Discrete - Time Signals and Systems

Z-Transform & IIR Filters 1

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Transfer function: 1st Order IIR Filters

Consider the difference equation of a general

1st order IIR system

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

Apply the Z-transform on both sides

$$Z\{y[n]\} = Z\{a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]\}$$

$$\Rightarrow Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

$$Y(z)(1 - a_1 z^{-1}) = X(z)(b_0 + b_1 z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 z^{-1})} = \frac{B(z)}{A(z)}$$

Unlike FIR, IIR filter transfer function is a ratio of polynomials

$B(z)$... *numerator polynomial is defined by the weighting coefficients* $\{b_k\}$ *that multiply* $x[n]$ *and its delayed versions*

$A(z)$... *denominator polynomial is defined by feedback coefficients*

An implementation view

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

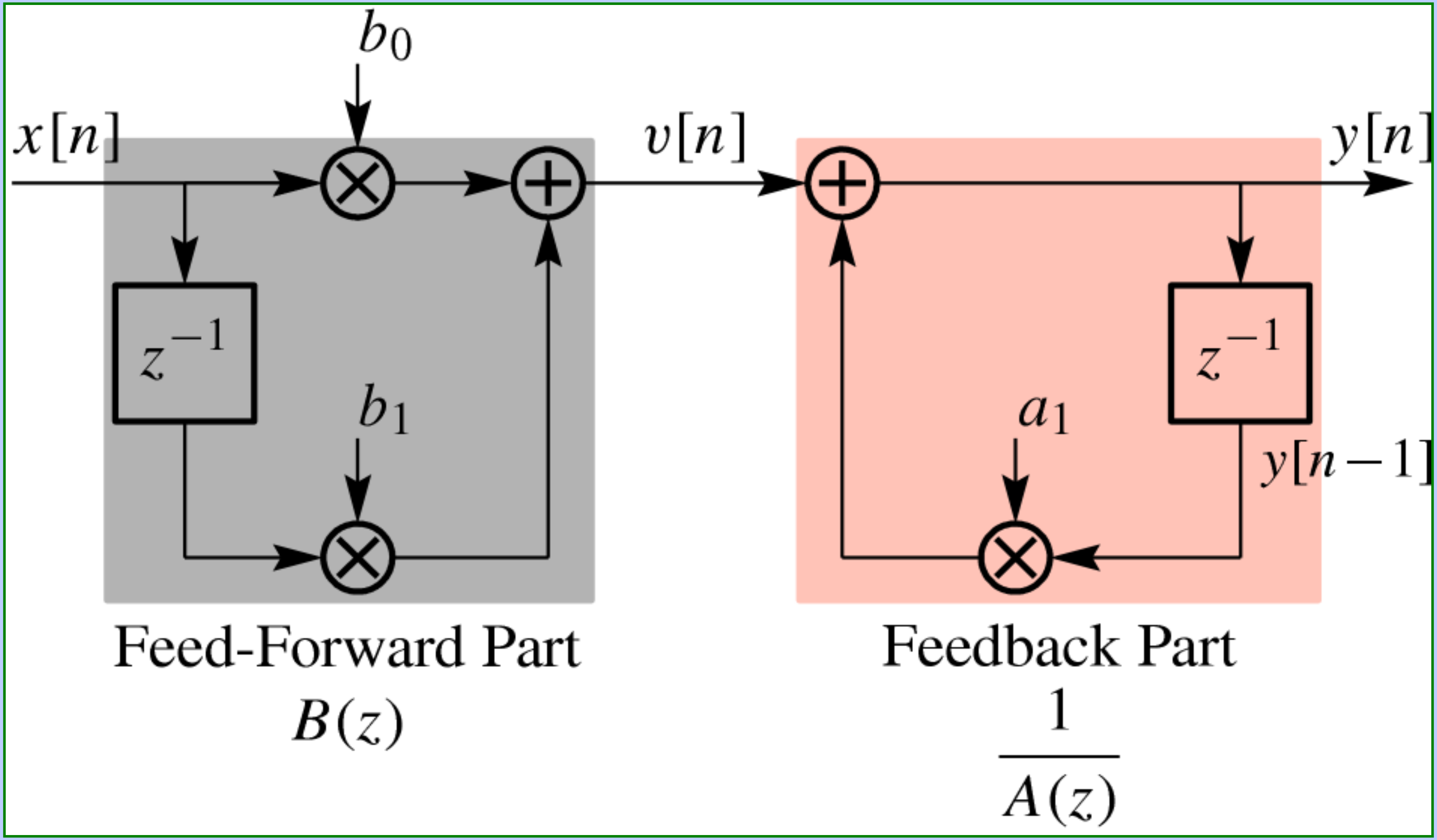
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 z^{-1})} = \frac{B(z)}{A(z)}$$

using the cascading property

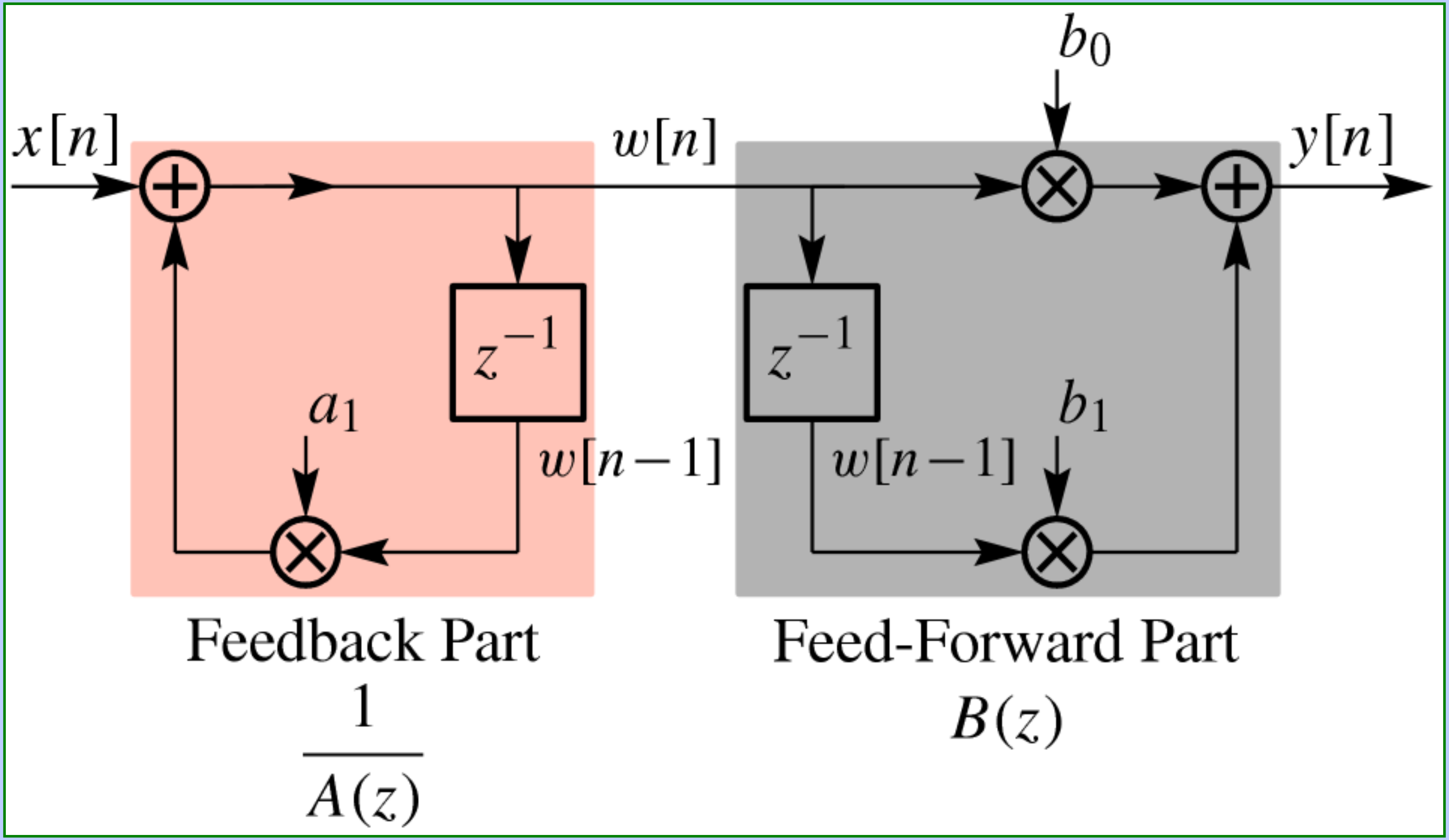
$$H(z) = H_1(z) H_2(z) = H_2(z) H_1(z)$$

$$H_1(z) = \frac{1}{A(z)}, \quad H_2(z) = B(z)$$

Direct form I



Direct form II



Example 1

Find the transfer function of the 1^st^ order IIR system

$$y[n] = 2y[n-1] - x[n] + \left(3/4\right) x[n-1]$$

Apply the Z – transform on both sides

$$Z\left\{y[n]\right\} = Z\left\{2y[n-1] - x[n] + \left(3/4\right) x[n-1]\right\}$$

$$\Rightarrow Y(z) = 2z^{-1}Y(z) - X(z) + \left(3/4\right) z^{-1} X(z)$$

$$Y(z)(1-2z^{-1}) = X(z)(-1+\left(3/4\right) z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(-1+\left(3/4\right) z^{-1})}{(1-2z^{-1})}$$

Example 2

Find the transfer function of the 1^{st} *order IIR system*

$$y[n] = ay[n-1] + bx[n] + cx[n-1]$$

Directly pick the numerator and denominator coefficients

coefficients related with $'x'$, *numerator*

coefficients related with $'y'$, *denominarator*

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b + cz^{-1})}{(1 + az^{-1})}$$

Transfer function: 2nd order IIR Filters

Consider the difference equation of a general 2^{nd} *order IIR system*

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

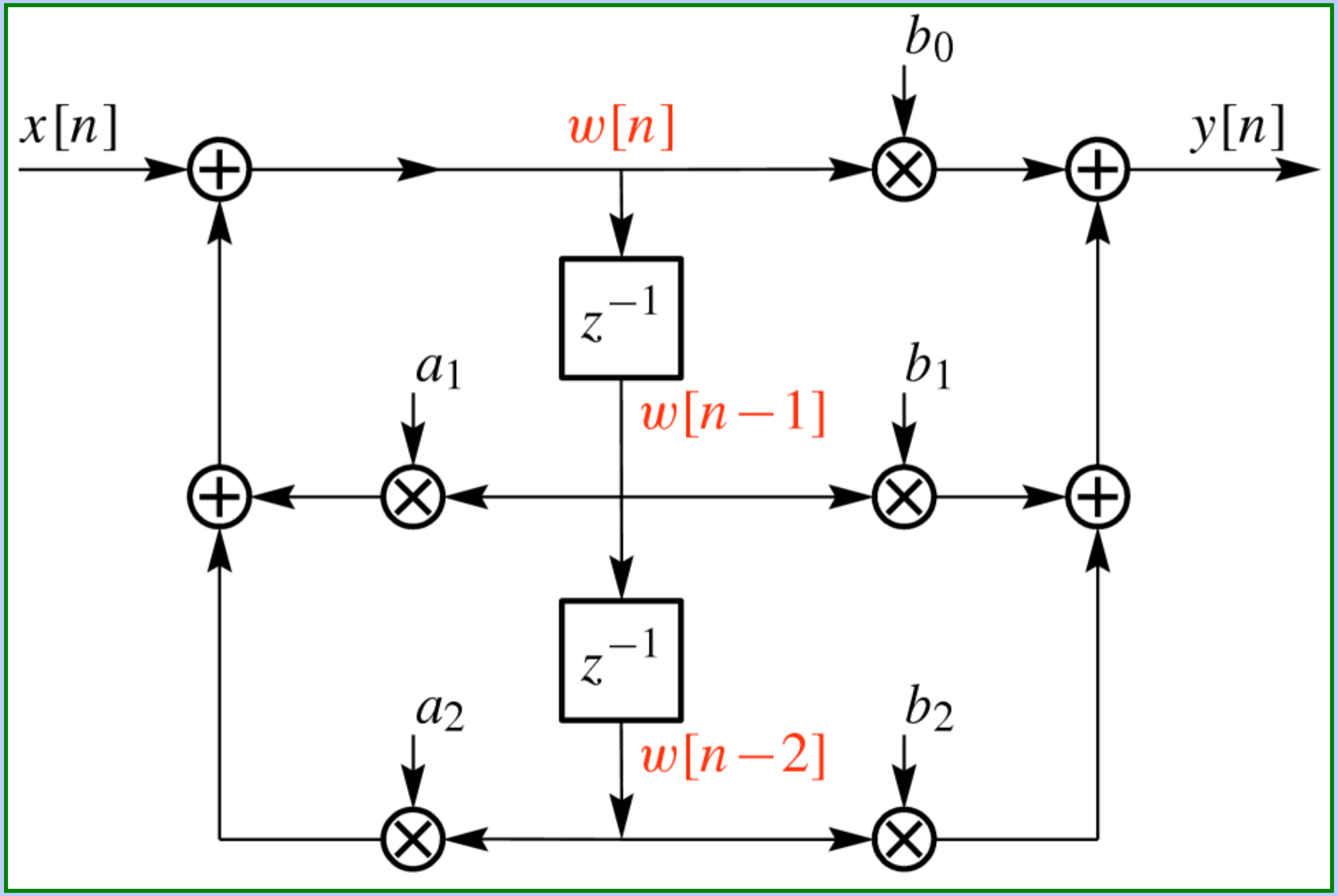
Apply the $Z-$*transform on both sides*

$$Z\left\{y[n]\right\} = Z\left\{\begin{array}{r} a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] \\ + b_2 x[n-2] \end{array}\right\}$$

$$\Rightarrow Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z)(1 - a_1 z^{-1} - a_2 z^{-2}) = X(z)(b_0 + b_1 z^{-1} + b_2 z^{-2})$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(b_0+b_1z^{-1}+b_2z^{-2})}{(1-a_1z^{-1}-a_2z^{-2})}$$



Example

Find the transfer function of a 2nd order IIR system

$$y[n] = 0.5y[n-1] + 0.3y[n-2] - x[n] + 3x[n-1] - 2x[n-2]$$

$$Y(z) = Z\left\{0.5y[n-1] + 0.3y[n-2] - x[n] + 3x[n-1] - 2x[n-2]\right\}$$

$$\Rightarrow Y(z) = 0.5z^{-1}Y(z) + 0.3z^{-2}Y(z) - X(z) + 3z^{-1}X(z) - 2z^{-2}X(z)$$

$$Y(z)(1 - 0.5z^{-1} - 0.3z^{-2}) = X(z)(-1 + 3z^{-1} - 2z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(-1 + 3z^{-1} - 2z^{-2})}{(1 - 0.5z^{-1} - 0.3z^{-2})}$$

Transfer function: General expression

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

$$Y(z) = \sum_{l=1}^{N} a_l z^{-l} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{l=1}^{N} a_l z^{-l}}$$

$a_l \ldots$ *feedback coefficients*

$b_k \ldots$ *feed – forward coefficients*

Transfer function: Impulse Response

Transfer function is also Z − transform of impulse response, $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$

From previous lecture Impulse response of ,

$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$ *is*

$h[n] = b_0 a_1^n u[n] + b_1 a_1^{n-1} u[n-1]$

$\because a^n u[n] \xleftrightarrow{z} \dfrac{1}{1 - az^{-1}}$

$$H(z) = b_0\left(\frac{1}{1-a_1z^{-1}}\right) + b_1z^{-1}\left(\frac{1}{1-a_1z^{-1}}\right)$$

$$H(z) = \frac{b_0 + b_1z^{-1}}{1-a_1z^{-1}} \ldots \textit{same result}$$

In summary, except for polynomial functions on both numerator and denominator, general properties of Z – transforms are same for IIR and FIR filters

Example 1: Application of inverse Z-transform

Find the impulse response of the feedback system

$$H(z) = \frac{1-3z^{-1}}{1+2z^{-1}}$$

$$H(z) = \frac{1}{1+2z^{-1}} - \frac{3z^{-1}}{1+2z^{-1}}$$

$$\because\ a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}} \quad \& \quad z^{-1}X(z) \overset{z}{\longleftrightarrow} x[n-1]$$

$$\frac{1}{1+2z^{-1}} \overset{z}{\longleftrightarrow} (-2)^n u[n] \quad \& \quad \frac{3z^{-1}}{1+2z^{-1}} \overset{z}{\longleftrightarrow} 3(-2)^{n-1} u[n-1]$$

$$h[n] = (-2)^n u[n] - 3(-2)^{n-1} u[n-1]$$

Example 2

Find the impulse response of the feedback system

$$H(z) = \frac{z^2 + 2}{(z^2 + 0.4z - .12)}$$

$$= \frac{z(z+2)}{(z-0.2)(z+0.6)}$$

$$= \frac{c_1 z}{(z-0.2)} + \frac{c_2 z}{(z+0.6)}$$

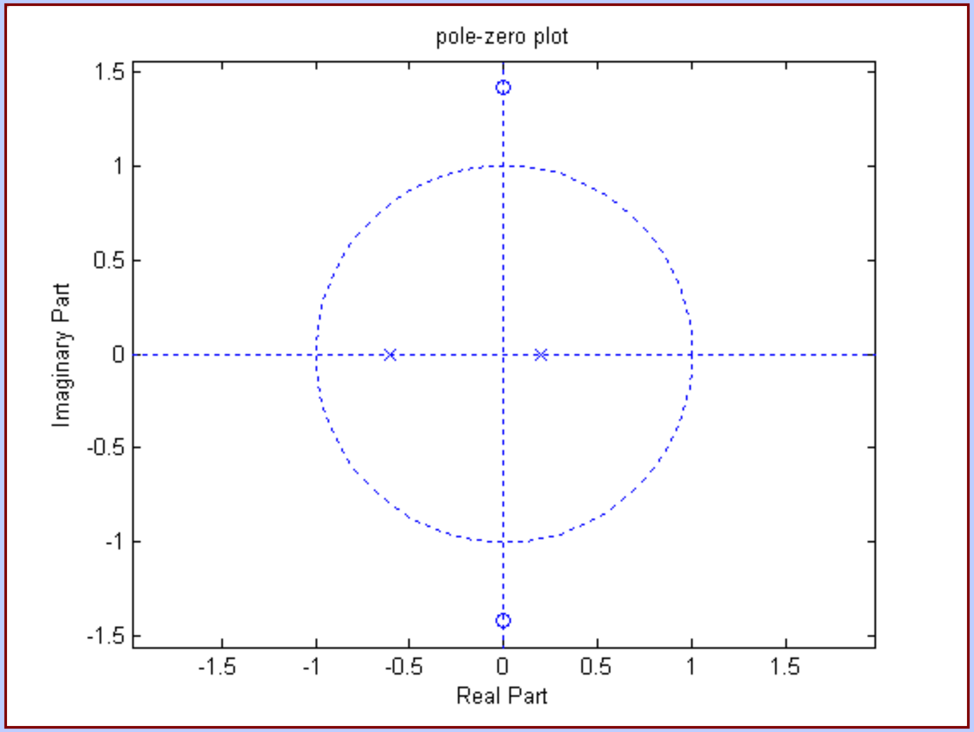
$$c_1 = X(z)\frac{(z-0.2)}{z}\bigg|_{z=0.2} = \frac{z+2}{(z+0.6)}\bigg|_{z=0.2} = 2.75$$

$$c_2 = X(z)\frac{(z+0.6)}{z}\bigg|_{z=-0.6} = \frac{z+2}{(z-0.2)}\bigg|_{z=-0.6} = -1.75$$

$$H(z) = \frac{2.75z}{(z-0.2)} - \frac{1.75z}{(z+0.6)}$$

$$\because a^n u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-a}$$

$$h[n] = \left\{ \begin{matrix} 2.75(0.2)^n u[n] \\ -1.75(-0.6)^n u[n] \end{matrix} \right\}$$



Poles & Zeros

FIR filters had poles only at $z = 0$. *However, IIR filters can have poles anywhere, because the poles are nothing but the roots of the denominator polynomial* $A(z)$

Consider a first order transfer function, $H(z) = \dfrac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$

$$\text{Zero}\ldots b_0 + b_1 z^{-1} = 0 \;\Rightarrow\; z_1 = -\frac{b_1}{b_0}$$

$$\text{Pole}\ldots 1 - a_1 z^{-1} = 0 \Rightarrow\; p_1 = a_1$$

Poles & Zeros — 2nd order IIR filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(b_0 + b_1 z^{-1} + b_2 z^{-2})}{(1 - a_1 z^{-1} - a_2 z^{-2})}$$

$$Convert\ the\ function\ as\ a\ function\ of\ \ 'z'$$

$$H(z) = \frac{(b_0 z^2 + b_1 z^1 + b_2)}{(z^2 - a_1 z^1 - a_2)}$$

$$A\ polynomial\ of\ degree\ 'N'\ has\ 'N'\ roots$$

$$\therefore\ order\ 2\ polynomial\ will\ have\ 2\ zeros\ and\ 2\ poles$$

Zeros, from numerator;

$$b_0 z^2 + b_1 z + b_2 = 0$$

$$[z_1, z_2] = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0 b_2}}{2b_0}$$

$$z_1 = \frac{-b_1 + \sqrt{b_1^2 - 4b_0 b_2}}{2b_0}, \quad z_2 = \frac{-b_1 - \sqrt{b_1^2 - 4b_0 b_2}}{2b_0}$$

$$if \;\; \left(b_1^2 - 4b_0 b_2\right) < 0 \;\; or \;\; b_1^2 < 4b_0 b_2$$

Then the z_1 *and* z_2 *will be a complex conjugate pair*

$$z_1 = z_2^*$$

Poles, from denominator;

$$z^2 - a_1 z - a_2 = 0$$

$$[p_1, p_2] = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$p_1 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2}, \quad p_2 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2}$$

$$if \left(a_1^2 + 4a_2\right) < 0 \ \ or \ \ a_1^2 < -4a_2$$

Then the p_1 *and* p_2 *will be a complex conjugate pair*

$$p_1 = p_2^*$$

Example 1

Find the poles and zeros of the feedback system

$$y[n] = 0.5y[n-1] + 2x[n]$$

$$H(z) = \frac{2}{1-0.5z^{-1}} = \frac{2z}{z-0.5}$$

$$Zero\ldots 2z = 0 \;\Rightarrow\; z = 0$$

$$Pole\ldots z - 0.5 = 0 \Rightarrow\; z = 0.5$$

Always convert the transfer function as a function of z

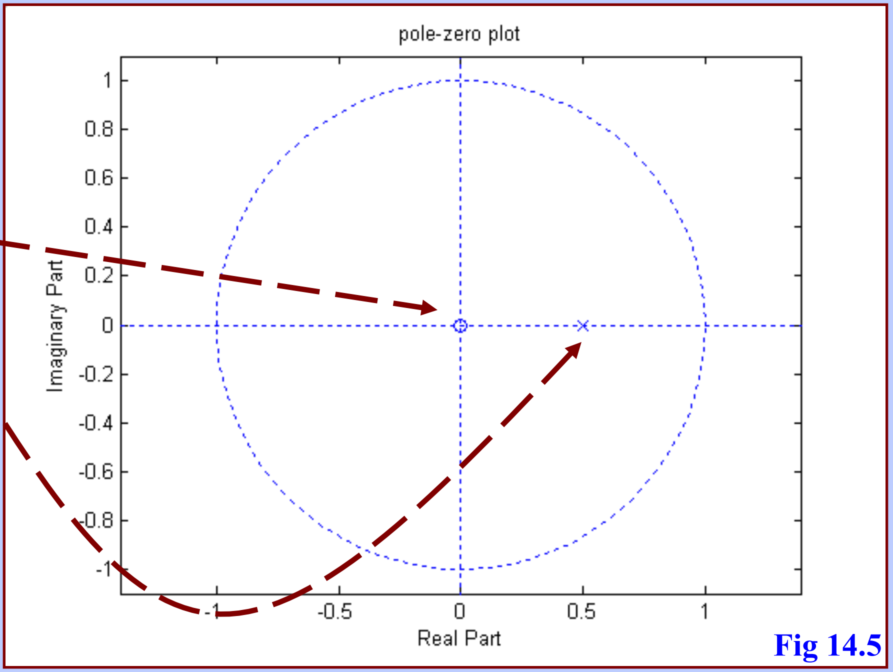


Fig 14.5

Example 2

Find the poles and zeros of the feedback system

$$y[n] = 0.5y[n-1] + 3x[n-1]$$

$$H(z) = \frac{3z^{-1}}{1-0.5z^{-1}} = \frac{3}{z-0.5}$$

$$Zero\ldots \lim_{z\to\infty} H(z) = 0 \;\Rightarrow\; z = \infty$$

$$Pole\ldots z - 0.5 = 0 \Rightarrow\; z = 0.5$$

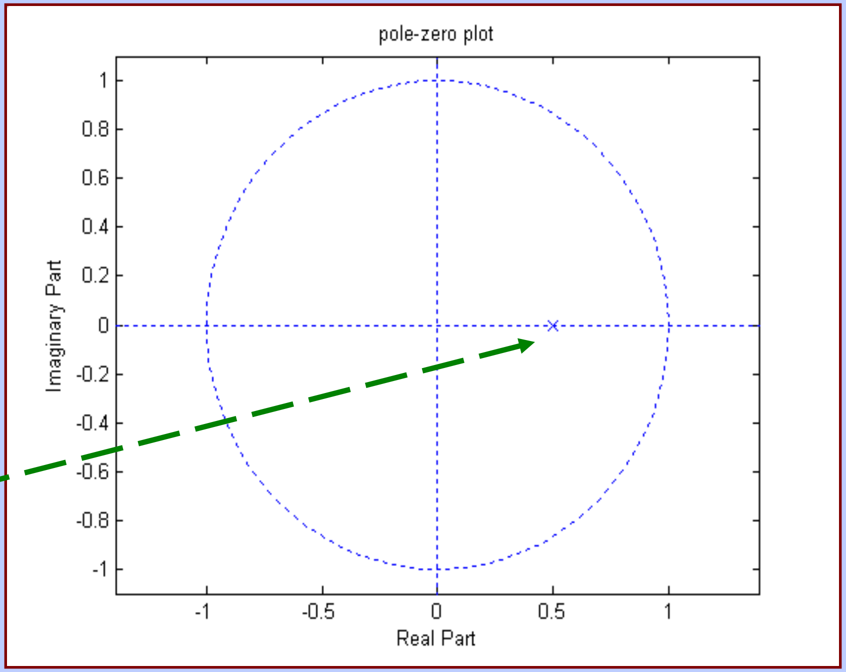


Fig 14.6

if all poles and zeros at

$z = 0$ *or* ∞ *are also counted, no.of poles = no.of zeros*

Example 3

Find the poles and zeros of the feedback system

$$H(z) = \frac{2 + 2z^{-1}}{1 - z^{-1} + z^{-2}}$$

Convert $H(z)$ *into a function of* z,

$$H(z) = \left(\frac{2 + 2z^{-1}}{1 - z^{-1} + z^{-2}} \right) \frac{z^2}{z^2} = \frac{2z^2 + 2z}{z^2 - z + 1}$$

Zeros... $2z^2 + 2z = 0; \quad 2z(z+1) = 0$

$$\Rightarrow \; z_1 = 0, \;\; z_2 = -1$$

$$poles...z^2 - z + 1 = 0$$

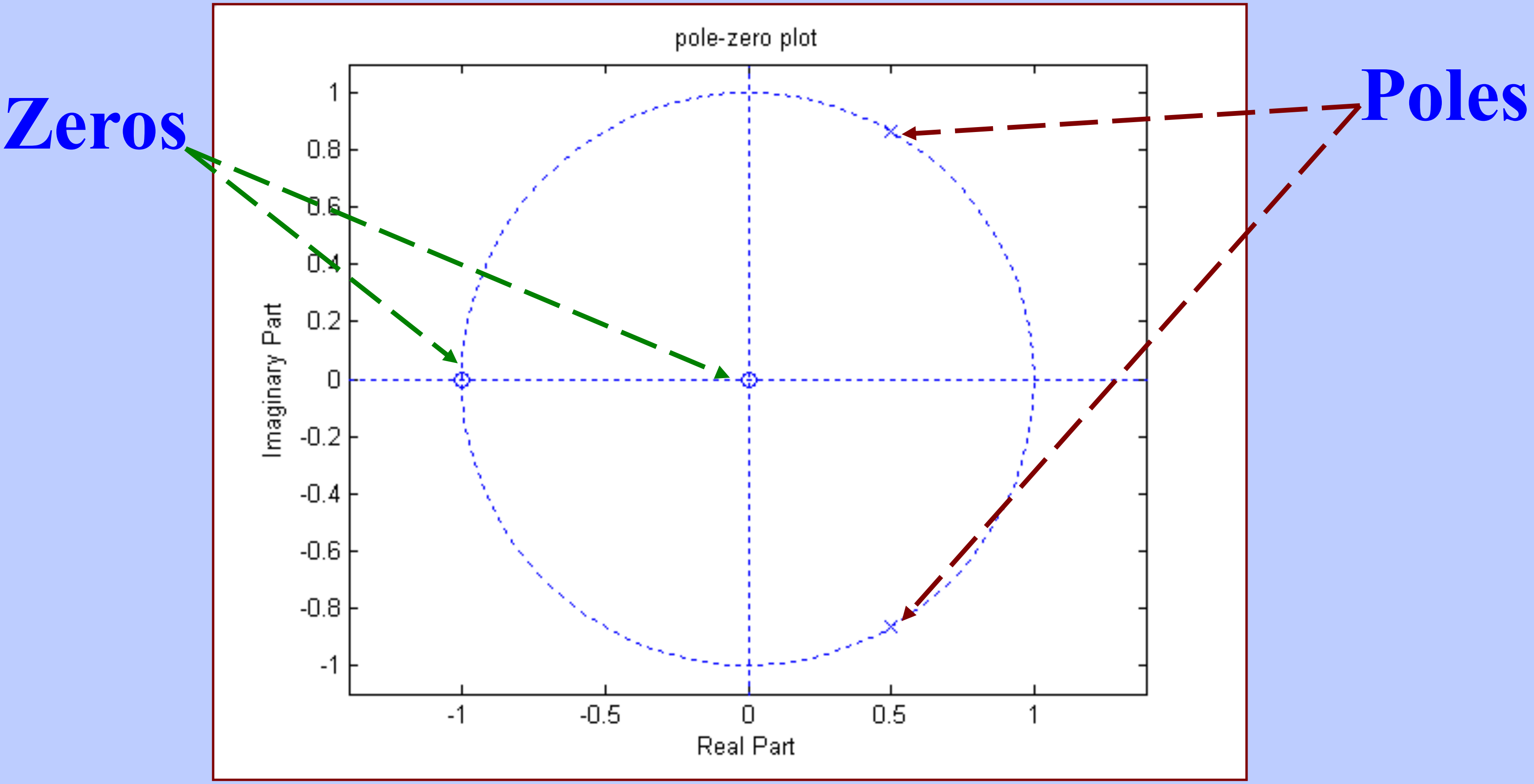
$$p_1 = \frac{1+\sqrt{1-4}}{2} = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\pi/3}$$

$$p_2 = \frac{1-\sqrt{1-4}}{2} = \frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\pi/3}$$

The system function can now be written in factored form also

$$H(z) = \frac{2z(z+1)}{(z-e^{j\pi/3})(z-e^{-j\pi/3})}$$

$$H(z) = \frac{2z(z+1)}{(z - e^{j\pi/3})(z - e^{-j\pi/3})}$$



Example 4

Find the poles and zeros of the feedback system

$$y[n] = y[n-1] + y[n-2] + x[n] + x[n-1] + 2x[n-2]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + z^{-1} + 2z^{-2})}{(1 - z^{-1} - z^{-2})}$$

Convert $H(z)$ *into a function of* z,

$$H(z) = \left(\frac{1 + z^{-1} + 2z^{-2}}{1 - z^{-1} - z^{-2}}\right)\frac{z^2}{z^2} = \frac{z^2 + z + 2}{z^2 - z - 1}$$

Zeros... $z^2 + z + 2 = 0$

$$z_1 = \frac{-1+\sqrt{1-8}}{2} = -\frac{1}{2} + \frac{j\sqrt{7}}{2}$$

$$z_2 = \frac{-1-\sqrt{1-8}}{2} = -\frac{1}{2} - \frac{j\sqrt{7}}{2}$$

$$Poles\ldots z^2 - z - 1 = 0$$

$$p_1 = \frac{1+\sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$$

$$p_2 = \frac{1-\sqrt{1+4}}{2} = \frac{1-\sqrt{5}}{2}$$

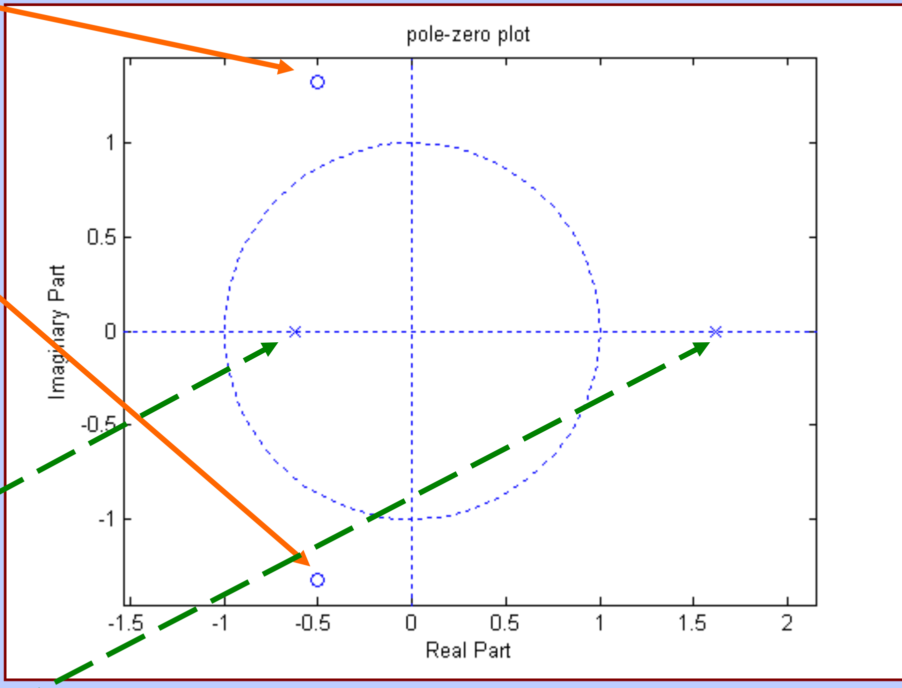


Fig 14.8

Example 5

Find the poles and zeros of the IIR filter

$$y[n] = 2y[n-1] - 3y[n-2] + 3x[n] + 2x[n-1] + x[n-2]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(3 + 2z^{-1} + z^{-2})}{(1 - 2z^{-1} + 3z^{-2})}$$

Convert $H(z)$ *into a function of* z,

$$H(z) = \left(\frac{3 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + 3z^{-2}} \right) \frac{z^2}{z^2} = \frac{3z^2 + 2z + 1}{z^2 - 2z + 3}$$

Zeros...$3z^2 + 2z + 1 = 0$

$$z_1 = \frac{-2 + \sqrt{4 - 12}}{2} = -1 + \frac{j2\sqrt{2}}{2} = -1 + j\sqrt{2}$$

$$z_2 = \frac{-2 - \sqrt{4 - 12}}{2} = -1 - \frac{j2\sqrt{2}}{2} = -1 - j\sqrt{2}$$

$$Poles ... z^2 - 2z + 3 = 0$$

$$p_1 = \frac{2 + \sqrt{4 - 12}}{2} = \frac{2 + j\sqrt{8}}{2} = 1 + j\sqrt{2}$$

$$p_2 = \frac{2 - \sqrt{4 - 12}}{2} = \frac{2 - j\sqrt{8}}{2} = 1 - j\sqrt{2}$$

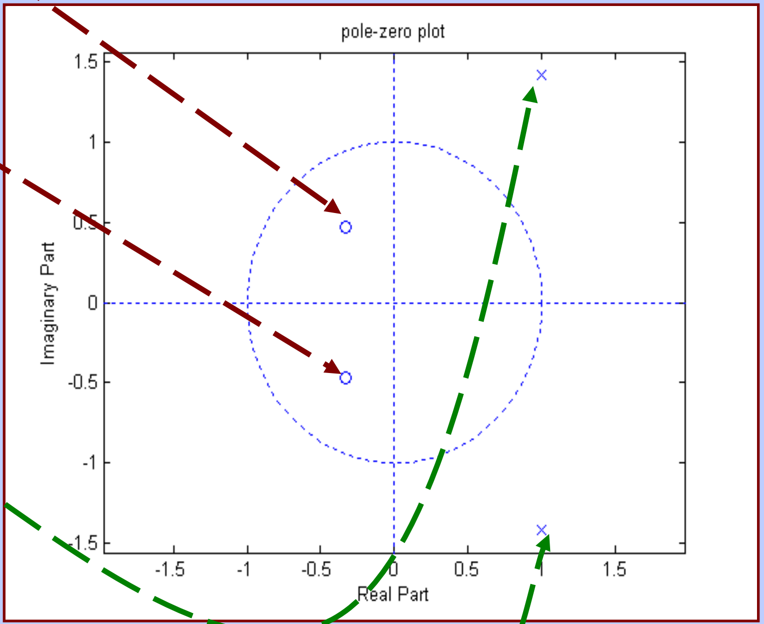


Fig 14.9

Poles locations & Stability

- **All FIR filters are stable**
- **The location of poles plays an important role in the stability of IIR filters due to recursion**

Consider the system function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \quad \textit{the impulse response,}$$

$$h[n] = b_0 a_1^n u[n] + b_1 a_1^{n-1} u[n-1]$$

$$h[n] = \begin{cases} 0 & for\ n < 0 \\ b_0 & for\ n = 0 \\ \left(b_0 + b_1 a_1^{-1}\right) a_1^n & for\ n \geq 1 \end{cases}$$

$Consider\ the\ part\ with\ n \geq 1,\ \left(b_0 + b_1 a_1^{-1}\right) a_1^n$

$let\ \left(b_0 + b_1 a_1^{-1}\right) = k$

$$\lim_{n \to \infty} k a_1^n \to 0 \quad if\ |a_1| < 1$$

$$\lim_{n\to\infty} k a_1^n \to \infty \quad if \ |a_1| \geq 1$$

It is desirable to have the impulse response die down as $n \to \infty$, otherwise an unbound output results even from a few input samples

$|a_1| < 1$ *implies that the pole will be inside the unit circle in z – plane*

Thus a causal LTI IIR system with initial rest conditions is stable only if all the poles lie inside the unit circle

IIR stability: All poles must be inside unit circle

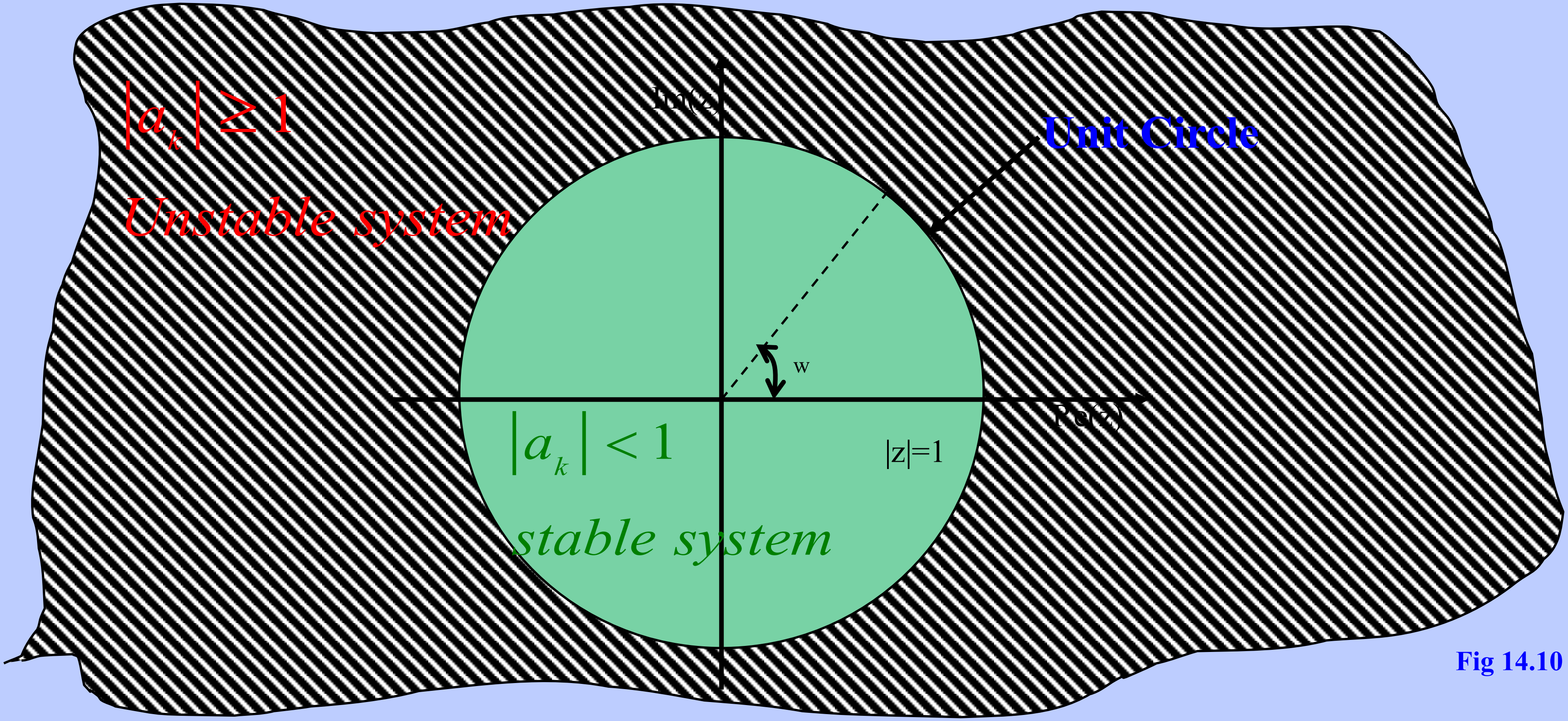


Fig 14.10

Location of Zeros has nothing to do with stability

Example 1

Comment on the stability of the IIR filter with transfer function

$$H(z) = \frac{1 - 2z^{-1}}{1 - 0.8z^{-1}}$$

$$zero \ldots 1 - 2z^{-1} = 0 \Rightarrow z = 2$$

$$pole \ldots 1 - 0.8z^{-1} = 0 \Rightarrow z = 0.8$$

$$|0.8| < 1, \; inside\ unit\ circle.$$

The filter is stable

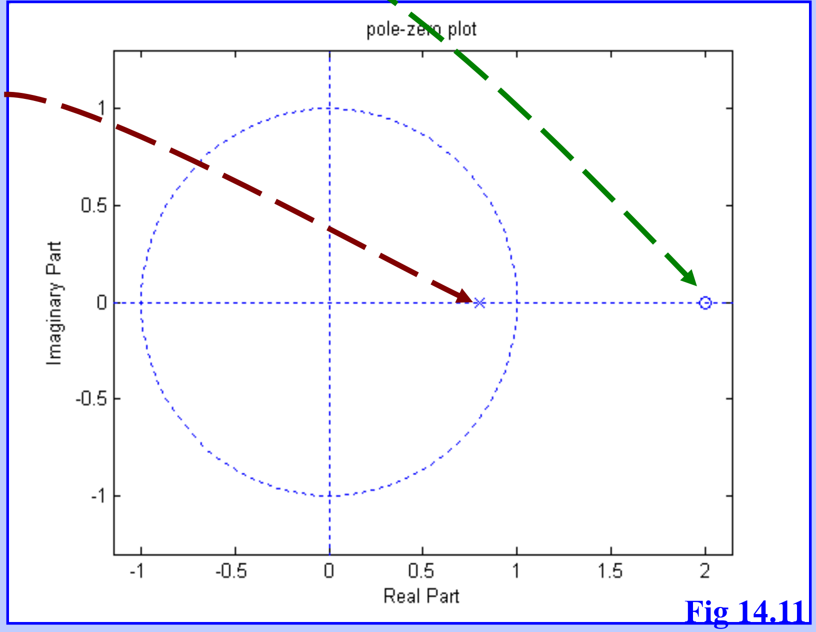


Fig 14.11

Example 2

Comment on the stability of the filter with diferrence equation

$$y[n] = 2y[n-1] + x[n] - 0.8x[n-1]$$

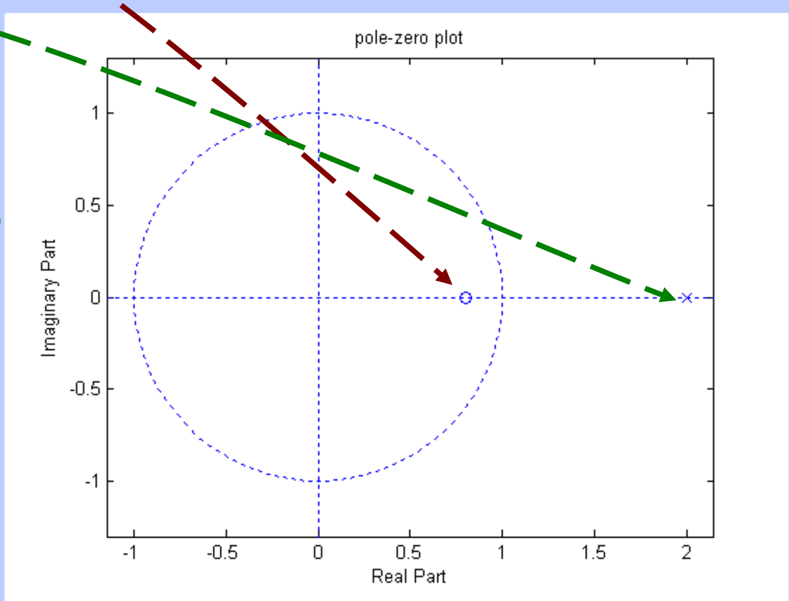
$$H(z) = \frac{1-0.8z^{-1}}{1-2z^{-1}}$$

$$zero \ldots 1-0.8z^{-1} = 0 \Rightarrow z = 0.8$$

$$pole \ldots 1-2z^{-1} = 0 \Rightarrow z = 2$$

$|2| > 1$, *outside unit circle.*

The filter is unstable



Example 3

Comment on the stability of the feedback system

$$y[n] = -y[n-2] + x[n]$$

$$H(z) = \frac{1}{1+z^{-2}}$$

$$= \left(\frac{1}{1+z^{-2}}\right)\frac{z^2}{z^2} = \frac{z^2}{z^2+1}$$

$$zeros...z^2 = 0 \Rightarrow \left[z_1, z_2\right] = [0,0]$$

$$poles...z^2 + 1 = 0$$

$$\Rightarrow z = \pm\sqrt{-1} = \left[p_1, p_2\right] = [-j, j]$$

$$\left|j\right| = 1, \; \textit{on the unit circle.}$$

The filter is unstable

2^{nd} order zero

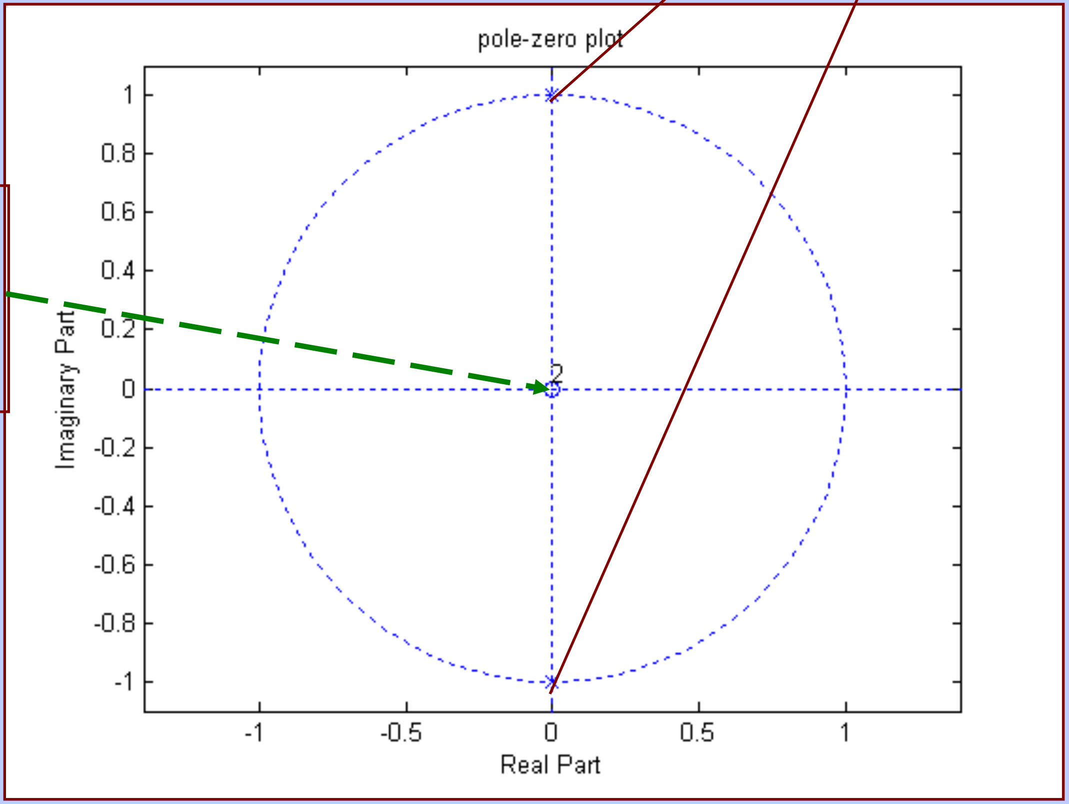


Fig 14.13

Example 4

Comment on the stability of the feedback system

$$y[n] = y[n-1] - 2y[n-2] + x[n] - 2x[n-1] + 3x[n-2]$$

$$H(z) = \frac{1 - 2z^{-1} + 3z^{-2}}{1 - z^{-1} + 2z^{-2}}$$

$$= \left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - z^{-1} + 2z^{-2}} \right) \frac{z^2}{z^2}$$

$$= \frac{z^2 - 2z + 3}{z^2 - z + 2}$$

$$zeros...z^2 - 2z + 3 = 0$$

$$z_1 = \frac{2+\sqrt{4-12}}{2} = 1 + j\sqrt{2}$$

$$z_2 = \frac{2-\sqrt{4-12}}{2} = 1 - j\sqrt{2}$$

$$poles...z^2 - z + 2 = 0$$

$$p_1 = \frac{1+\sqrt{1-8}}{2} = \frac{1}{2} + \frac{j\sqrt{7}}{2}$$

$$p_2 = \frac{1-\sqrt{1-8}}{2} = \frac{1}{2} - \frac{j\sqrt{7}}{2}$$

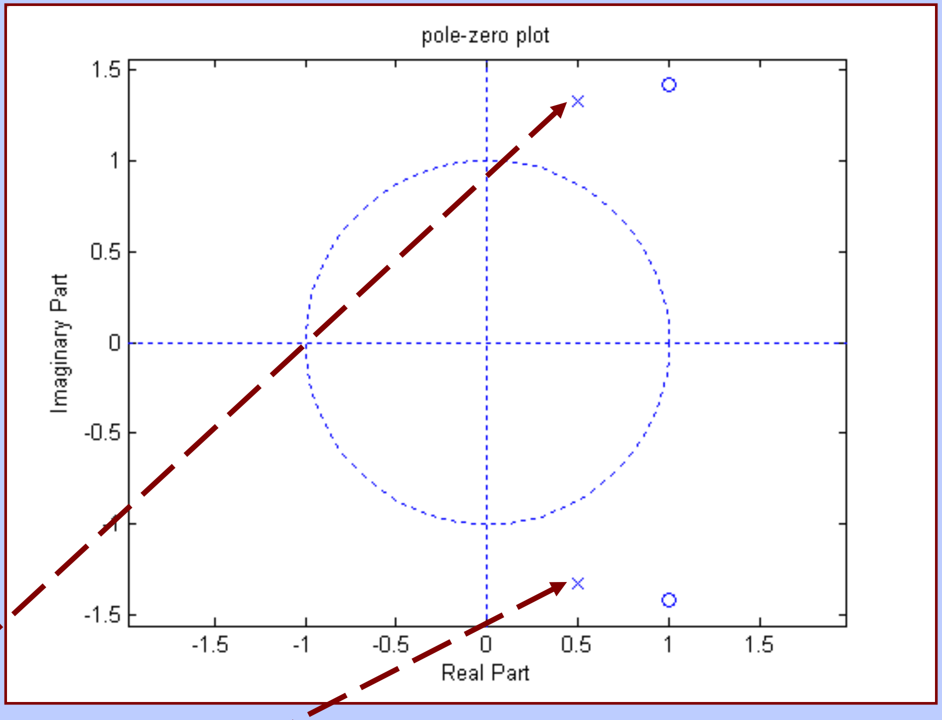


Fig 14.14

Unstable system

Example 5

Comment on the stability of the IIR system

$$4y[n] = y[n-1] - 2y[n-2] + x[n] - 5x[n-1] + 6x[n-2]$$

$$H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{4 - z^{-1} + 2z^{-2}}$$

$$= \left(\frac{1 - 5z^{-1} + 6z^{-2}}{4 - z^{-1} + 2z^{-2}} \right) \frac{z^2}{z^2}$$

$$= \frac{z^2 - 5z + 6}{4z^2 - z + 2}$$

$$Zeros ... z^2 - 5z + 6 = 0, \quad (z-2)(z-3) = 0$$

$$\Rightarrow [z_1, z_2] = [2, 3]$$

$$Poles ... 4z^2 - z + 2 = 0$$

$$p_1 = \frac{1 + \sqrt{1-32}}{8} = \frac{1}{8} + j\frac{\sqrt{31}}{8}$$

$$p_2 = \frac{1 - \sqrt{1-32}}{8} = \frac{1}{8} - j\frac{\sqrt{31}}{8}$$

$$Magnitude\ of\ poles = 0.707$$

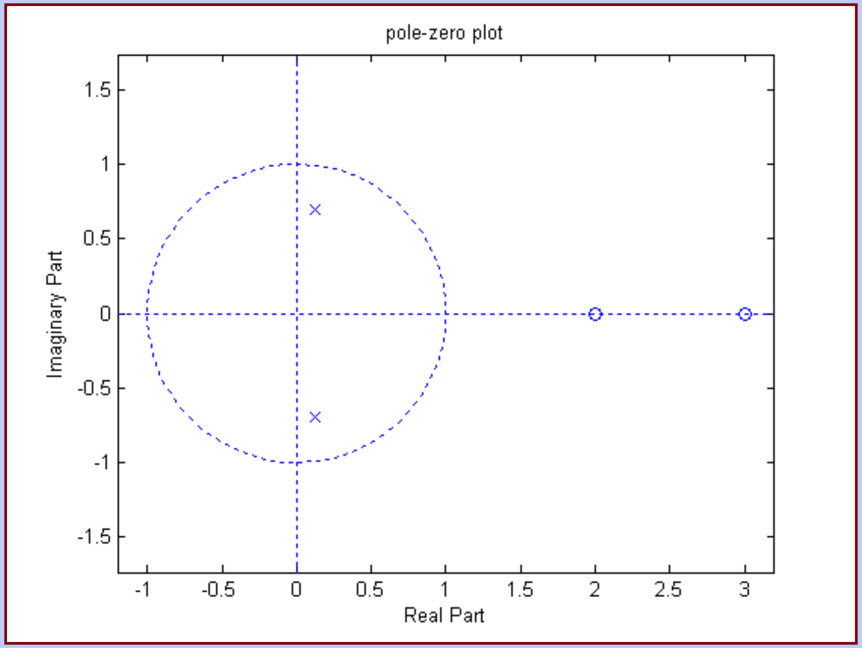


Fig 14.15

Stable system

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 8.3, 8.4, and 8.9 “Signal Processing First”, Prentice Hall, 2003