Discrete - Time Signals and Systems

IIR Filters
Introduction

Yogananda Isukapalli

General form of *Difference equations*

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

Recursive equation; Represents IIR filters

$a_l \ldots$ *feedback coefficients*

$b_k \ldots$ *feed − forward coefficients*

$N + M + 1 \ldots$ *Total coefficients*

$N \ldots$ *Order of IIR filters*

IIR: Block Diagram Representation

Consider a first – order general equation

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

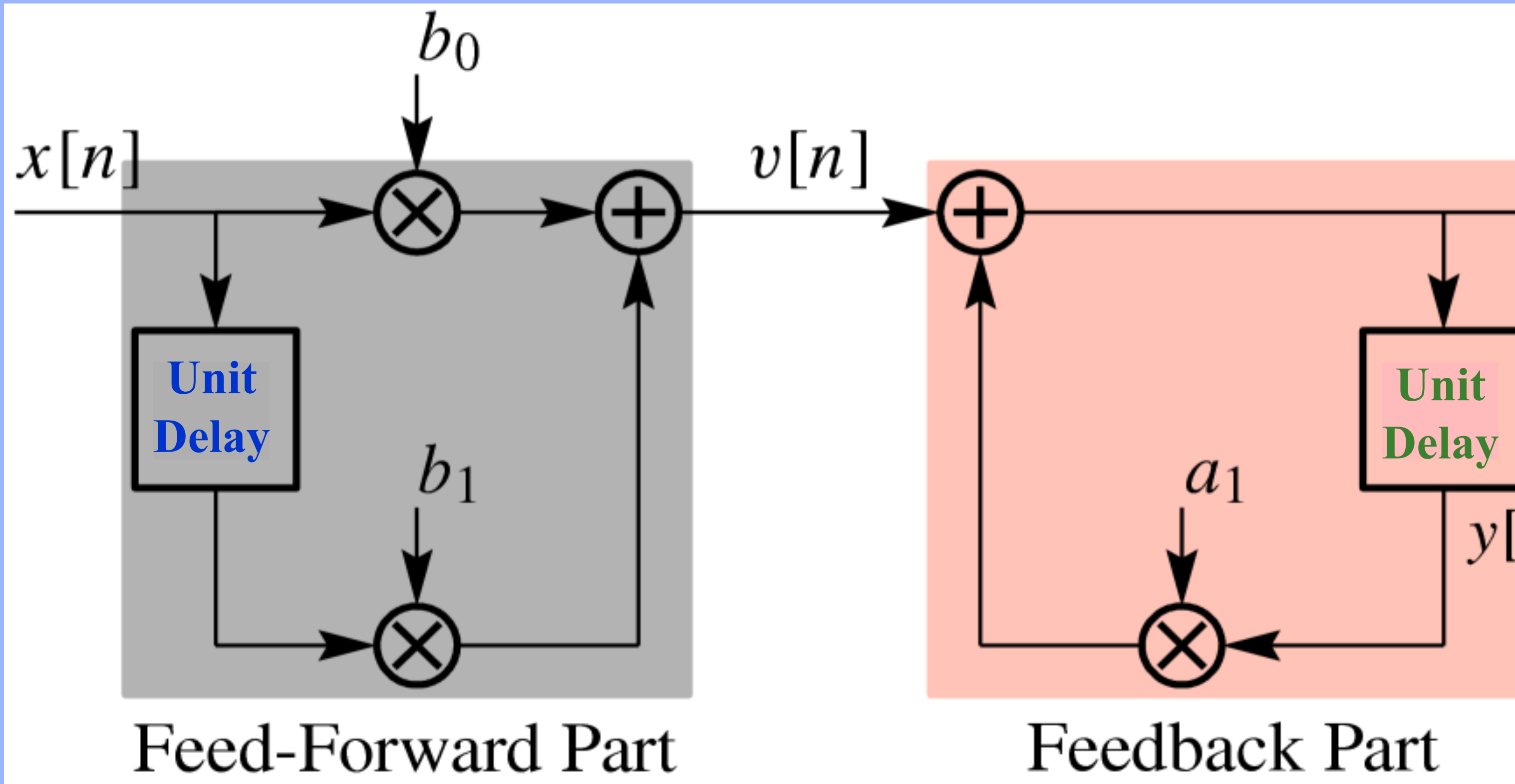


Fig.13.1

Example: IIR Filter

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Let, \quad a_1 = 0.8$$

$$b_0 = 5$$

$$y[n] = 0.8 y[n-1] + 5x[n]$$

Initial rest conditions:

$$x[n] = 0 \quad for\ n < 0$$

$$y[n] = 0 \quad for\ n < 0$$

Input

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$

Total of 4 samples

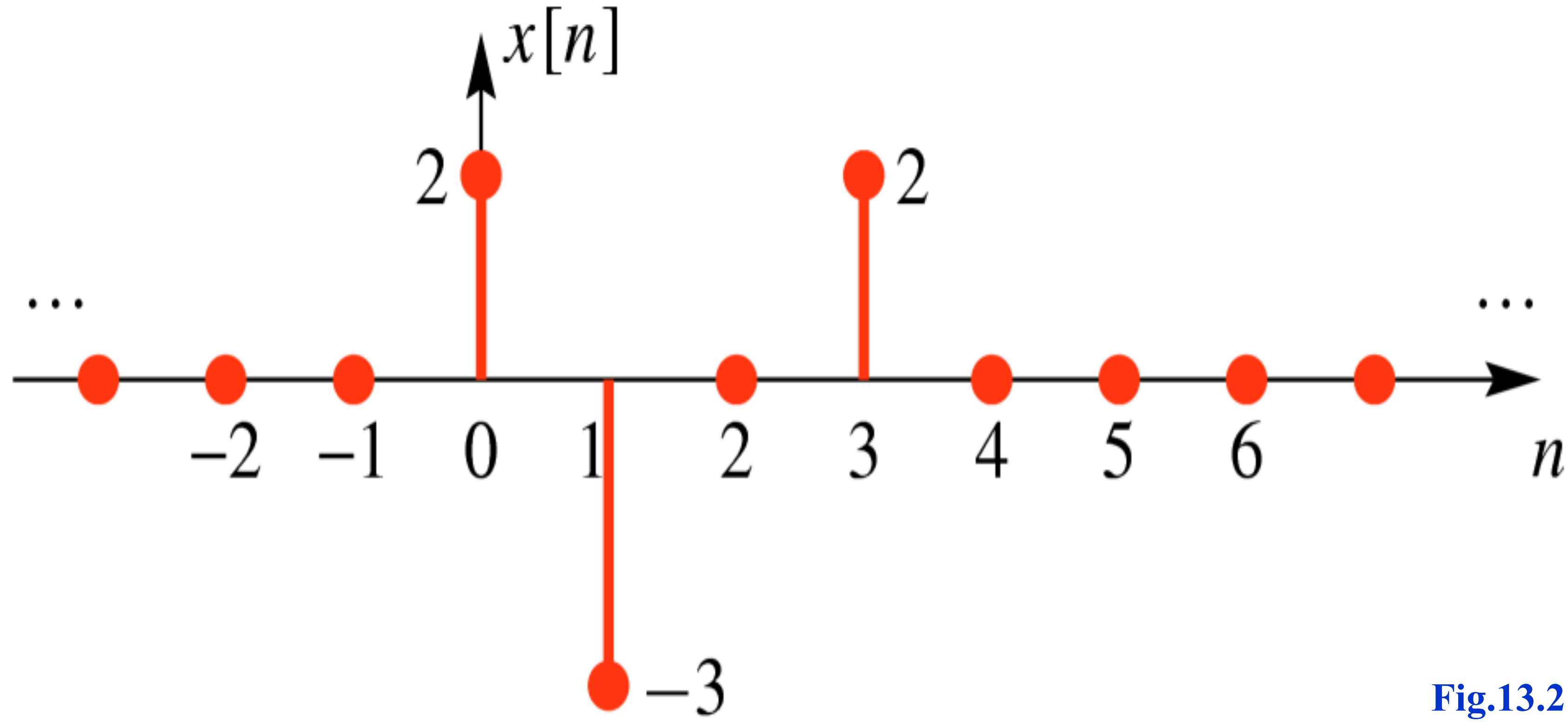


Fig.13.2

$$y[0] = 0.8y[-1] + 5x[0]$$

$$y[-1] = 0 \ldots \textit{initial rest conditions}$$

$$y[0] = 5x[0] = 5.2 = 10$$

Continue the recursion;

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

$$\because\ x[n] = 0 \qquad \text{for } n > 3$$

$$y[n] = 0.8\,y[n-1] \qquad \text{for } n > 3$$

$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$

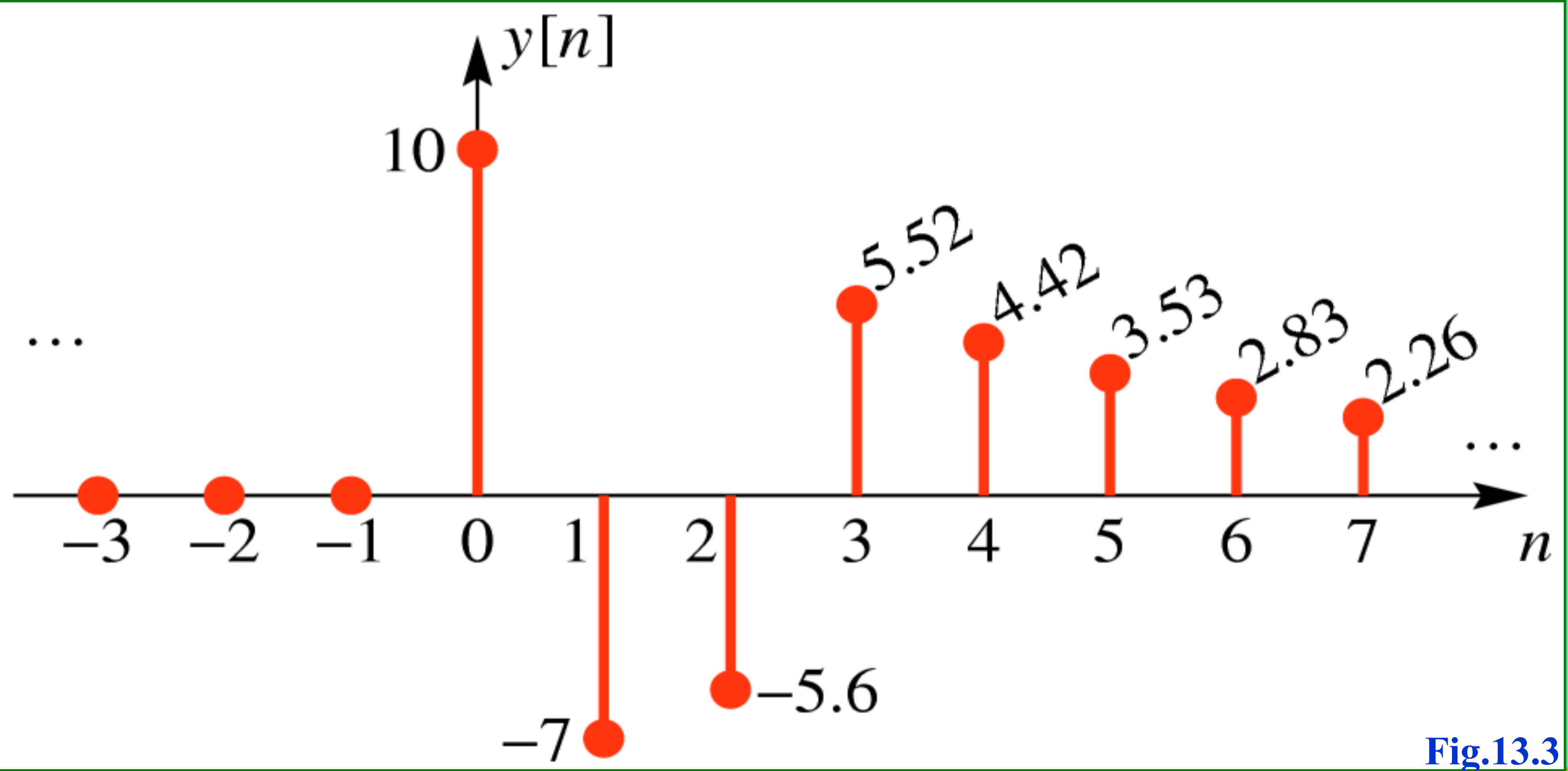


Fig.13.3

Impulse response of first order IIR system

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

Solution:

$n = 1$ $\quad y[1] = a_1 y[0] + b_0 x[1]$

$n = 2$ $\quad y[2] = a_1 y[1] + b_0 x[2]$

$$= a_1 \left(a_1 y[0] + b_0 x[1] \right) + b_0 x[2]$$

$$= a_1^2 y[0] + a_1 b_0 x[1] + b_0 x[2]$$

$n = 3$ $\quad y[3] = a_1 y[2] + b_0 x[3]$

$$= a_1\left(a_1^2 y[0] + a_1 b_0 x[1] + b_0 x[2]\right) + b_0 x[3]$$

$$y[3] = a_1^3 y[0] + a_1^2 b_0 x[1] + a_1 b_0 x[2] + b_0 x[3]$$

Generalizing the 1^{st} *order discrete − time system,*

$$y[n] = a_1^n y[0] + b_0\left[a_1^{n-1} x[1] + a_1^{n-2} x[2] + ... + a_1^0 x[n]\right]$$

$$y[n] = a_1^n y[0] + b_0 \sum_{m=1}^{n} a_1^{(n-m)} x[m]$$

$$If\ \ x[n]=\delta[n]\ \ then\ y[n]=h[n]$$

$$y[n]=a_1 y[n-1]+b_0 x[n]$$

$$h[n]=a_1 y[n-1]+b_0 \delta[n]$$

$$y[n]=a_1^n y[0]+b_0\sum_{m=1}^{n} a_1^{(n-m)} x[m]$$

$$\because\ y[0]=a_1 y[-1]+b_0 x[0]=a_1.0+b_0.1=b_0$$

$$h[n]=b_0 a_1^n+b_0\sum_{m=1}^{n} a_1^{(n-m)}\,0$$

$$h[n]=\begin{cases} b_0 a_1^n & for\ \ n\geq 0 \\ 0 & for\ \ n<0 \end{cases}$$

Tabular form: Impulse Response

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

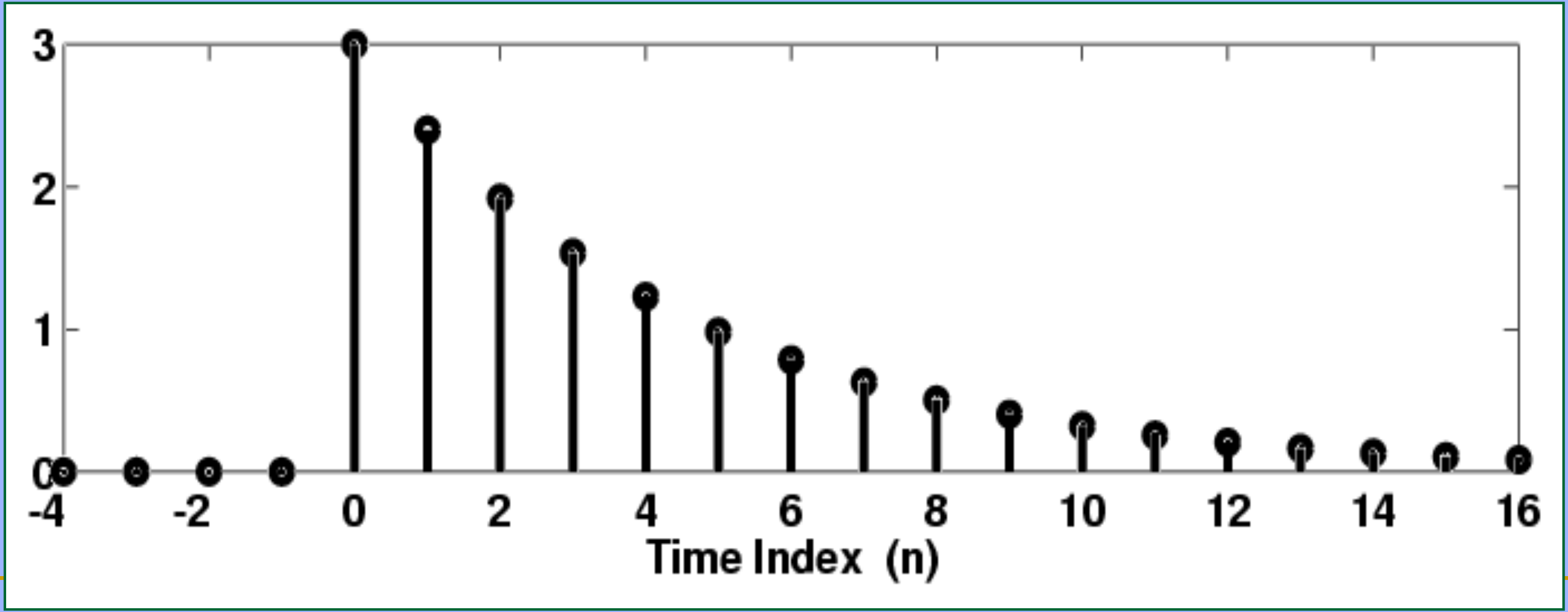
Example: Impulse Response

Find the impulse response of the system;

$$y[n] = 0.8y[n-1] + 3x[n]$$

$$h[n] = b_0(a_1)^n u[n]$$

$$h[n] = 3(0.8)^n u[n]$$



Impulse response of a more general 1^{st} *order IIR system*

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = a_1 h[n-1] + b_0 \delta[n] + b_1 \delta[n-1]$$

Using the principle of superposition and previous result;

$$h[n] = b_0 (a_1)^n u[n] \text{ for the system}$$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

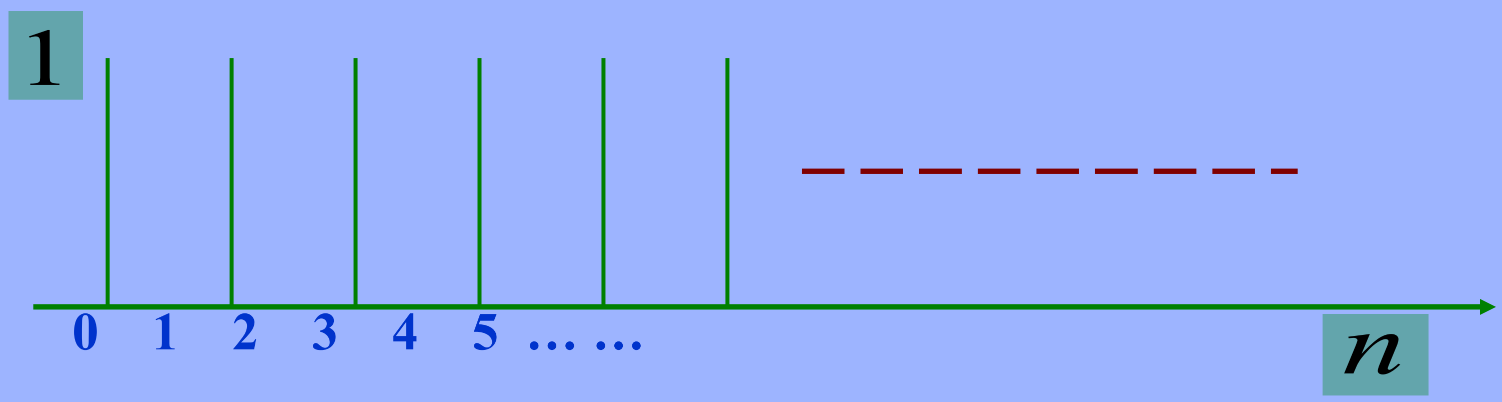
For the present system

$$h[n] = b_0 a_1^n u[n] + b_1 a_1^{n-1} u[n-1]$$

Unit Step Sequence Input

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$Let\ x[n] = u[n] = 1 \quad for\ all\ n, \quad n \geq 0$$



$$y[n] = a_1^n y[0] + b_0 \left[a_1^{n-1} x[1] + a_1^{n-2} x[2] + \ldots\ldots + a_1^0 x[n] \right]$$

$$y[0] = a_1 y[-1] + b_0 x[0] = a_1 . 0 + b_0 . 1 = b_0$$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

$$y[n] = a_1^n b_0 + b_0 \left[a_1^{n-1} x[1] + a_1^{n-2} x[2] + \ldots\ldots + a_1^0 x[n] \right]$$

$$= b_0 \left[a_1^n + a_1^{n-1} + a_1^{n-2} + \ldots\ldots + 1 \right]$$

$$y[n] = b_0 \left(\frac{1 - a_1^{n+1}}{1 - a_1} \right) \qquad a_1 \neq 1$$

if $a_1 > 1$ a_1^{n+1} *will dominate the denominator*

$$\lim_{n \to \infty} a_1^{n+1} \to \infty$$

$y[n]\ldots$ *Will be unstable*

$$if\ \ a_1 < 1\ \ then\ \ \lim_{n\to\infty} a_1^{n+1} \to 0;$$

$$y[n] \ldots Will\ be\ stable;\quad y[n] = \frac{b_0}{1-a_1}$$

$$if\ \ a_1 = 1;\quad y[n] = \frac{0}{0},\ unbounded\ \ output$$

$$\lim_{a_1\to 1} b_0\left(\frac{1-a_1^{n+1}}{1-a_1}\right) = b_0 \lim_{a_1\to 1} \frac{\frac{d}{da_1}\left(1-a_1^{n+1}\right)}{\frac{d}{da_1}\left(1-a_1\right)}$$

$$= b_0 \lim_{a_1\to 1} \frac{(n+1)a_1^n}{1} = b_0 (n+1) 1^n = b_0\left(n+1\right)$$

Example: Unit Step Response

$$y[n] = 0.8y[n-1] + 3x[n]$$

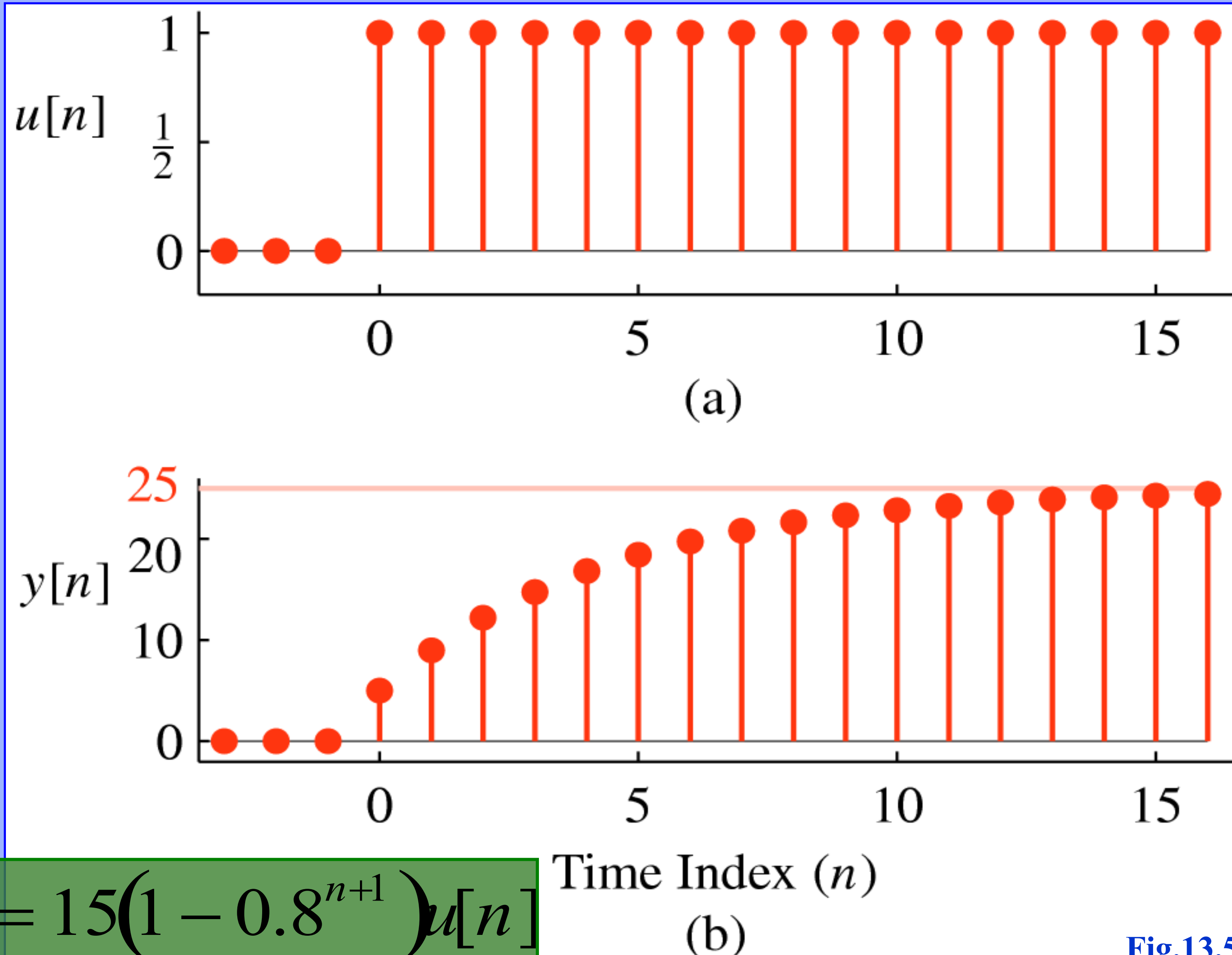
$$a_1 = 0.8$$

$$b_0 = 3$$

$$y[n] = b_0\left(\frac{1-a_1^{n+1}}{1-a_1}\right) \qquad a_1 \neq 1$$

$$= 3\left(\frac{1-0.8^{n+1}}{1-0.8}\right) = 3\left(\frac{1-0.8^{n+1}}{0.2}\right)$$

$$= 15\left(1-0.8^{n+1}\right)$$



$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$

Fig.13.5

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 8.1 and 8.2 “Signal Processing First”, Prentice Hall, 2003