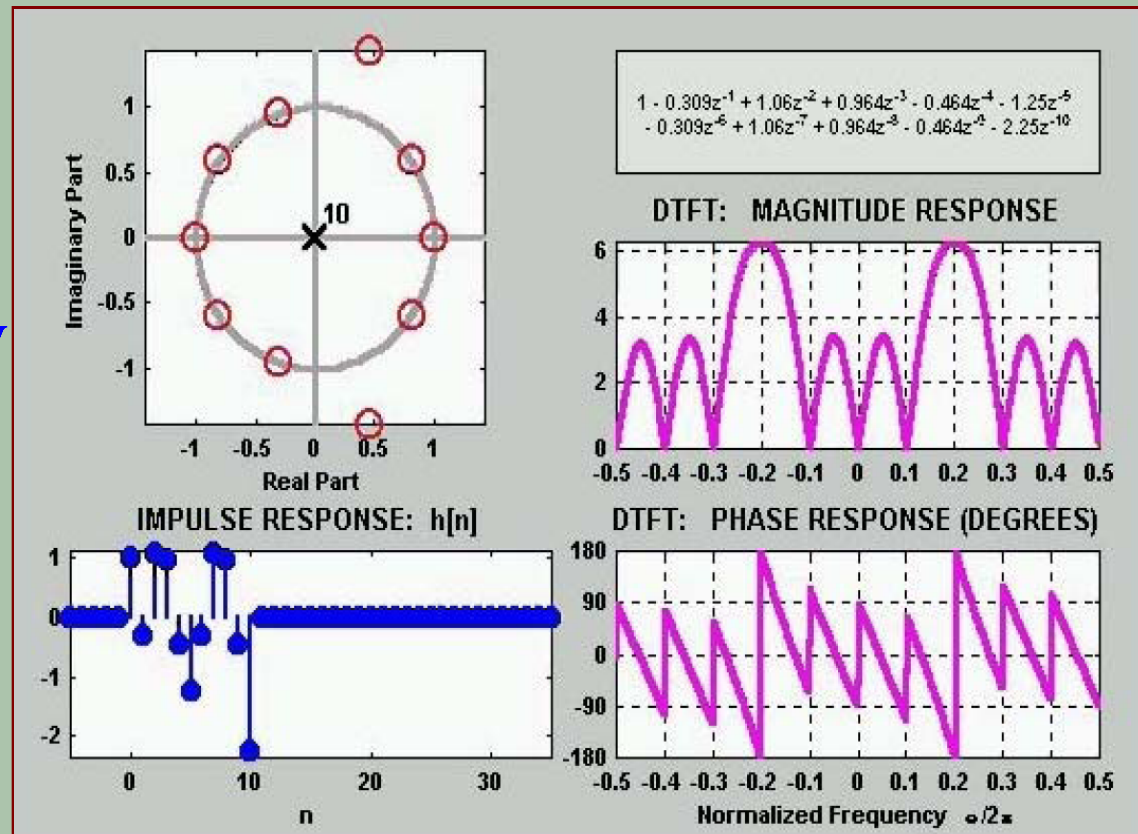


Fig 12.25

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “7.5-7.8 --Signal Processing First”, Prentice Hall, 2003