Discrete - Time Signals and Systems

Continuous-Time signals & systems
Impulse Response

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Simple Real Continuous-Time System

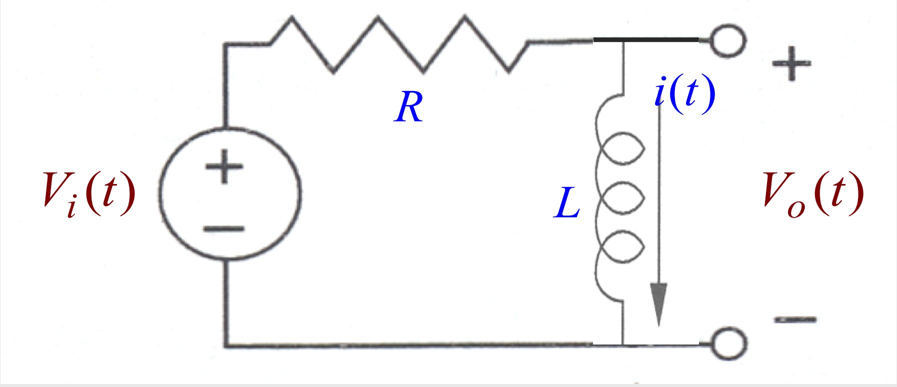


Fig 17.1

$$V_i(t) = Ri(t) + L\frac{di}{dt} \qquad (1)$$

$$V_o(t) = L\frac{di}{dt} \qquad (2)$$

$$\frac{di}{dt} = \frac{V_o(t)}{L} \qquad (3)$$

Substitute (3) *in* (1)

$$V_i(t) - Ri(t) - V_o(t) = 0; \quad KVL \quad theorem$$

General Continuous-Time Systems

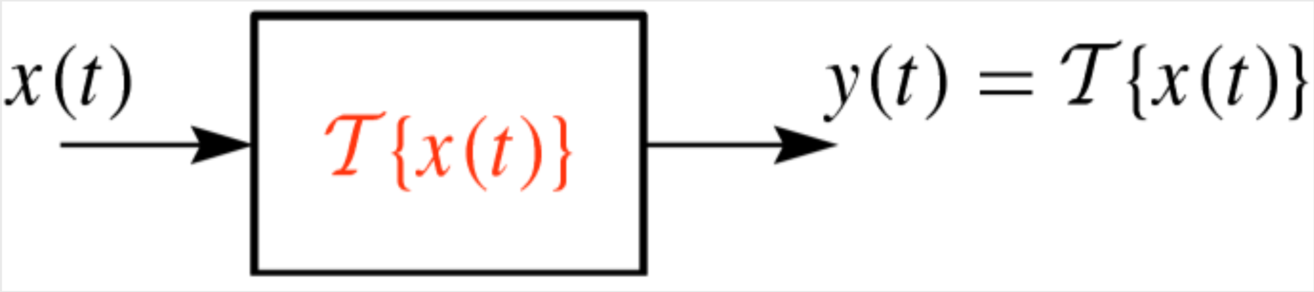


Fig 17.2

$$x(t) \xrightarrow{\tau} y(t)$$

a) Linear System Property

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{\tau} \alpha y_1(t) + \beta y_2(t)$$

b) Time - Invariant System Property

$$x(t - t_0) \xrightarrow{\tau} y(t - t_0)$$

Filters satisfying both (a) and (b) are known as Linear Time Invariant systems, these are an important class of filters

Example 1

$y(t) = \left[x(t)\right]^2$

Linear System Property

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{\tau} \alpha y_1(t) + \beta y_2(t)$$

$x(t) = \alpha x_1(t) + \beta x_2(t)$

$y(t) = \left[\alpha x_1(t) + \beta x_2(t)\right]^2$

$\quad = \alpha^2 \left[x_1(t)\right]^2 + \beta^2 \left[x_2(t)\right]^2 + 2\alpha\beta x_1(t) x_2(t)$

$\alpha y_1(t) + \beta y_2(t) = \alpha \left[x_1(t)\right]^2 + \beta \left[x_2(t)\right]^2$

Thus, $y(t) \neq \alpha y_1(t) + \beta y_2(t)$; *The system is nonlinear*

If the input is $x(t - t_0)$, *then output* $w(t) = \left[x(t - t_0)\right]^2$

$w(t) = y(t - t_0)$; *Time invariant*

Example 2

$$y(t) = \frac{dx(t)}{dt}$$

Linear System Property

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{\tau} \alpha y_1(t) + \beta y_2(t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = \frac{d}{dt}\left[\alpha x_1(t) + \beta x_2(t)\right]$$

$$= \frac{d}{dt}\alpha x_1(t) + \frac{d}{dt}\beta x_2(t) = \alpha \frac{d}{dt} x_1(t) + \beta \frac{d}{dt} x_2(t)$$

$$\alpha y_1(t) + \beta y_2(t) = \alpha \frac{dx_1(t)}{dt} + \beta \frac{dx_2(t)}{dt}$$

Thus, $y(t) = \alpha y_1(t) + \beta y_2(t);$ *The system is linear*

Example 2 *contd…*

$$y(t) = \frac{dx(t)}{dt}$$

$b)$ *Time - Invariant System Property*

$$x(t - t_0) \xrightarrow{\tau} y(t - t_0)$$

$$x(t) = x(t - t_0)$$

$$w(t) = \frac{d}{dt} x(t - t_0)$$

$$y(t - t_0) = \frac{d}{dt} x(t - t_0)$$

$$w(t) = y(t - t_0)$$

Thus the system is Time – Invariant

This system belongs to LTI (Linear & Time – Invariant) type of systems

Example 3

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Linear System Property

$$\alpha x_1(t) + \beta x_2(t) \xrightarrow{F} \alpha y_1(t) + \beta y_2(t)$$

$$x(t) = \alpha x_1(t) + \beta x_2(t)$$

$$y(t) = \int_{-\infty}^{t} \left[\alpha x_1(\tau) + \beta x_2(\tau)\right] d\tau$$

$$= \int_{-\infty}^{t} \alpha x_1(\tau)d\tau + \int_{-\infty}^{t} \beta x_2(\tau)d\tau$$

$$\alpha y_1(t) + \beta y_2(t) = \alpha \int_{-\infty}^{t} x_1(\tau)d\tau + \beta \int_{-\infty}^{t} x_2(\tau)d\tau$$

Thus, $y(t) = \alpha y_1(t) + \beta y_2(t)$; *The system is linear*

Example 3 *contd…*

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

b) Time - Invariant System Property

$$x(t - t_0) \xrightarrow{F} y(t - t_0)$$

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau$$

let, $\tau - t_0 = \sigma; \quad d\tau = d\sigma,$ *The limits have to be changed*

lower limit $'-\infty - t_0 = -\infty'$; *upper limit* $'t - t_0'$

$$w(t) = \int_{-\infty}^{t-t_0} x(\sigma)d\sigma = y(t - t_0)$$

Thus the system is Time – Invariant; LTI system

Example 4: Time-Varying systems

$y(t) = x(t)\cos(\omega_c t),\ \ amplitude\ modulator\ system$

$Time\text{-}Invariant\ \ System\ Property$

$x(t-t_0) \xrightarrow{\text{F}} y(t-t_0)$

$If\ \ the\ input\ is\ 'x(t-t_0)';$

$amplitude\ modulator\ results\ in\ x(t-t_0)\cos(\omega_c t)$

$\because\ y(t-t_0) \neq\ x(t-t_0)\cos(\omega_c t)$

$the\ system\ is\ a\ time\ varying\ system$

$Also\ this\ system\ is\ not\ a\ linear\ one,\ verify$

Unit Impulse Response

Fig 17.3

If the input x(t) is $\delta(t)$ the output y(t) is known as impulse response $h(t)$

Just like the discrete-time system, continuous LTI system can be characterized by its impulse response.

Unit Impulse Response *contd…*

Linear
Time - Invariant
System (LTI)

Discrete-time systems are described by difference equations. Continuous-time systems are described by differential equations

Consider a continuous − time system described by,

$$\frac{d^2 y(t)}{dt} + 2\frac{dy(t)}{dt} + y(t) = x(t)$$

$$y(0^-) = \dot{y}(0^-) = 0$$

$x(t) = \delta(t)$, *solving the equation leads to*; $h(t) = te^{-t}u(t)$

Unit Impulse Response - Stability & Causality

$a)$ *A continuous-time LTI system is stable if and only if*

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Example:

$$h(t) = u(t+10) - u(t-10) = \begin{cases} 1 & -10 \le t < 10 \\ 0 & otherwise \end{cases}$$

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-10}^{10} 1 dt = 20$$

$\therefore h(t)$ *is stable*

$b)$ *A continuous-time LTI system is causal if and only if*

$$h(t) = 0 \text{ for } t < 0$$

Example

$a)\ Find\ the\ impulse\ response\ of\ an\ integrator$

$$y(t)=\int_{-\infty}^{t} x(\tau)d\tau$$

$$x(t)=\delta(t), \quad y(t)=h(t)$$

$$h(t)=\int_{-\infty}^{t} \delta(\tau)d\tau=u(t),\ recall\ the\ previous\ result$$

$Thus\ unit\ step\ function\ is\ the\ impulse\ response\ of\ integrator$

$b)\ Find\ the\ impulse\ response\ of\ a\ differentiator;\ y(t)=\dfrac{dx(t)}{dt}$

$$x(t)=\delta(t), \quad y(t)=h(t)$$

$$h(t)=\frac{d\delta(t)}{dt}=\delta'(t)$$

$\delta'(t)\ is\ a\ mathematical\ function\ known\ as\ a\ 'doublet'$

Unit Impulse Response & Unit Step Response

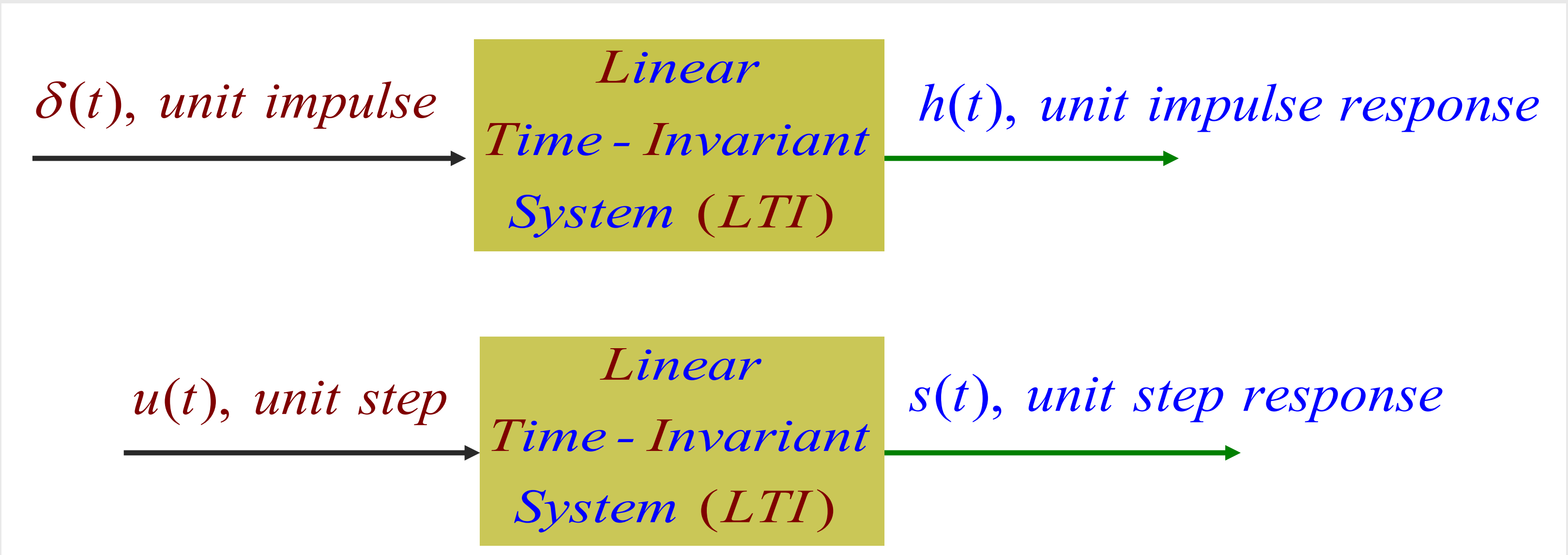


Fig 17.4

$$\because \delta(t) = \frac{du(t)}{dt}$$

$$follows\ that;\ h(t) = \frac{ds(t)}{dt}$$

Example

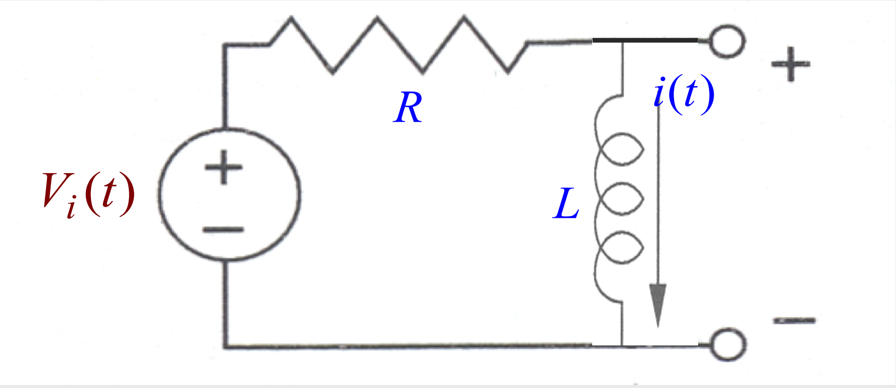


Fig 17.5

$$V_i(t) = Ri(t) + L\frac{di}{dt}$$

Unit step response (solve the differential equation);

$$s(t) = \frac{1}{R}\left(1 - e^{-Rt/L}\right)u(t)$$

$$\because h(t) = \frac{ds(t)}{dt}$$

$$h(t) = \frac{1}{L}\left(e^{-Rt/L}\right)u(t)$$

LTI system output - Convolution integral

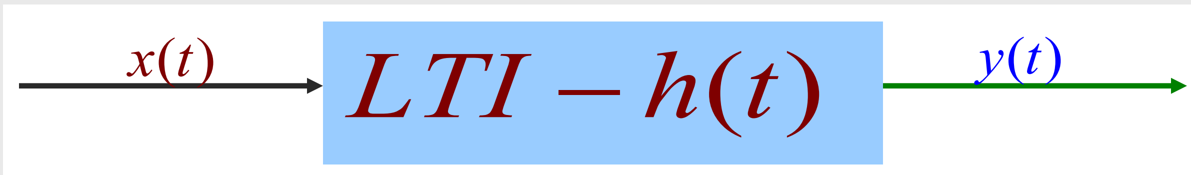


Fig 17.6

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

The output of every LTI system can be described by the above convolutional integral

$y(t) = x(t) * h(t)$; *standard notation*

if $x(t) = \delta(t)$;

$$y(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = \int_{0^-}^{0^+} \delta(\tau)h(t-\tau)d\tau = h(t)$$

Examples

$$a)\ x(t) = \delta(t - t_0)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau - t_0)h(t-\tau)d\tau$$

$$= \int_{t_0^-}^{t_0^+} \delta(\tau - t_0)h(t-\tau)d\tau = h(t - t_0)$$

$$b)\ x(t) = u(t), \quad h(t) = \delta(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{t^-}^{t^+} u(\tau)\delta(t-\tau)d\tau = u(t)$$

Examples

$c)\ h(t) = \delta(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$= \int_{t^-}^{t^+} x(\tau)\delta(t-\tau)d\tau = x(t) = x(t) * \delta(t)$$

$d)\ x(t) = u(t), \quad h(t) = u(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$$

$$= \int_{0}^{\infty} u(t-\tau)d\tau = \int_{0}^{t} 1\, d\tau = \begin{vmatrix} t & t \geq 0 \\ 0 & t < 0 \end{vmatrix}$$

$\because u(t-\tau) = 1 \ \ for\ \ t-\tau \geq 0 \ \ or \ \ \tau \leq t$

Example

$$e)\ x(t) = tu(t); \quad h(t) = e^{-t}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \tau u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_{-\infty}^{\infty} \tau e^{-(t-\tau)}u(\tau)u(t-\tau)d\tau$$

$$\left(u(t) = \begin{vmatrix} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{vmatrix}\right) \cap \left(u(t-\tau) = \begin{vmatrix} 1 & \tau \leq t \\ 0 & \tau > t \end{vmatrix}\right), \textit{for limits}$$

$$= \int_0^t \tau e^{-(t-\tau)}d\tau = \int_0^t \tau e^{-t}e^{\tau}d\tau = e^{-t}\int_0^t \tau e^{\tau}d\tau = e^{-t}\int_0^t \tau e^{\tau}d\tau$$

$$= e^{-t}\left[e^{\tau}(\tau-1)\right]\Big|_0^t = e^{-t}\left[te^{t} - e^{t} + 1\right]u(t) = \left[t - 1 + e^{-t}\right]u(t)$$

Example

$$f)\ x(t)=\delta(t-t_1);\quad h(t)=\delta(t-t_2)$$

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$=\int_{-\infty}^{\infty} \delta(\tau-t_1)\delta(t-\tau-t_2)d\tau$$

$$=\int_{t_1^-}^{t_1^+} \delta(\tau-t_1)\delta(t-\tau-t_2)d\tau=\delta(t-t_1-t_2)$$

$$\delta(t-t_1)*\delta(t-t_2)=\delta(t-(t_1+t_2))$$

$Practice,$ $\left[\delta(t-5)+\delta(t-5)\right]*\left[\delta(t-0.5)+3\delta(t)\right]=$

$$=\delta(t-5.5)-3\delta(t-5)+\delta(t+4.5)-3\delta(t+5)$$

Example

$$g)\ x(t) = u(t) - u(t-1); \quad h(t) = e^{-t}u(t)$$

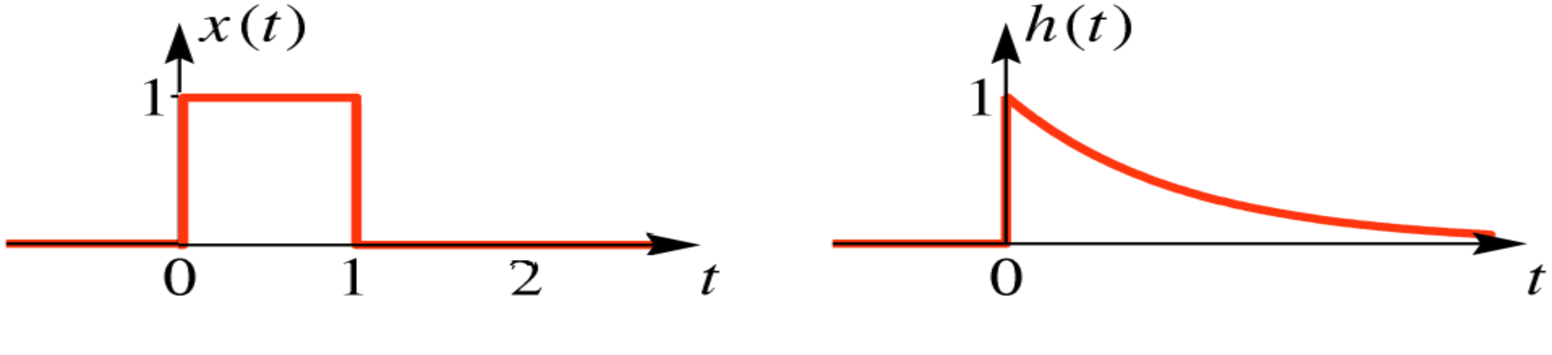
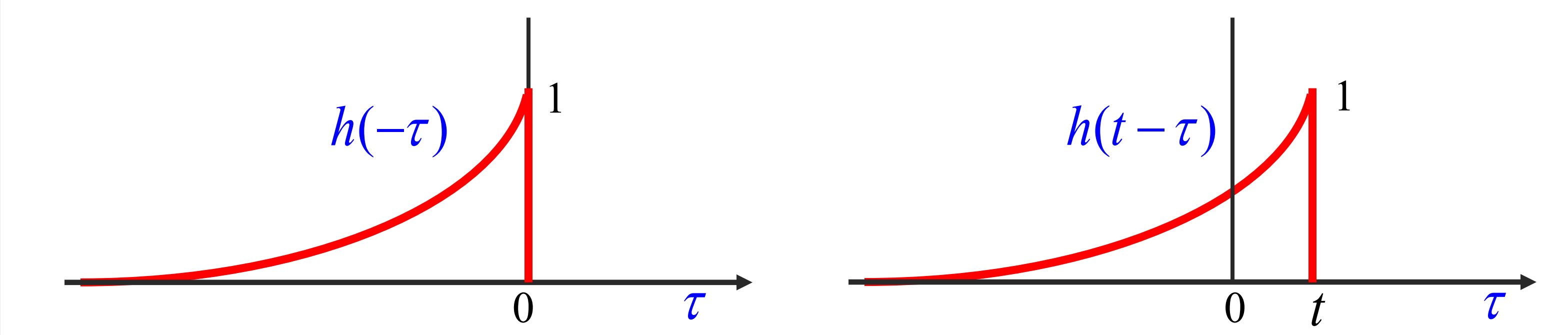


Fig 17.7

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



The problem has 3 parts to it;

$t < 0$; $y(t) = 0 \;\; for \;\; t < 0$;

since there is no overlap between the two functions

$0 \le t \le 1$

$$y(t) = \int_0^t e^{-(t-\tau)} d\tau = e^{-t} \int_0^t e^{\tau} d\tau$$

$$= e^{-t} \left[e^{\tau} \right]_0^t = e^{-t} \left[e^{t} - 1 \right] = 1 - e^{-t}$$

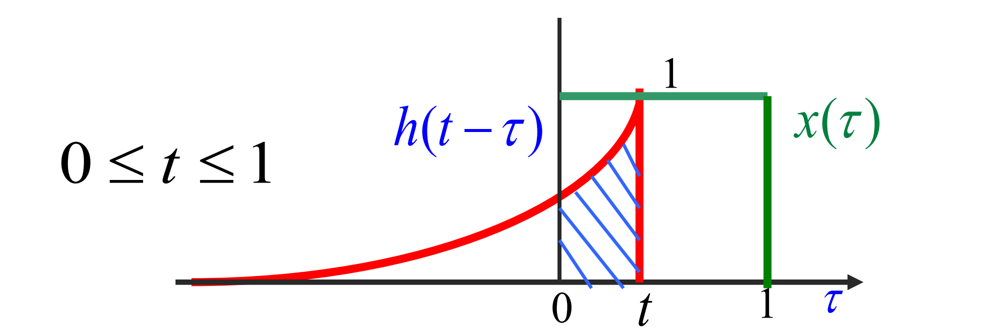


Fig 17.9

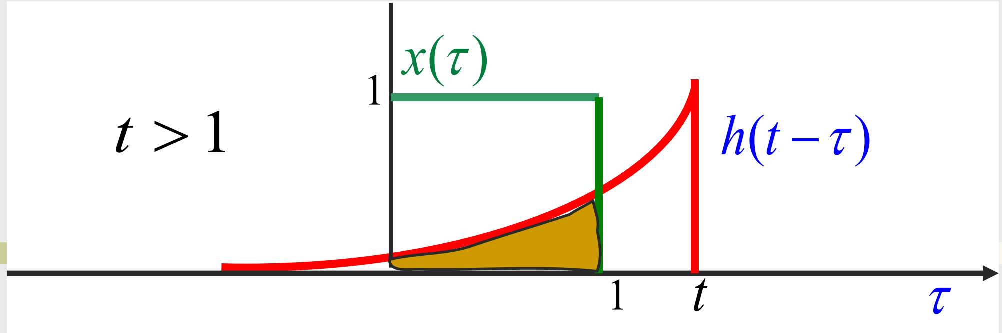


Fig 17.10

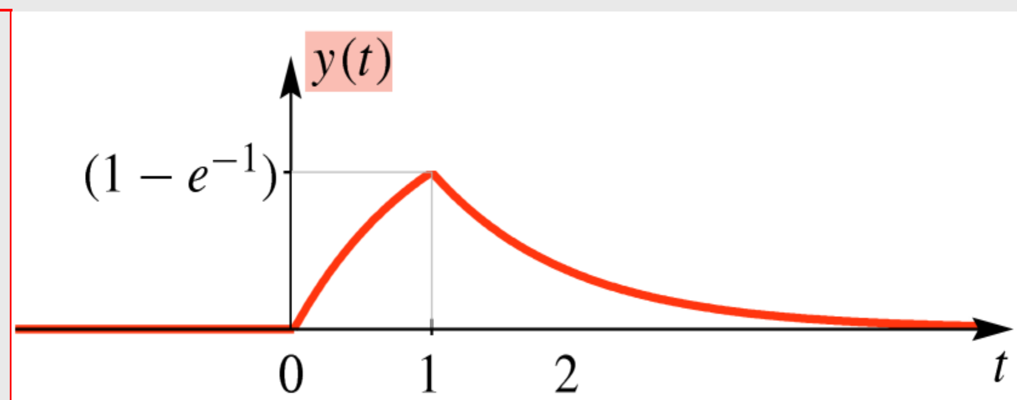
$t > 1$

$$y(t) = \int_0^1 e^{-(t-\tau)} d\tau + \int_0^t 0 d\tau$$

$$= e^{-t} \int_0^1 e^{\tau} d\tau = e^{-t} \left[e^{\tau} \right]_0^1$$

$$= e^{-t} \left[e^1 - 1 \right]$$

$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & 0 \le t \le 1 \\ e^{-t}\left[e - 1\right] & t > 1 \end{cases}$$



Practice Example 1

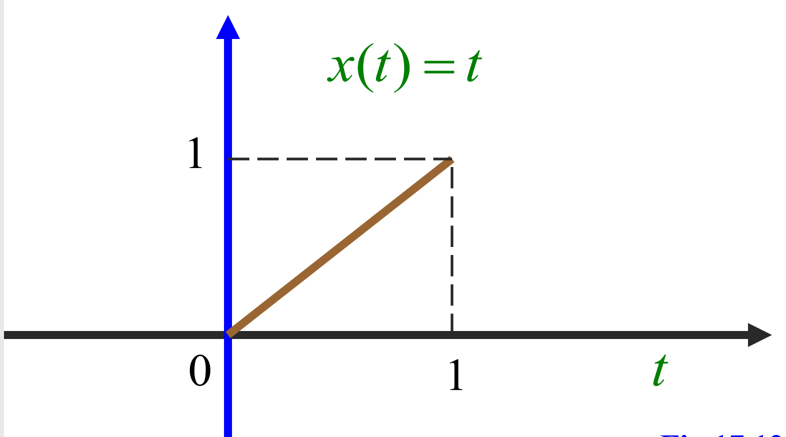


Fig 17.12

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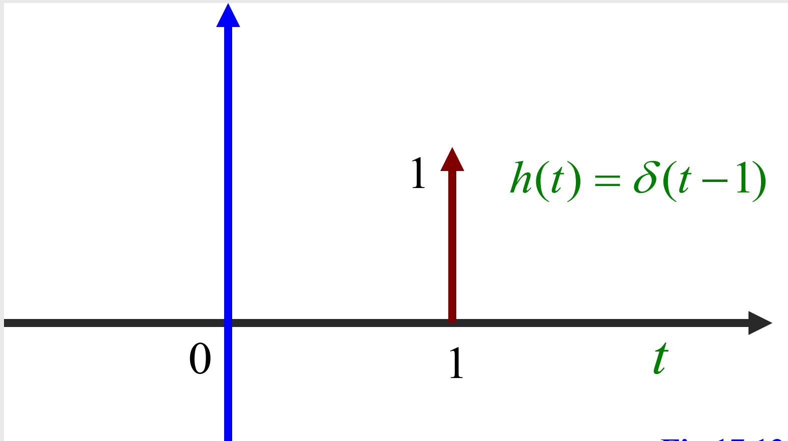
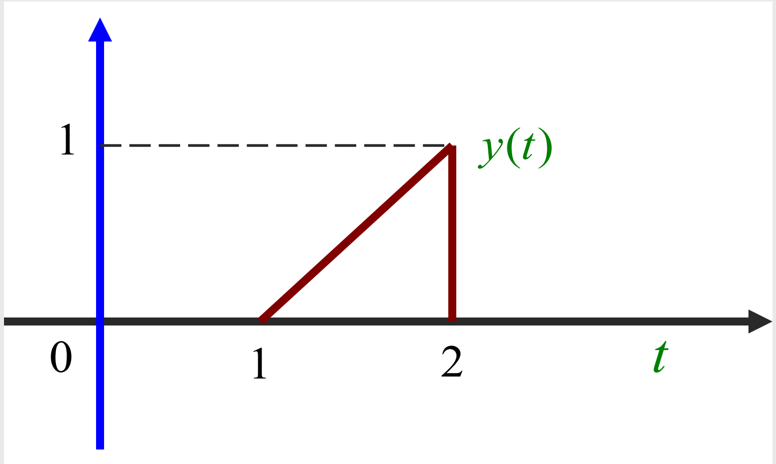


Fig 17.13

$$y(t) = \begin{cases} 0 & t < 1 \\ t-1 & 1 \le t \le 2 \\ 0 & t > 2 \end{cases}$$

Output is shifted version of input



Practice Example 2

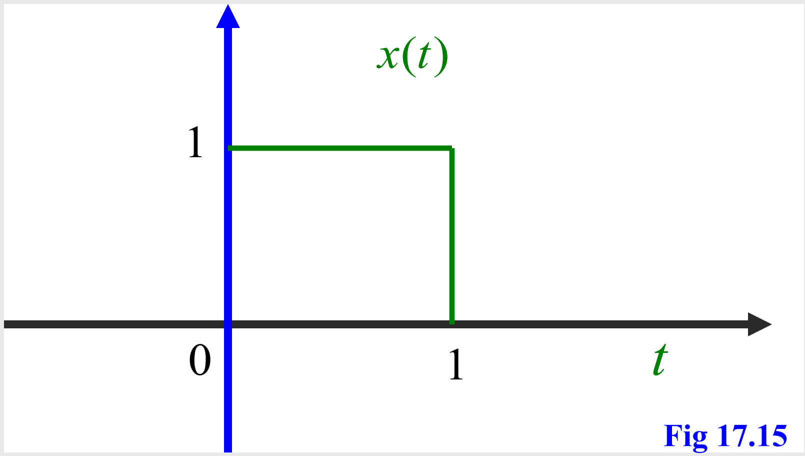


Fig 17.15

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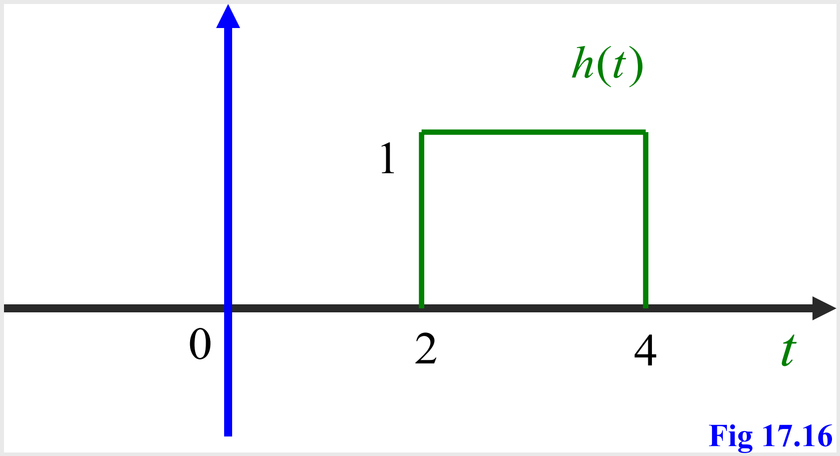
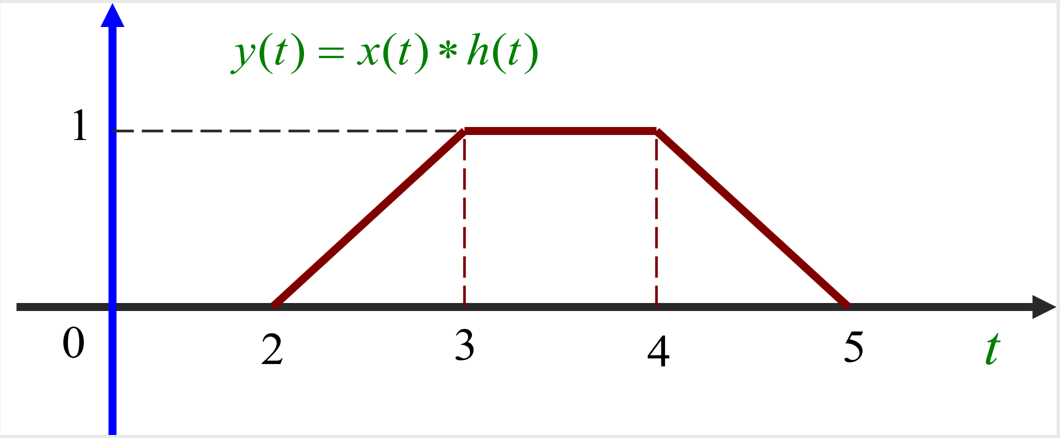


Fig 17.16



A look at Discrete Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad for \ \ -\infty < n < \infty$$

The format of getting the output is same for both continuous and discrete − time sequences

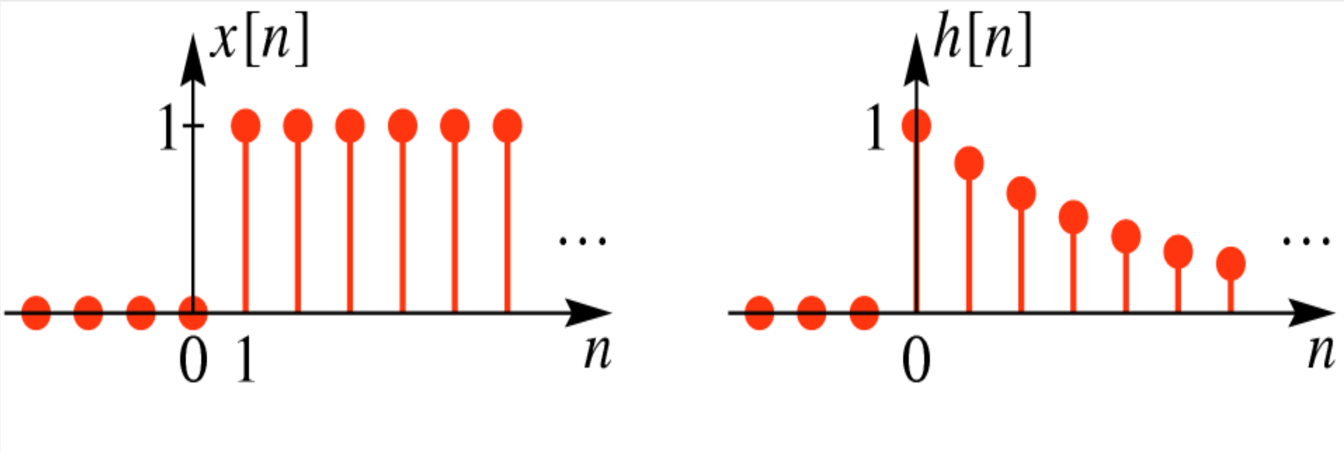
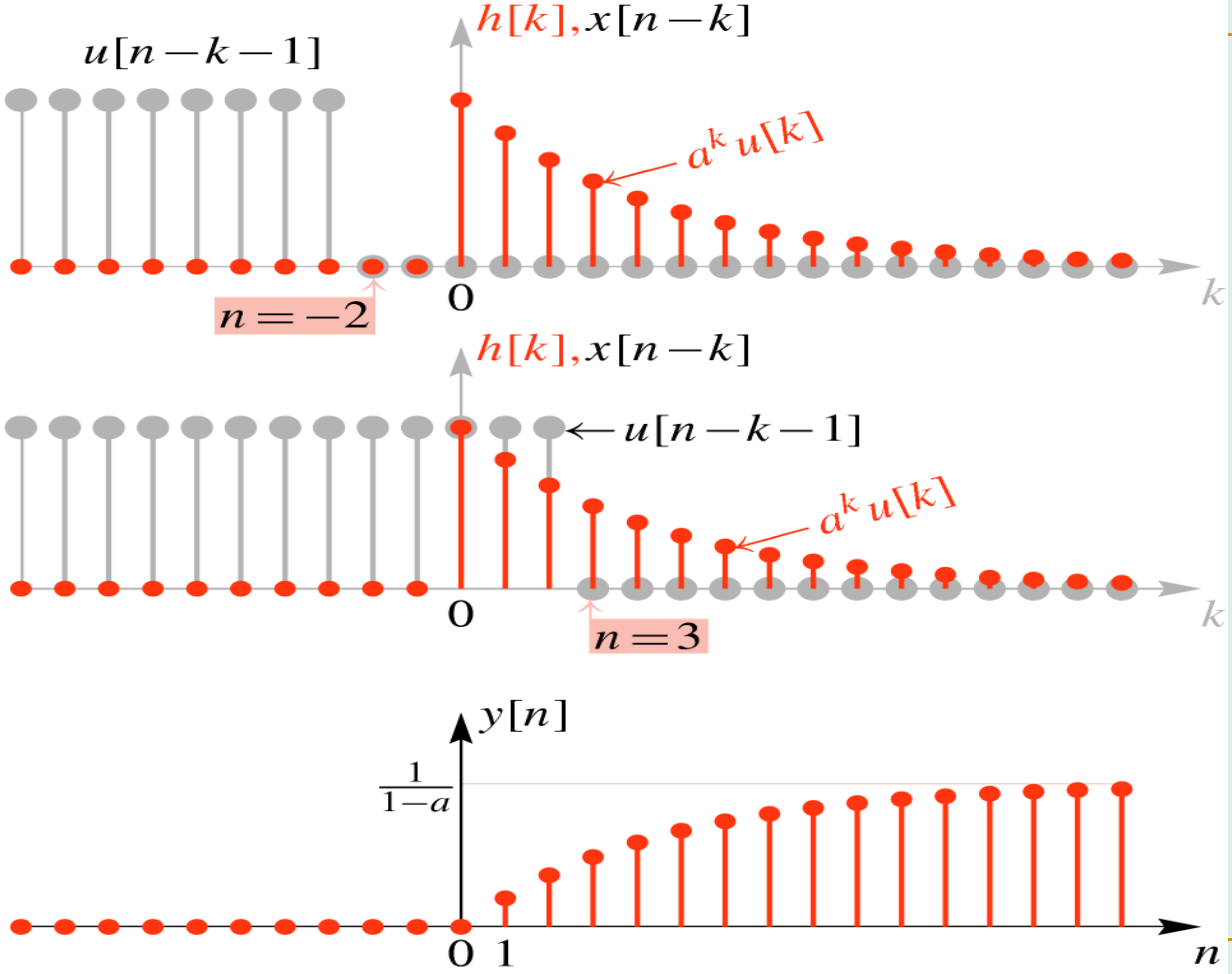


Fig 17.18



h[k], x[n − k]
u[n − k − 1]
a^k u[k]
0
k
n = −2
h[k], x[n − k]
u[n − k − 1]
a^k u[k]
0
k
n = 3
y[n]
1/(1−a)
0 1
n

Demo: Discrete-Time Convolution

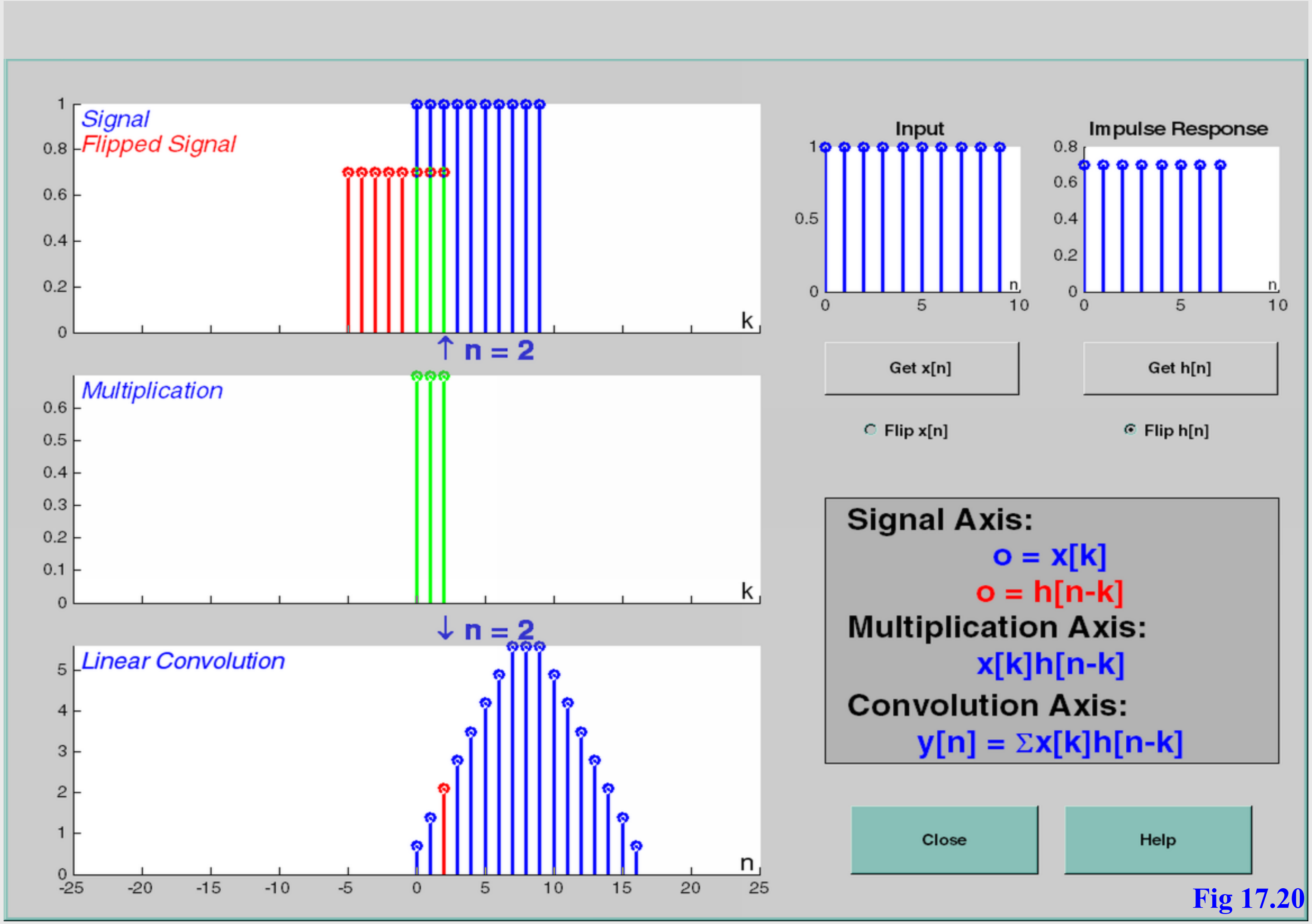


Fig 17.20

command: dconvdemo

Demo: Continuous-Time Convolution

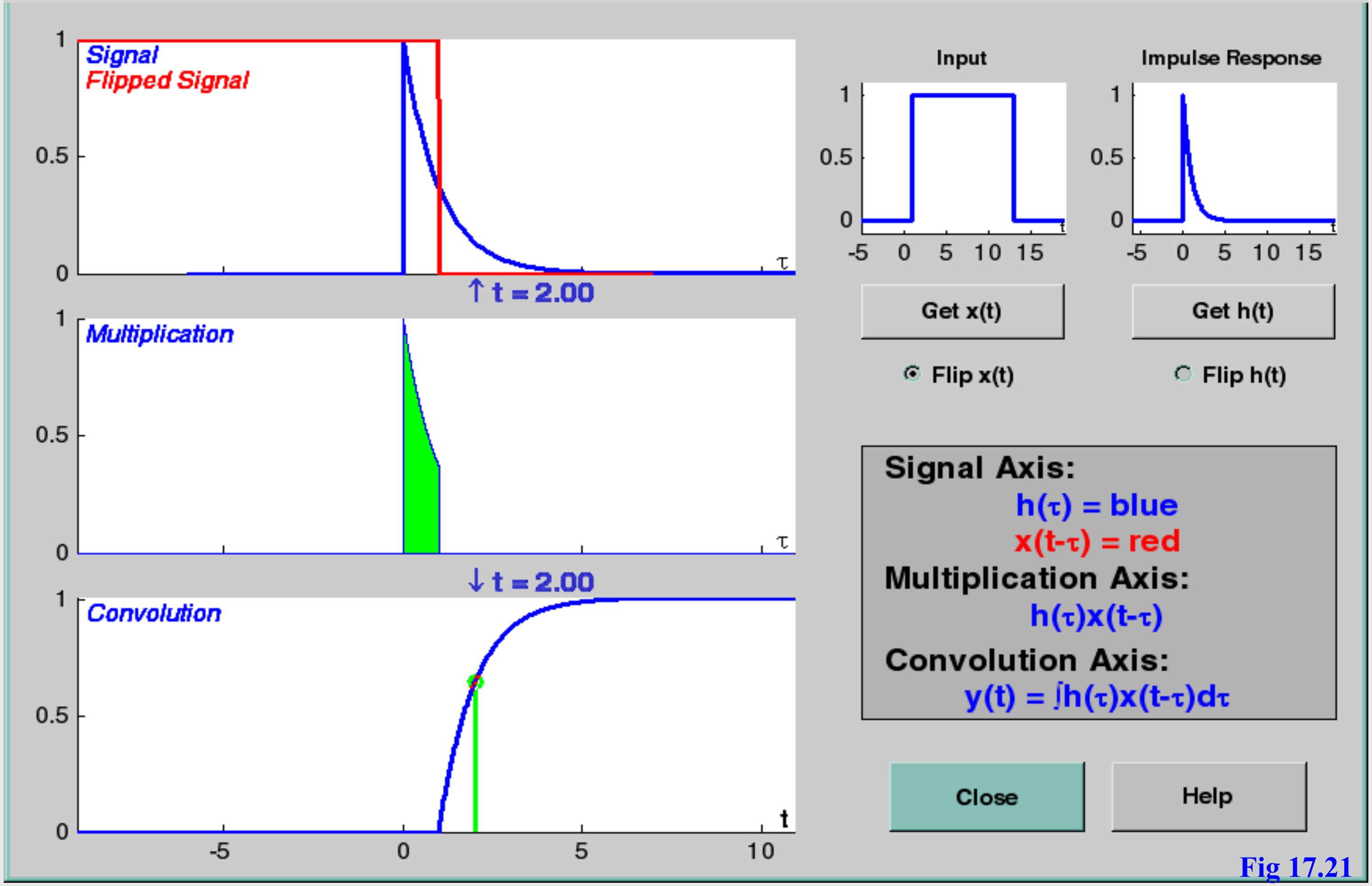


Fig 17.21

command: cconvdemo

Summary

Linear Time-Invariant System (LTI)

Continuous Time (LTI) systems

Differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Output : Convolution integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Discrete-Time (LTI) Systems

Differenceequations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Output : Convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$for \quad -\infty < n < \infty$$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 9.3-9.8 “Signal Processing First”, Prentice Hall, 2003