Discrete - Time Signals and Systems

Z-Transform & IIR Filters 2

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Frequency Response—IIR filters

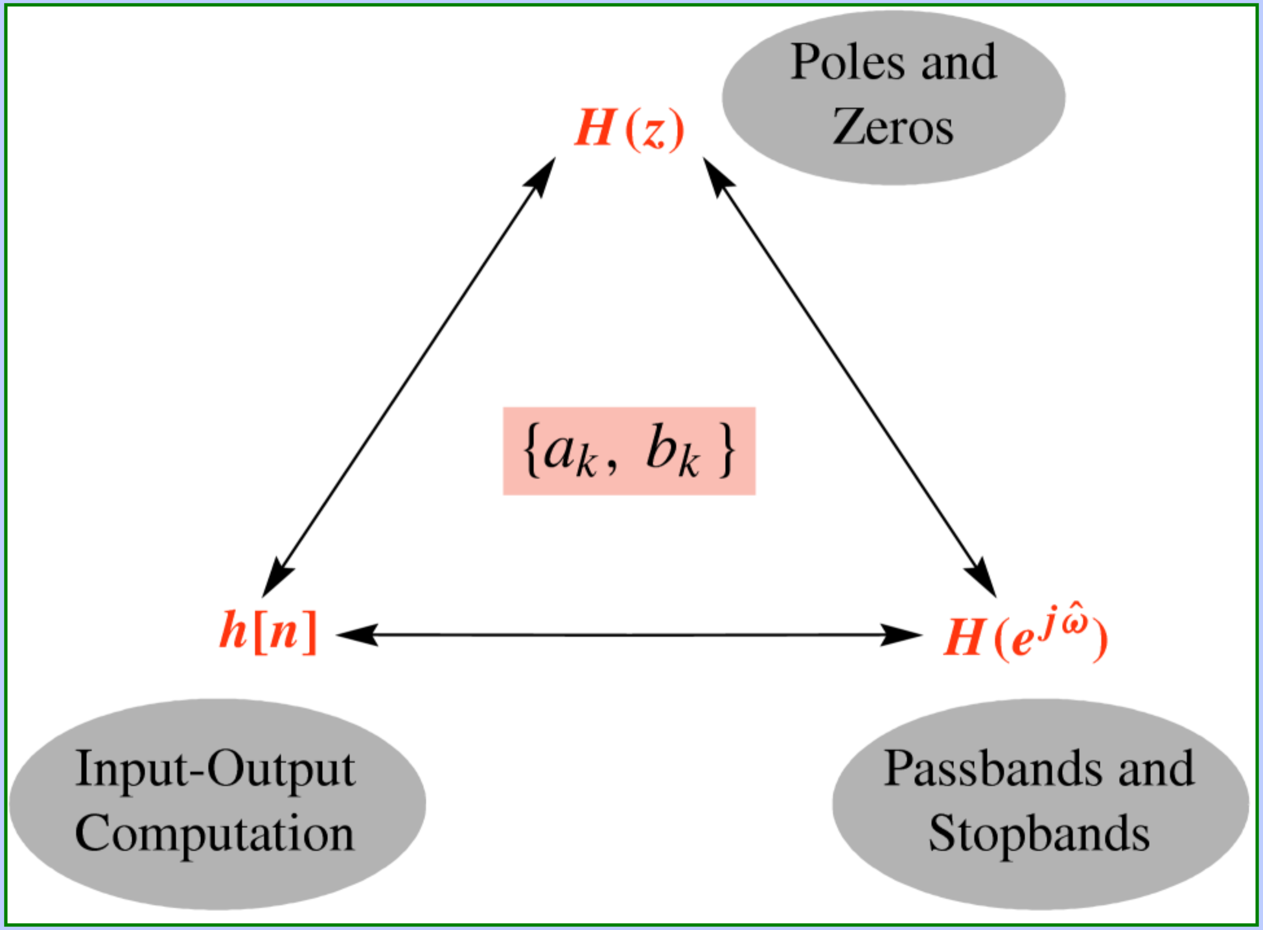
Recall that for an FIR system,

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

i.e., system response evaluated along the unit circle in the z – plane is equl to the frequency response

The relation holds true for IIR filters, the only condition being that the IIR filter should be stable i.e., all poles must be inside the unit circle

IIR Filters: Relation between the three domains



Steady-State Response & Stability

Consider the 1st order IIR system

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0}{1 - a_1 e^{-j\hat{\omega}}}$$

Apply a complex exponential input to the filter

$$x[n] = e^{j\hat{\omega}_0 n} \Rightarrow y[n] = \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}}} \right) e^{j\hat{\omega}_0 n} \qquad -\infty < n < \infty$$

An interesting situation develpos when the input is applied suddenly, there will be two different outputs from an IIR filter; known as 'transient response' and 'steady state response'

$Z-$transform is most useful to analyze this kind of situation

Applying the input suddenly

$x[n]=e^{j\hat{\omega}_0 n}u[n]$, *notice that in previous case input existed in the range* $[-\infty,\infty]$

$$Y(z) = X(z)H(z)$$

$$e^{j\hat{\omega}_0 n}u[n] \stackrel{n}{\longleftrightarrow} \frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$Y(z) = \left(\frac{b_0}{1 - a_1 z^{-1}} \right) \left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}} \right)$$

$$= \frac{b_0}{\left(1 - a_1 z^{-1}\right)\left(1 - e^{j\hat{\omega}_0} z^{-1}\right)}$$

$$Y(z)=\frac{A}{\left(1-a_1 z^{-1}\right)}+\frac{B}{\left(1-e^{j\hat{\omega}_0} z^{-1}\right)}$$

Using the partial fraction

$$Y(z)=\frac{\left(\dfrac{b_0 a_1}{a_1-e^{j\hat{\omega}_0}}\right)}{1-a_1 z^{-1}}+\frac{\left(\dfrac{b_0}{1-a_1 e^{-j\hat{\omega}_0}}\right)}{\left(1-e^{j\hat{\omega}_0} z^{-1}\right)}$$

$$y[n]=\underbrace{\left(\frac{b_0 a_1}{a_1-e^{j\hat{\omega}_0}}\right)(a_1)^n u[n]}_{\text{Transient Component}}+\underbrace{\left(\frac{b_0}{1-a_1 e^{-j\hat{\omega}_0}}\right)e^{j\hat{\omega}_0 n}u[n]}_{\text{Steady State component}}$$

'Transient Component' only depends on the pole location

'Steady State component' depends on the applied input signal

It is desirable to have the pole inside the unit circle otherwise the transient component dominates the output

so stability concept is extremely important for IIRfilter frequency response

Example: Transient & Steady State

Consider the 1st order IIR system

$$y[n] = -0.8\,y[n-1] + 5x[n]$$

$$b_0 = 5,$$

$$a_1 = -0.8$$

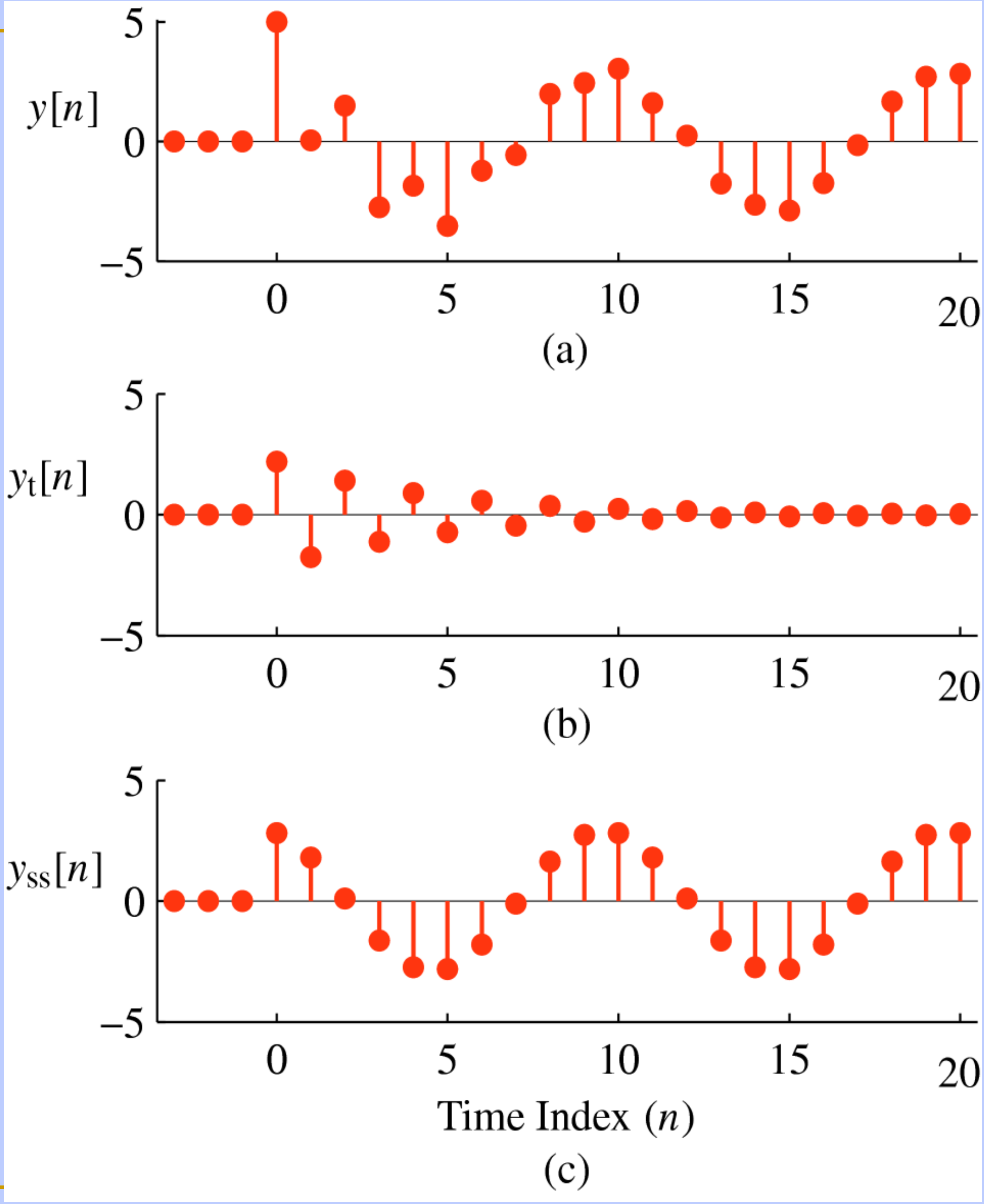
Let the input be, $x[n] = e^{j(2\pi/10)n}$

$$y[n] = \underbrace{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n]}_{Transient\ Component} + \underbrace{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]}_{Steady\ State\ component}$$

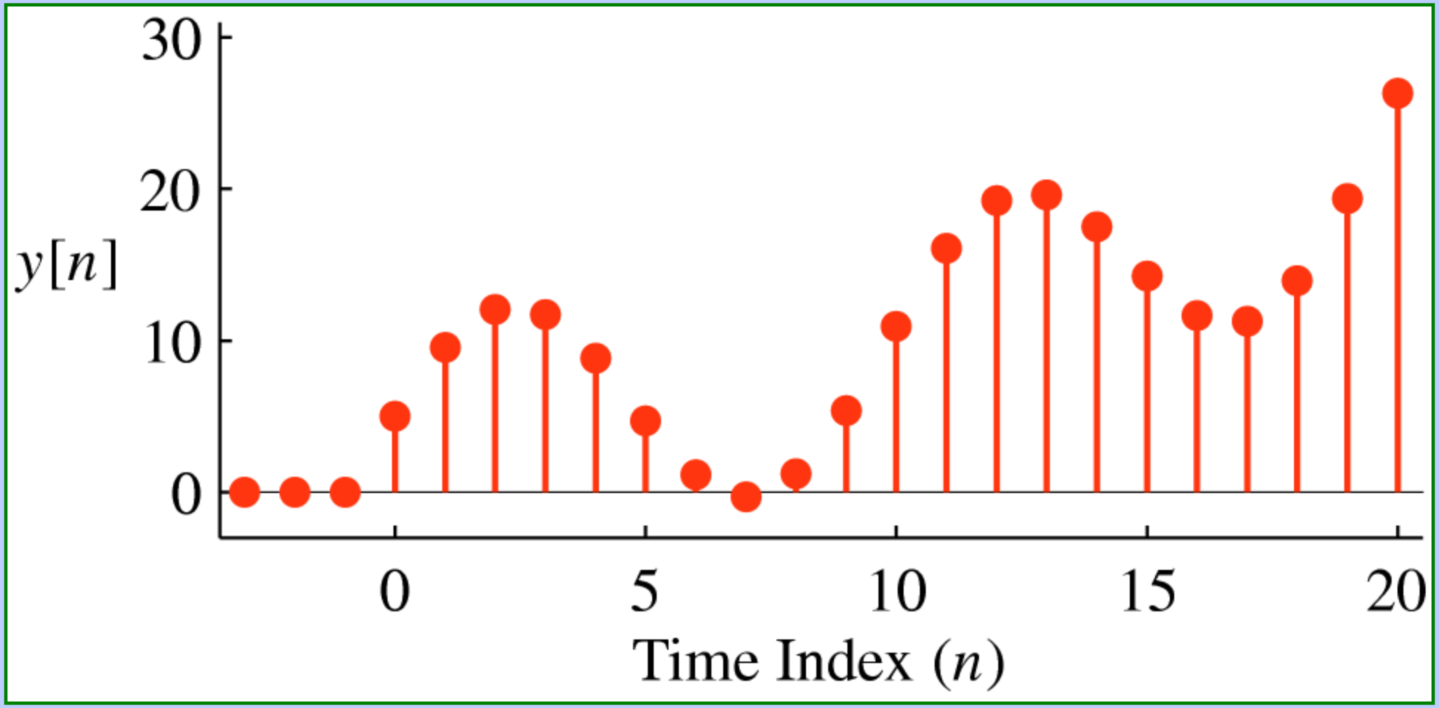
$$y_t[n] = \left(\frac{-4}{-0.8 - e^{j0.2\pi}} \right) (-0.8)^n \, u[n]$$

The transient component is decaying as the pole is inside the unit circle

$$y_{ss}[n] = \left(\frac{5}{1 + 0.8e^{-j0.2\pi}} \right) e^{j0.2\pi n} u[n]$$



Plot of output alone with pole at $a_1 = 1.1$, *the output is totally dominated by transient component, it is an unstable system*



Calculation of frequency response

Consider the 1st order IIR system

$$H(z) = \frac{(b_0 + b_1 z^{-1})}{(1 - a_1 Z^{-1})}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{(b_0 + b_1 e^{-j\hat{\omega}})}{(1 - a_1 e^{-j\hat{\omega}})}$$

Magnitude square $\left|H(e^{j\hat{\omega}})\right|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})$

$$= \frac{(b_0 + b_1 e^{-j\hat{\omega}})}{(1 - a_1 e^{-j\hat{\omega}})}\left(\frac{(b_0 + b_1 e^{-j\hat{\omega}})}{(1 - a_1 e^{-j\hat{\omega}})}\right)^*$$

$$= \frac{(b_0 + b_1 e^{-j\hat{\omega}})}{(1 - a_1 e^{-j\hat{\omega}})}\left(\frac{(b_0^* + b_1^* e^{+j\hat{\omega}})}{(1 - a_1^* e^{+j\hat{\omega}})}\right)$$

$$= \frac{(|b_0|^2 + |b_1|^2 + b_0 b_1^* e^{+j\hat{\omega}} + b_0^* b_1 e^{-j\hat{\omega}})}{(1 + |a_1|^2 - a_1^* e^{+j\hat{\omega}} - a_1 e^{-j\hat{\omega}})}$$

$$|H(e^{j\hat{\omega}})|^2 = \frac{|b_0|^2 + |b_1|^2 + 2\Re e\left\{b_0^* b_1 e^{-j\hat{\omega}}\right\}}{1 + |a_1|^2 - 2\Re e\left\{a_1 e^{-j\hat{\omega}}\right\}}$$

The phase response is given by,

$$\phi(\hat{\omega}) = \tan^{-1}\left(\frac{-b_1 \sin\hat{\omega}}{b_0 + b_1 \cos\hat{\omega}}\right) - \tan^{-1}\left(\frac{-a_1 \sin\hat{\omega}}{1 - a_1 \cos\hat{\omega}}\right)$$

Obviously these equations don't tell much about the nature of frequency response

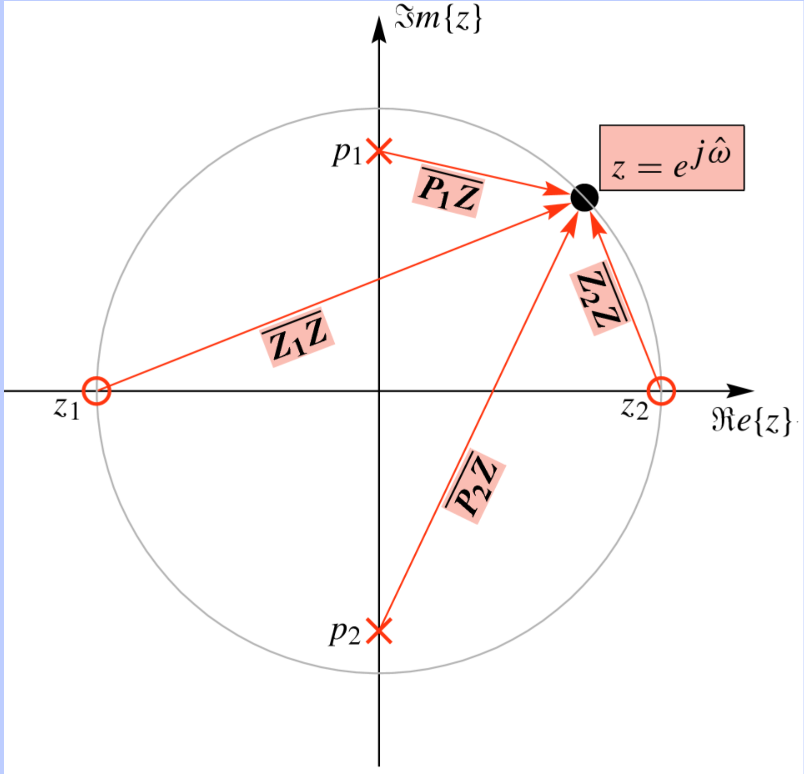
MATLAB command 'freqz' gives a good visualization

Frequency Response: Graphical Evaluation

Given a pole-zero plot, frequency response can be evaluated graphically

$$\left| H(e^{j\hat{\omega}}) \right| = G \frac{\overline{Z_1 Z} \, . \, \overline{Z_2 Z}}{\overline{P_1 Z} \, . \, \overline{P_2 Z}}$$

Phase is the difference between sum of angles of zeros and sum of angles of poles

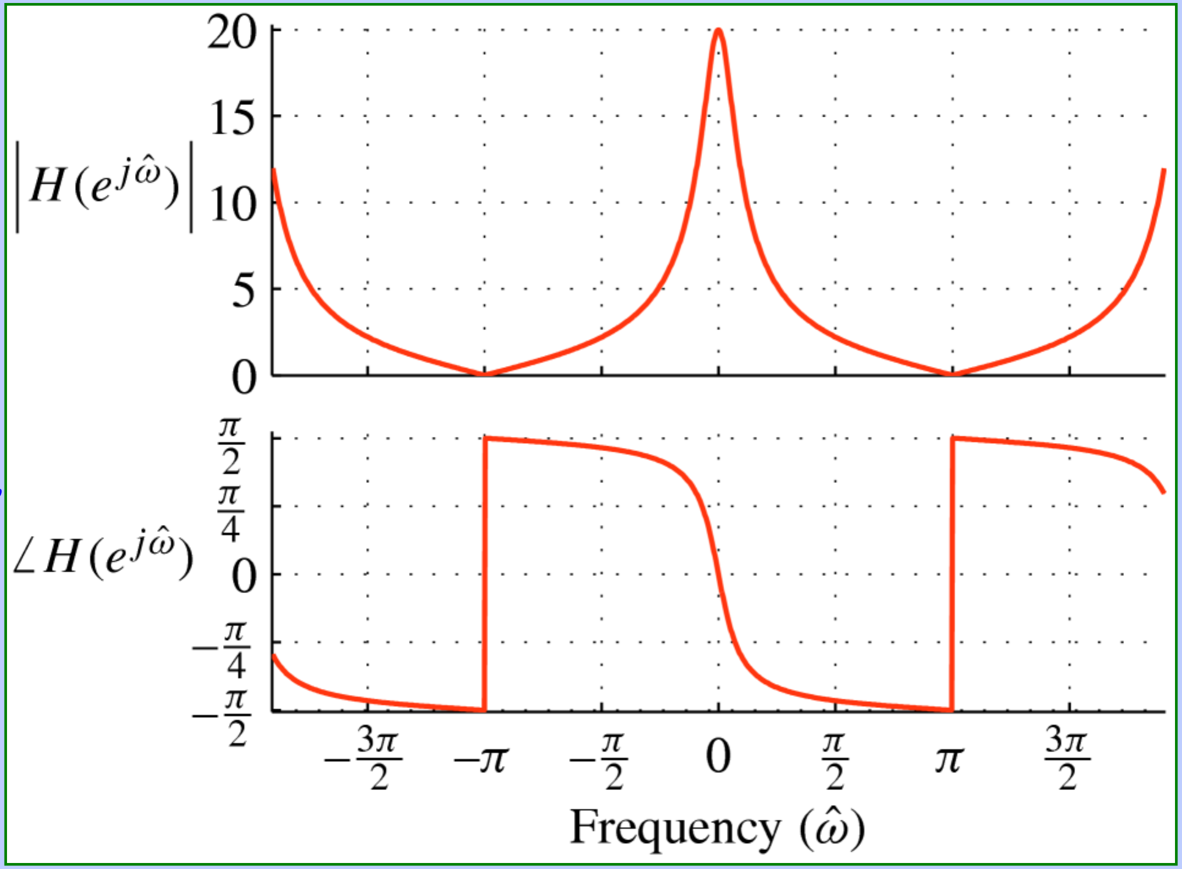


Example 1: Frequency Response

Consider the 1st order IIR system

$$y[n] = 0.8y[n-1] + 2x[n] + 2x[n-1]$$

$$H(z) = \frac{(2 + 2z^{-1})}{(1 - 0.8Z^{-1})}$$



for MATLAB

$b = [2, 2]$, *numerator*

$a = [1, -0.8]$,

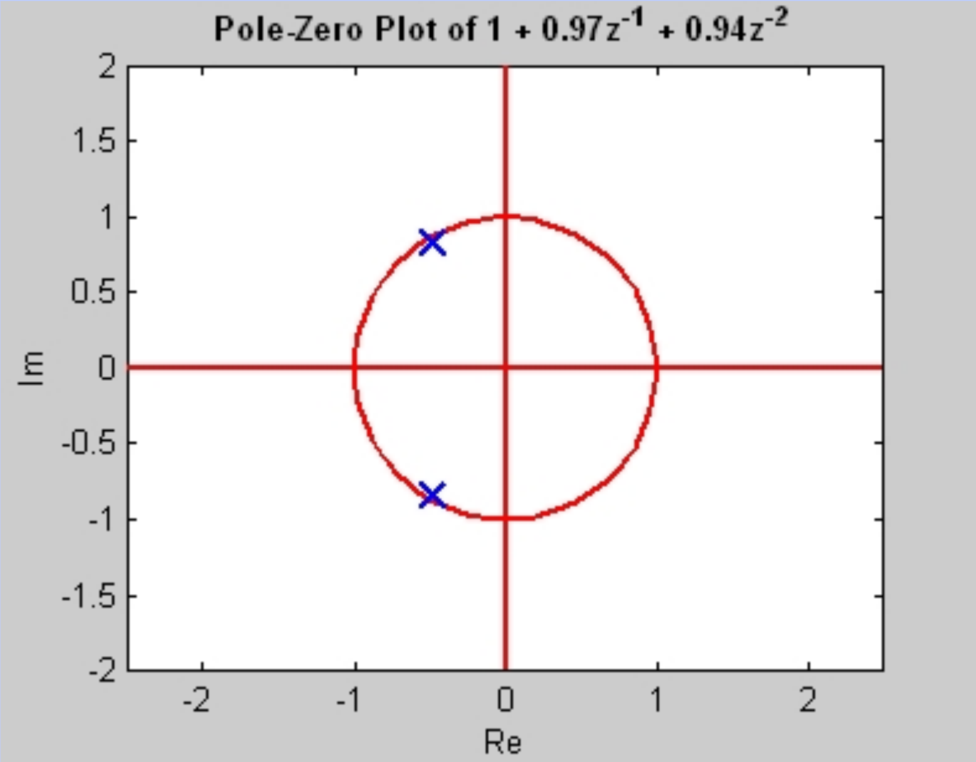
denominator

$freqz(b, a)$

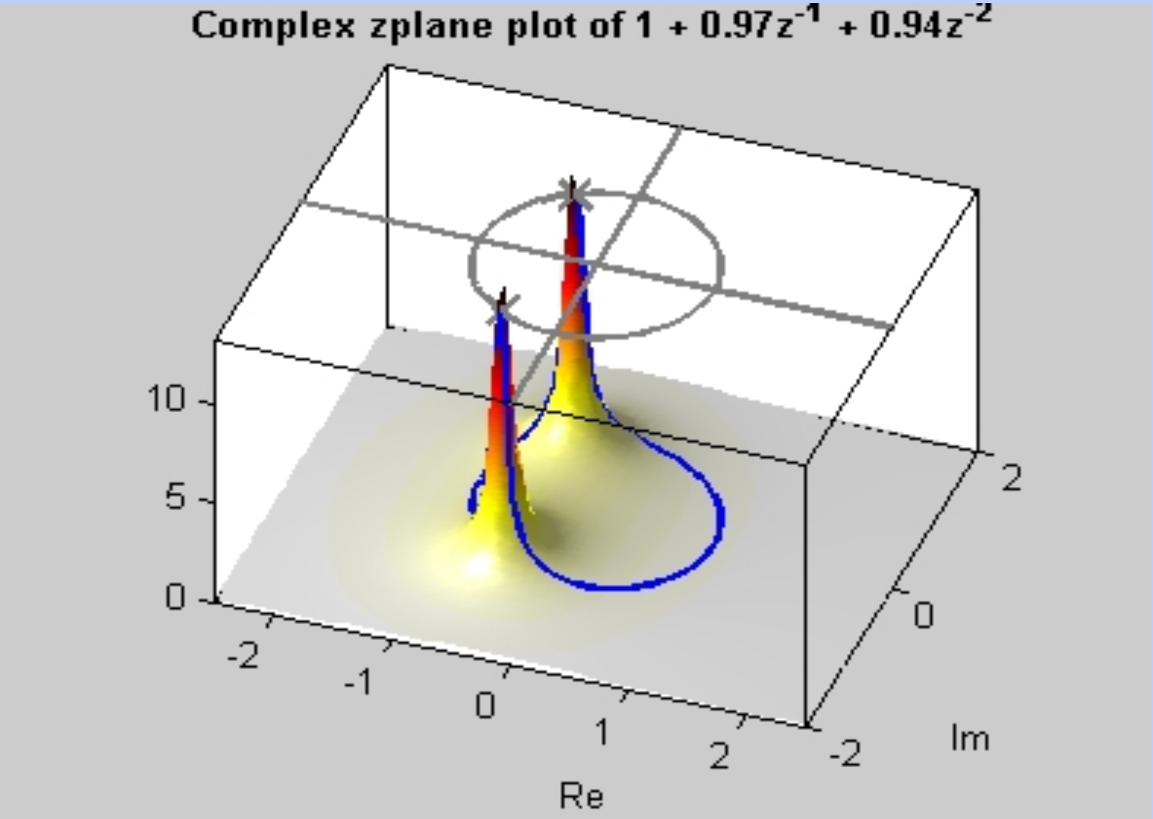
Example 2

Consider the feedback system with transfer function

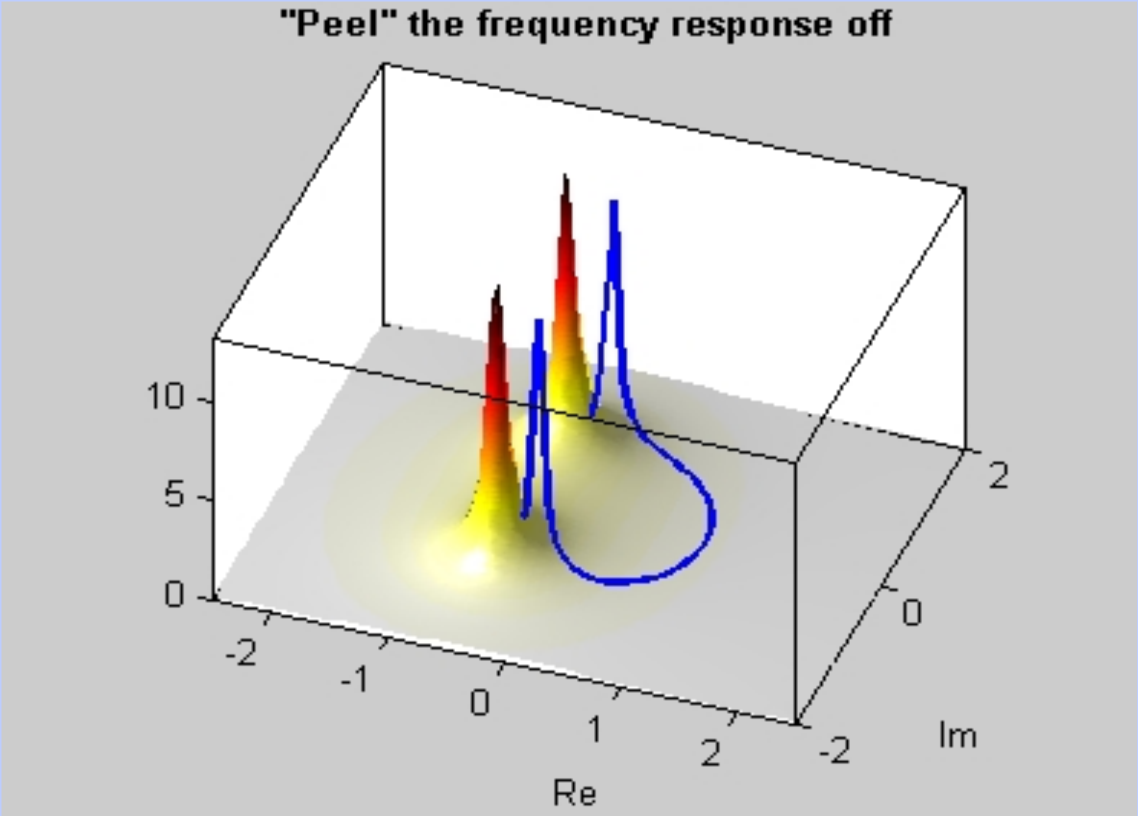
$$H(z) = \frac{1}{1 + 0.97z^{-1} + 0.94z^{-2}}$$



Frequency response plot of the system

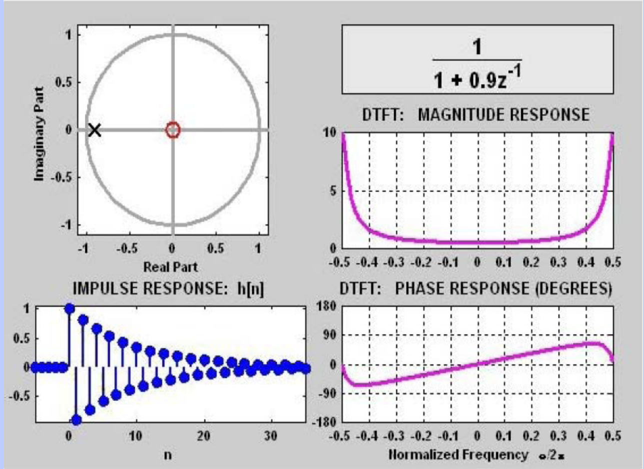


Frequency response continued…



Example 3: IIR filter with one pole and a zero at z = 0

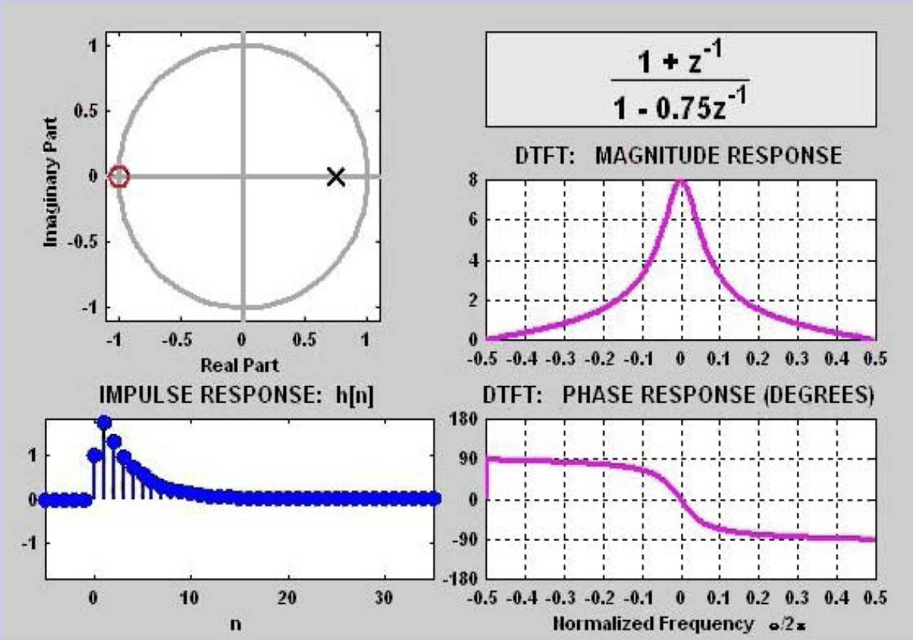
The pole is moved from left to right, starting at z = -0.9 and ending at z = +0.9. Notice that the impulse response decays very little when the pole is near the unit circle. The frequency response also changes from a highpass filter when the pole is near z = -1 to t a lowpass filter when the pole approaches z = +1



Example 4: IIR filter with one pole and one zero

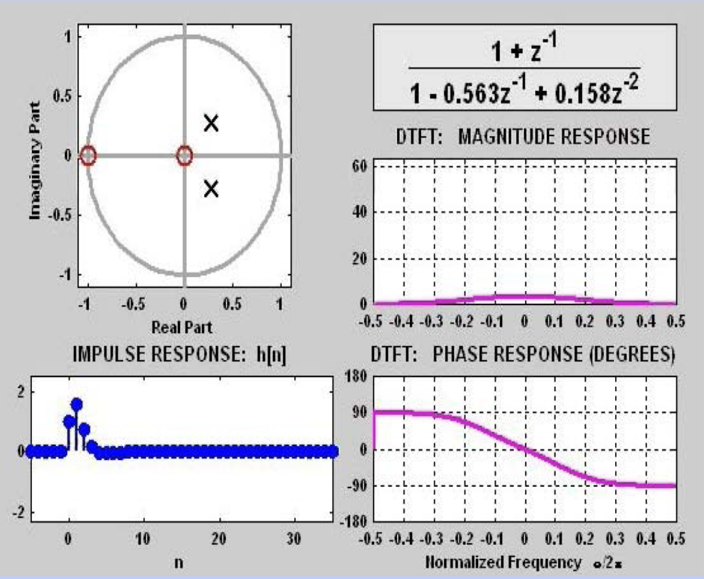
First the zero is moved from left to right, starting at z = -1 and ending at z = +1; then the pole is moved from right to left, starting at z = +0.75 and ending a z = -0.75.

Notice that the frequency response changes from a lowpass filter to a notchfilter and then to a highpass filter.



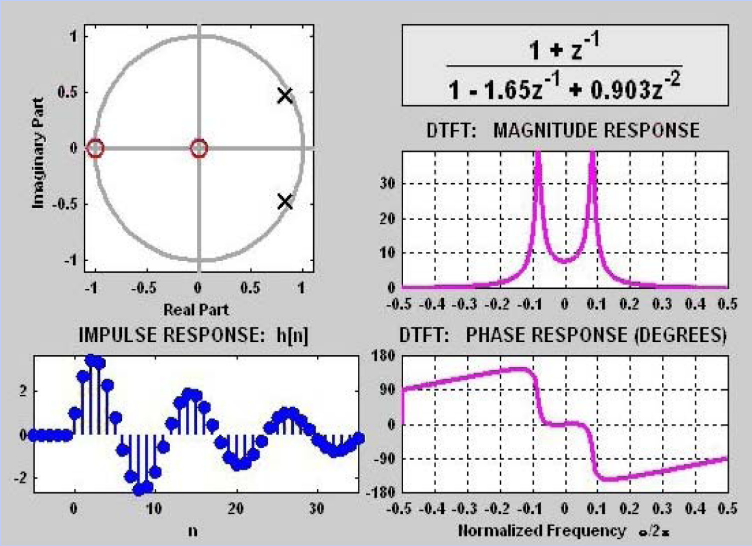
Example 5: Movement of Poles

IIR filter with two poles and a zero at z = -1. The pole pair is complex, at an angle of 45 degrees. During the movie, the radius of the pole pair is varied to illustrate the effect of moving the poles very close to the unit circle. Notice that the decay rate of the impulse response changes, but the oscillation period remains the same. The frequency response becomes a very sharp bandpass filter as the poles approach the unit circle.



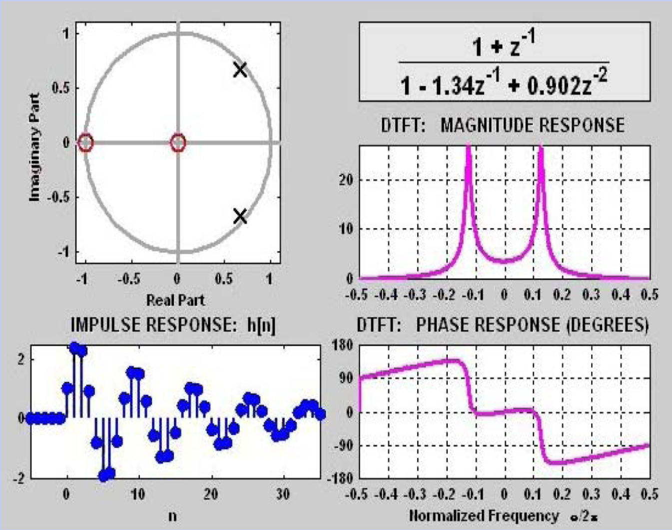
Example 6: Angular Movement of Poles

IIR filter with two poles and a zero at z = -1. The pole pair is complex, at a radius of |z| = 0.95. During the movie the angle of the pole pair is varied, to illustrate the effect of moving the poles around the unit circle. Notice that the decay rate of the impulse response stays the same, but the oscillation period changes. The frequency response is a very sharp bandpass filter whose center frequency changes as the poles move in angle



Example 7: Movement of Zeros in an IIR Filter

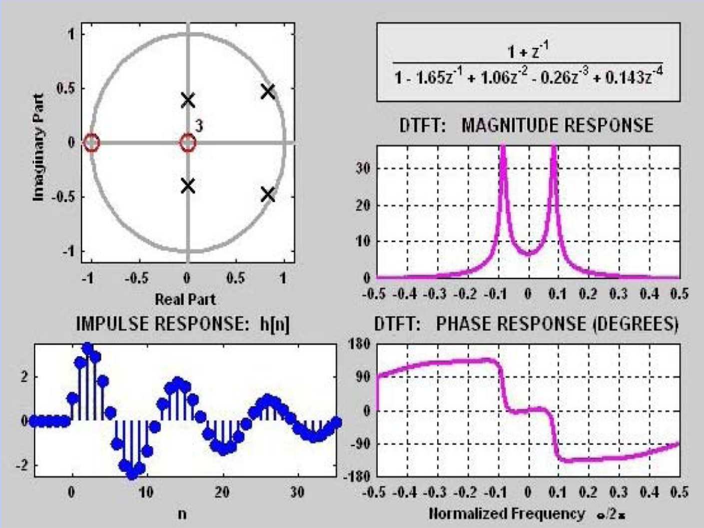
IIR filter with two poles and a zero at z = -1. The zero is moved from z = -1 to z = +1 during the movie. The pole pair is complex, at an angle of 45 degrees and radius 0.95. Notice that the decay rate and oscillation rate of the impulse response remain the same. The zero should have no effect on the "tail" of h[n], but the second sample in h[n] does change, which sets up an initial condition for the rest of h[n] Likewise, the frequency response is not very sensitive to changes in the zero location; its center frequency and bandwidth remain more or less the same, although the peak height does vary noticeably



Example 8: Radial Movement of Two out of Four Poles

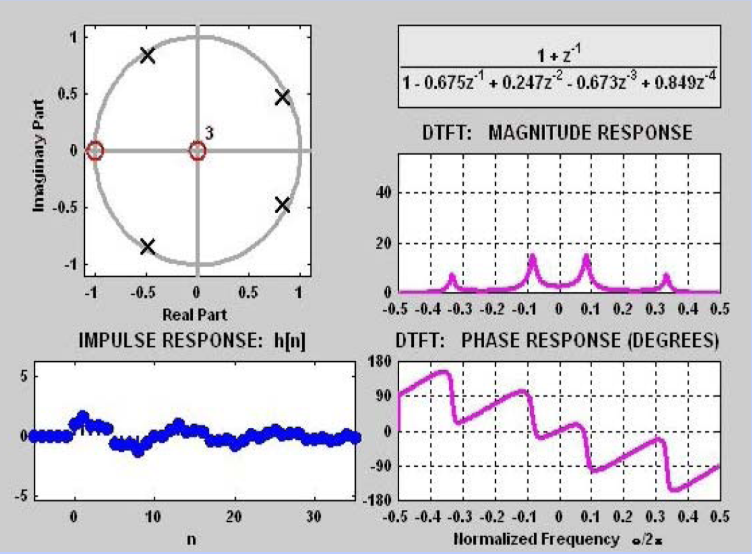
IIR filter with four poles and a zero at z = -1. One complex pole pair is fixed, at a radius of 0.95 and an angle of 30 degrees. The other pole pair has a fixed angle of 90 degrees, but during the movie the radius of this pole pair is varied, to study the interaction of the poles and the resulting impact on the frequency response.

The fixed pole pair causes a peak in the frequency response, but that peak does not change significantly as the other pole pair is varied.



Example 9: Angular Movement of Two out of Four Poles

IIR filter with four poles and a zero at z = -1. One complex pole pair is fixed, at a radius of 0.95 and an angle of 30 degrees. The other pole pair has a fixed radius of 0.9, but during the movie the angle of this pole pair is varied, to illustrate the effect of moving the poles close to one another.



Notice the interaction of the poles and the resulting impact on the frequency response.

The fixed pole pair causes a peak in the frequency response, and that peak does not change significantly until the other pole pair gets relatively close

Bandwidth: Definition

3-dB width in the frequency domain is a popular definition for bandwidth

$$H(z) = \frac{1 - z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

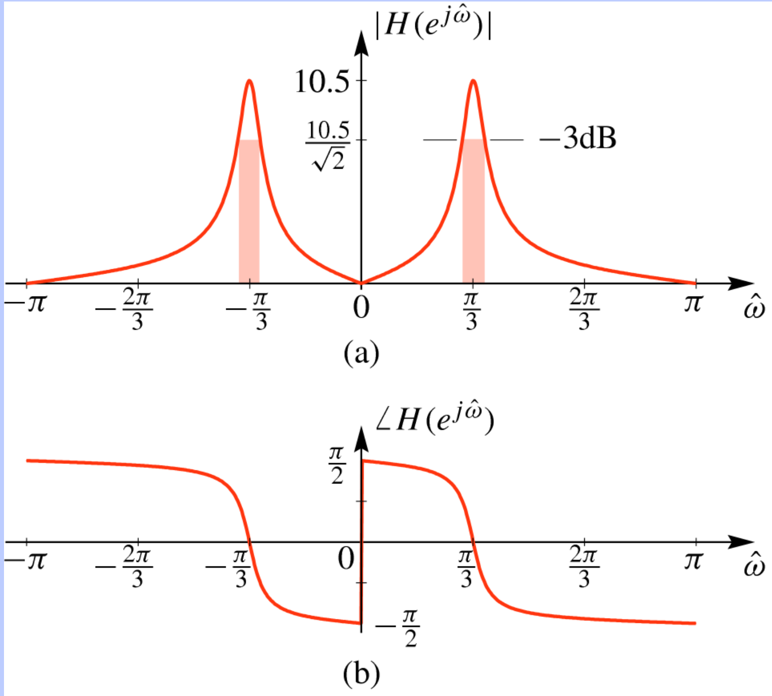
$$zeros..[z_1, z_2] = [1, -1]$$

$$Poles...[p_1, p_2]$$

$$= [0.9e^{j\pi/3}, 0.9e^{-j\pi/3}]$$

$$B.W = 2\pi(0.0335)$$

$$= 0.2105\ rad$$



Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 8.5, and 8.8-8.10 “Signal Processing First”, Prentice Hall, 2003