Discrete - Time Signals and Systems

Continuous-Time signals & systems

Fourier Transform

Yogananda Isukapalli

Fourier Transform

Motivation:

We need to define the frequency spectrum for a more general class of continuous time signals. Fourier Transform is at the heart of modern communication systems

We have defined the spectrum for a limited class of signals such as sinusoids and periodic signals. These kind of spectrum are part of Fourier series representation of signals

Fourier Transform : Definition

Forward continuous – time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (1)$$

Inverse continuous – time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt \qquad (2)$$

Time domain *Frequency domain*

$$x(t) \quad \overset{F}{\longleftrightarrow} \quad X(j\omega)$$

Example

$$Determine\ the\ Fourier\ Transform\ of\ the\ one\text{-}sided$$

$$exponential\ signal; \quad x(t) = e^{-7t}u(t)$$

$$Forward\ continuous-time\ Fourier\ Transform$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-7t}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-7t}e^{-j\omega t}dt \quad (notice\ the\ change\ in\ limits)$$

Example *contd…*

$$= \int_0^\infty e^{-(7+j\omega)t}\,dt = \left.\frac{e^{-(7+j\omega)t}}{-(7+j\omega)}\right|_0^\infty$$

$$= \frac{\left(e^{-(7+j\omega)\infty} - e^{-(7+j\omega)0}\right)}{-(7+j\omega)}$$

$$= \frac{1}{(7+j\omega)} \quad \left(\because e^{-(7+j\omega)\infty} = 0\right)$$

Time domain *Frequency domain*

$$e^{-7t} \xleftrightarrow{F} \frac{1}{(7+j\omega)}$$

Fourier Series & Fourier Transform

Interpretation of Fourier Transform from Fourier Series

$x_{T_0}(t)$ periodic signal, with period T_0

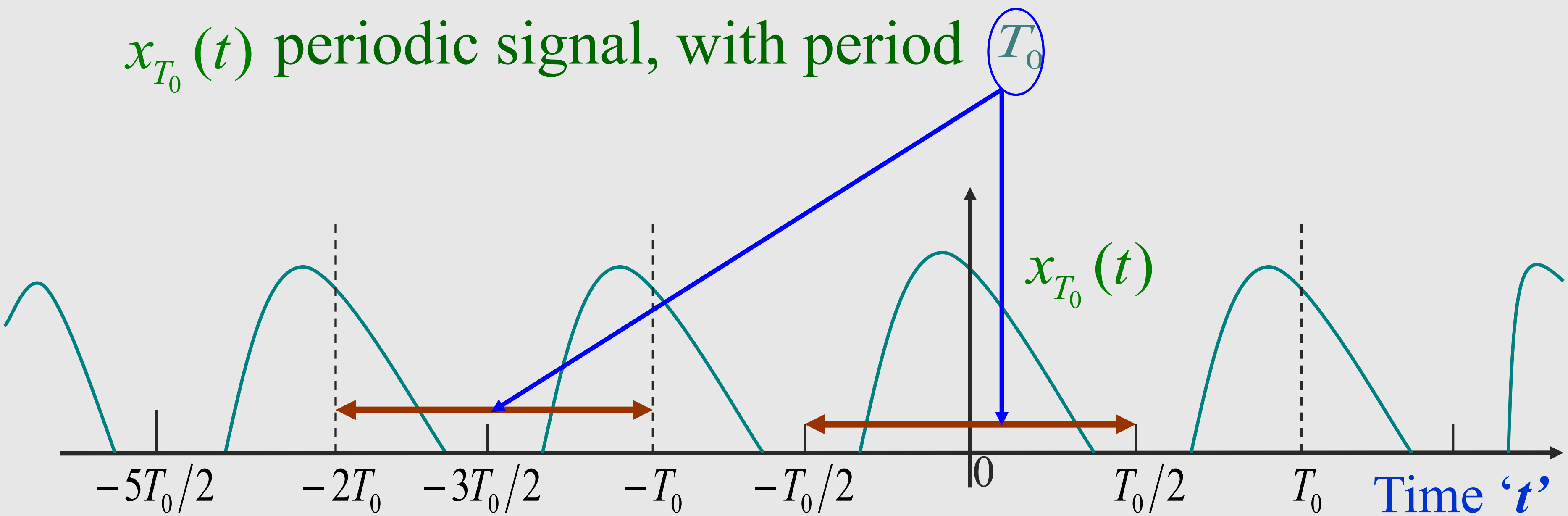


Fig.18.1

Note that the shape of the signal is drawn arbitrarily

Fourier Series & Fourier Transform *contd…*

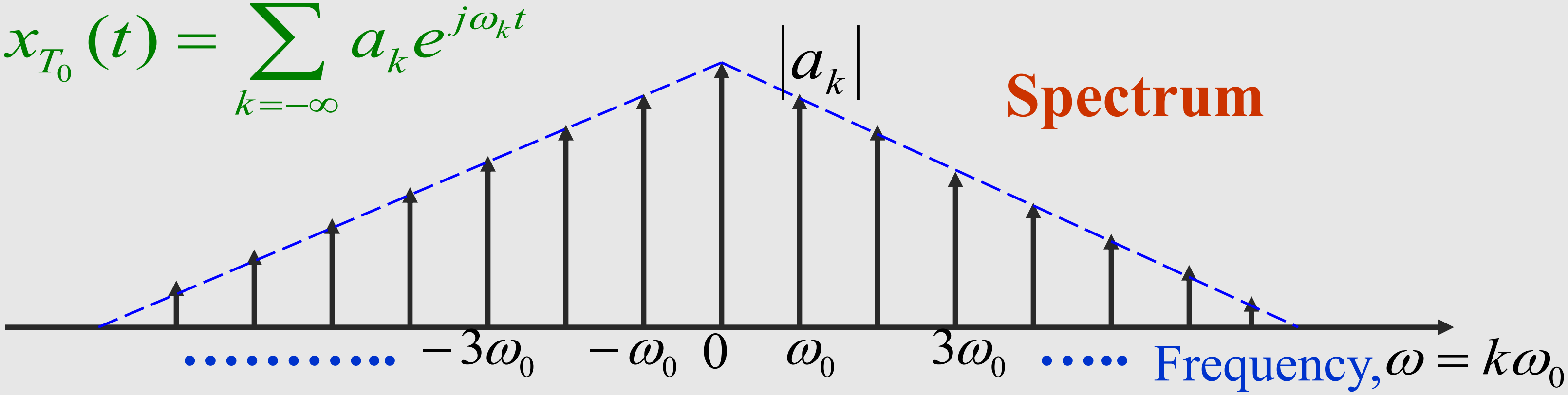
The periodic signal can be approximated by summing up harmonically related periodic exponential signals

Fourier Analysis equation;

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-j\omega_k t}dt, \quad \omega_k = k\omega_0 = \frac{k2\pi}{T_0}$$

And the Fourier synthesis equation,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}$$



Now, assume that the period of the signal x(t) is infinite

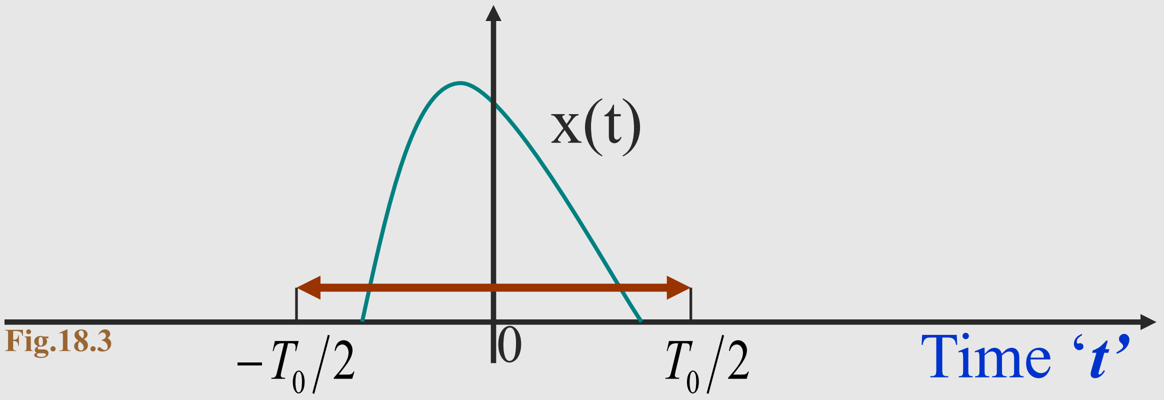


Fig.18.3

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

Note that the above equation also implies that all non-periodic signals can be considered periodic with a period of '∞'

Fourier Series & Fourier Transform *contd…*

Consider the Fourier Analysis equation

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_k t} dt$$

Now define;

$$X(j\omega_k) = X(jk\omega_0) = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_k t} dt = T_0 a_k$$

$$\therefore X(j\omega_k) = T_0 a_k,$$

$$a_k = \frac{X(j\omega_k)}{T_0}$$

Fourier Series & Fourier Transform *contd…*

Then from the Fourier synthesis equation,

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{X(j\omega_k)}{T_0} \right) e^{j\omega_k t}$$

pay attention to the term, $e^{j\omega_k t} = e^{jk\left(\frac{2\pi}{T_0}\right)t}; \quad \lim_{T_0 \to \infty} e^{jk\left(\frac{2\pi}{T_0}\right)t} \to 1$

$$\therefore x_{T_0}(t) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} X(j\omega_k) e^{j\omega_k t}$$

Fourier Series & Fourier Transform *contd…*

$$x_{T_0}(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} X(j\omega_k)e^{j\omega_k t}$$

Define;

$$\Delta\omega_k = \omega_{k+1} - \omega_k = \frac{2\pi(k+1)}{T_0} - \frac{2\pi k}{T_0} = \frac{2\pi}{T_0}$$

$$T_0 = \frac{2\pi}{\Delta\omega_k}$$

substitute the result in the sysnthesis equation,

$$x_{T_0}(t) = \frac{\Delta\omega_k}{2\pi}\sum_{k=-\infty}^{\infty} X(j\omega_k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(j\omega_k)e^{j\omega_k t}\Delta\omega_k$$

Fourier Series & Fourier Transform *contd…*

$$\lim_{T_0 \to \infty} T_0 \quad \Rightarrow \quad \Delta\omega_k \to 0 \quad \left(\because \Delta\omega_k = \frac{2\pi}{T_0} \right)$$

$$\omega_k = \frac{2\pi k}{T_0} = k\Delta\omega_k$$

$$\lim_{T_0 \to \infty} T_0 \quad \Rightarrow \quad \omega_k \to \omega$$

The above equation implies that the discrete-variable 'ω_k' can now be replaced with a continuous variable 'ω'

Going back to the original equation connecting the periodic and non-periodic signals; $\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$

Fourier Series & Fourier Transform *contd…*

$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(j\omega_k)e^{j\omega_k t}\Delta\omega_k$$

$$\lim_{T_0\to\infty} x_{T_0}(t) = x(t) \quad \Rightarrow \quad \lim_{\Delta\omega_k\to 0} x_{T_0}(t) = x(t)$$

$$\lim_{\Delta\omega_k\to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(j\omega_k)e^{j\omega_k t}\Delta\omega_k$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$Because;\int_{-\infty}^{\infty} x(u)du = \lim_{\Delta u\to 0}\sum_{k=-\infty}^{\infty} X(k\Delta u)\Delta u$$

Fourier Series & Fourier Transform *contd…*

Fourier Transform integrals

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}d\omega \quad : \quad Analysis \qquad (1)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad : \quad Synthesis \qquad (2)$$

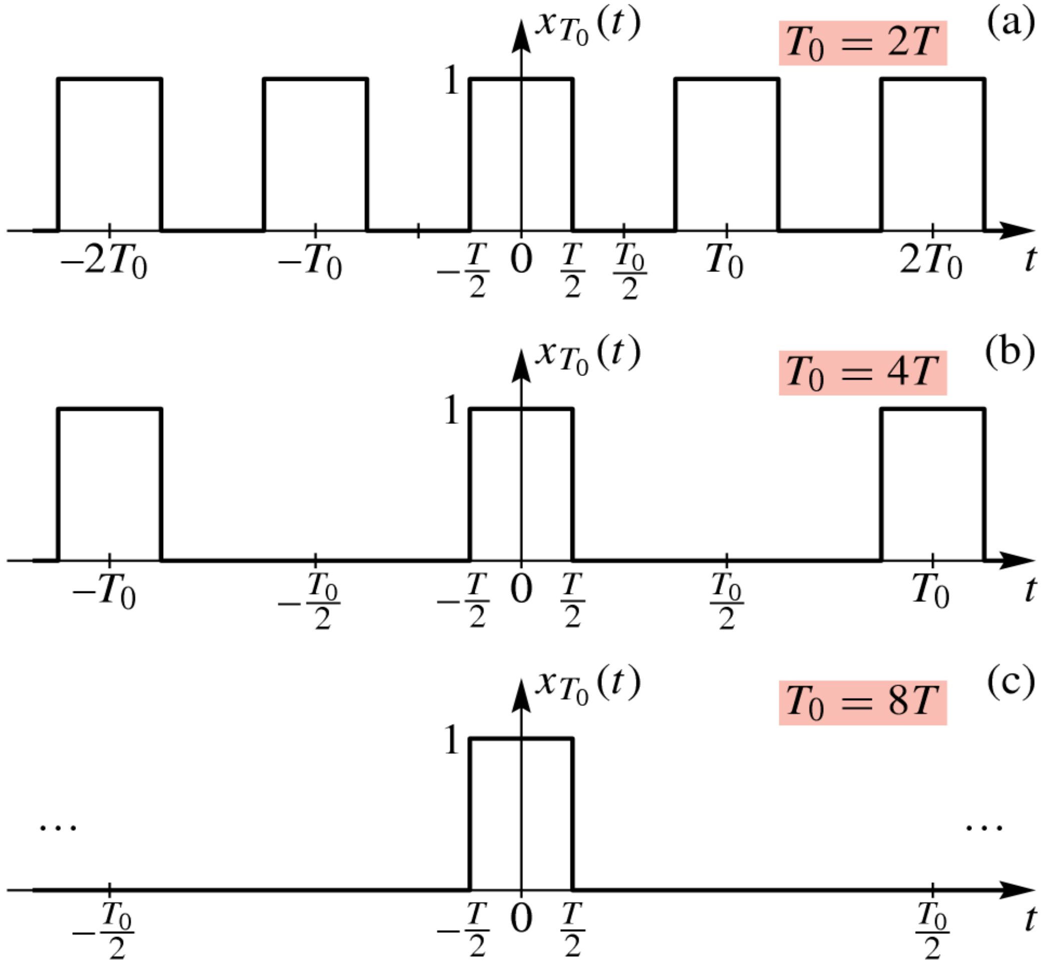
equations (1) *and* (2) *exist if and only if*, $\lim_{t\to\infty} x(t) = 0;$

that is $\int_{-\infty}^{\infty} |x(t)|dt < \infty$

Sufficient condition for the existance of $X(j\omega)$

Example

The periodicity is increased to show the effect of higher period. Notice that for fig.c, no repetition is visible in the plot. Also the spectrum plot for the three cases shows an interesting phenomenon.



Example *contd…*

Fourier Analysis equation;

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_k t} dt$$

$$a_k T_0 = \int_{-T/2}^{T/2} e^{-j\omega_k t} dt$$

(notice the change in limits for the present case)

$$a_k T_0 = \left. \frac{e^{-j\omega_k t}}{-j\omega_k} \right|_{-T/2}^{T/2} = \left[\frac{e^{-j\omega_k T/2} - e^{j\omega_k T/2}}{-j\omega_k} \right]$$

$$= \frac{\sin(\omega_k T/2)}{\omega_k / 2}$$

Example *contd…*

The frequecncies, $k\omega_0 = \omega_k$*, get closer and closer as* $T_0 \to \infty$ *and eventually become dense in the interval* $-\infty < \omega < \infty$*. The quatities '*$a_k T_0$*' approach a continuous envelope function*

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(\omega_k T/2)}{\omega_k/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

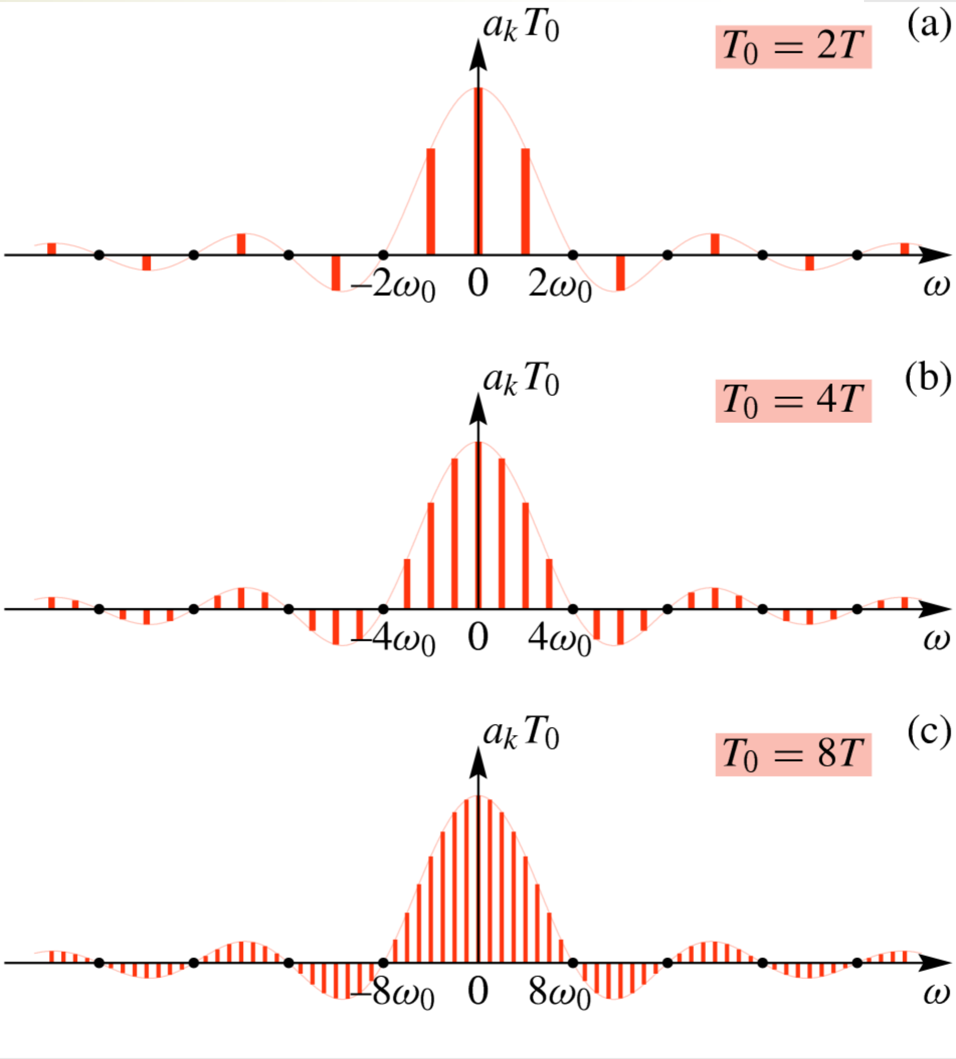


Fig.18.5

Fourier Transform Pair 1

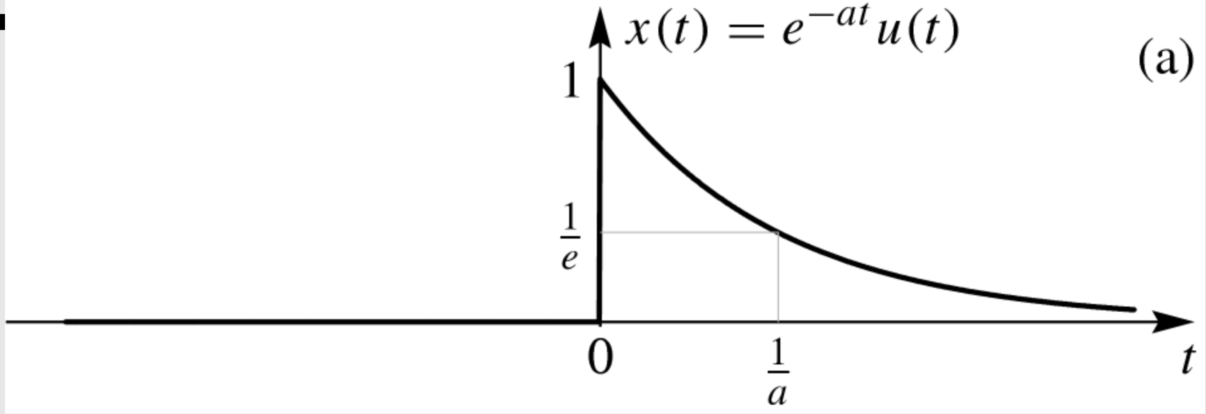


Fig.18.6

$$x(t) = e^{-at}u(t)$$

Forward continuous – time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \quad \textit{(notice the change in limits)}$$

Fourier Transform Pair 1 *contd….*

$$= \int_0^\infty e^{-(a+j\omega)t} dt = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^\infty$$

$$= \frac{\left(e^{-(a+j\omega)\infty} - e^{-(a+j\omega)0}\right)}{-(a+j\omega)}$$

$$= \frac{1}{(a+j\omega)} \quad \left(\because e^{-(a+j\omega)\infty} = 0\right)$$

Time domain *Frequency domain*

$$e^{-at}u(t) \quad \overset{F}{\longleftrightarrow} \quad \frac{1}{(a+j\omega)}, \quad \textit{Fourier transform is unique}$$

Fourier Transform Pair 1 *contd….*

$$X(jw) = \frac{1}{(a + j\omega)}$$

$$|X(jw)| = \frac{1}{\left(a^2 + \omega^2\right)};$$

$$\angle X(jw) = -\tan^{-1}\left(\frac{1}{\omega}\right)$$

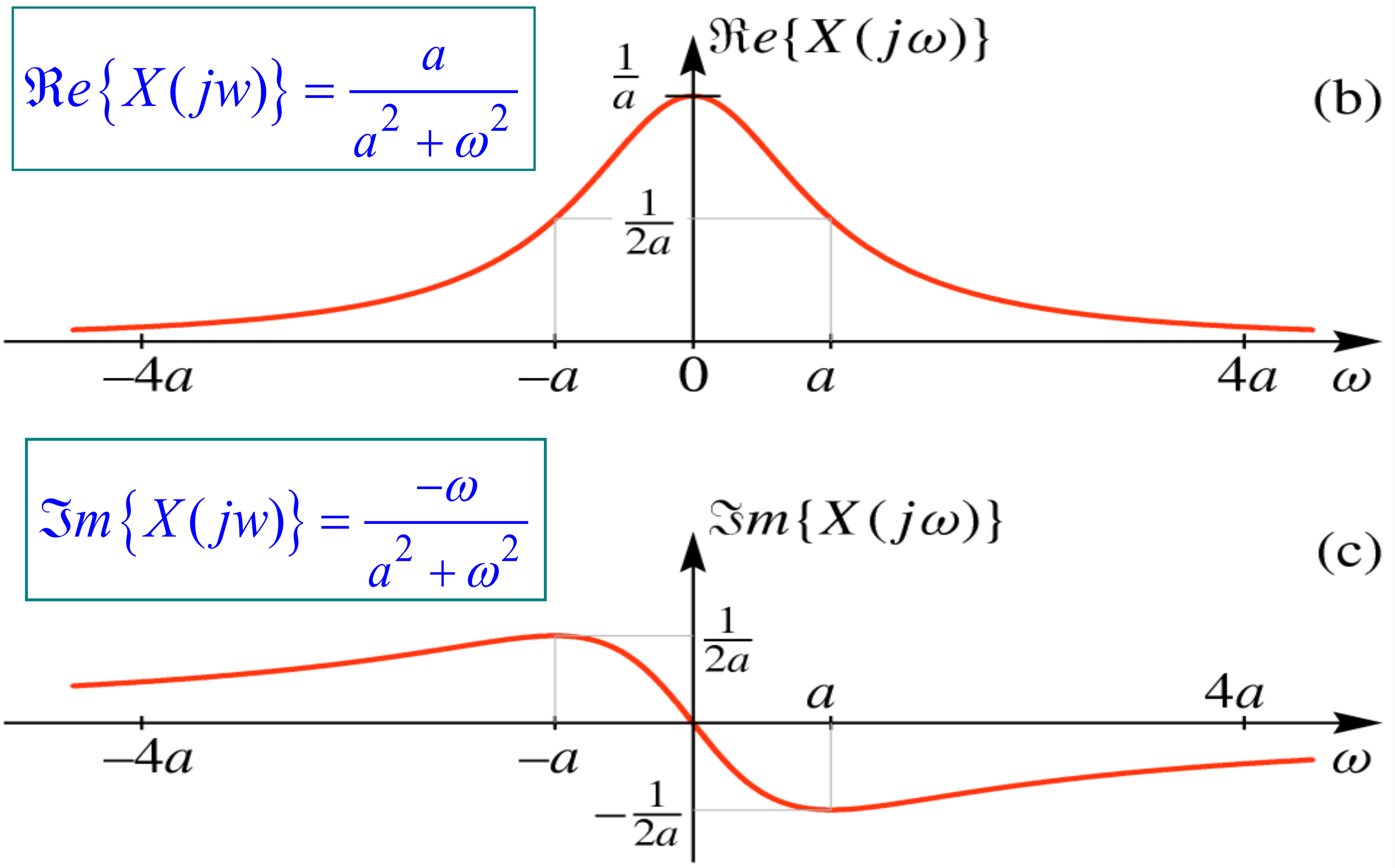
$$X(jw) = \frac{1}{(a + j\omega)}\left(\frac{a - j\omega}{a - j\omega}\right) = \frac{a - j\omega}{\left(a^2 + \omega^2\right)}$$

$$\Re e\{X(jw)\} = \frac{a}{a^2 + \omega^2}; \quad \Im m\{X(jw)\} = \frac{-\omega}{a^2 + \omega^2}$$

Fourier Transform Pair 1 *contd….*

$$\Re e\{X(jw)\} = \frac{a}{a^2 + \omega^2}$$

$$\Im m\{X(jw)\} = \frac{-\omega}{a^2 + \omega^2}$$



Fourier Transform Pair 2

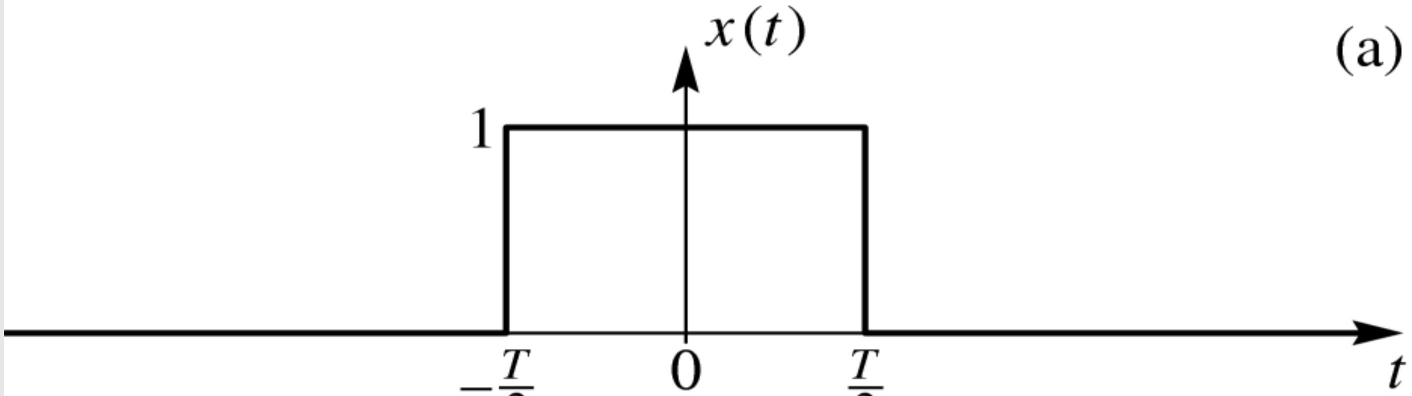


Fig.18.8

$$x(t) = \left[u(t+T/2) - u(t-T/2)\right]$$

Forward continuous – time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[u(t+T/2) - u(t-T/2)\right] e^{-j\omega t} dt$$

Fourier Transform Pair 2 *contd….*

$$= \int_{-T/2}^{T/2} e^{-(j\omega t)} dt = \left. \frac{e^{-(j\omega t)}}{(-j\omega)} \right|_{-T/2}^{T/2}$$

$$= \frac{\left(e^{-(j\omega T/2)} - e^{(j\omega T/2)}\right)}{-(j\omega)}$$

$$= \frac{-2j\sin(\omega T/2)}{-j\omega} = \frac{\sin(\omega T/2)}{\omega/2}, \quad \text{'sinc' function; } \frac{\sin(\pi\theta)}{\pi\theta}$$

Time domain — *Frequency domain*

$$\left[u(t+T/2) - u(t-T/2)\right] \overset{F}{\longleftrightarrow} \frac{\sin(\omega T/2)}{\omega/2}$$

Fourier Transform Pair 2 *contd….*

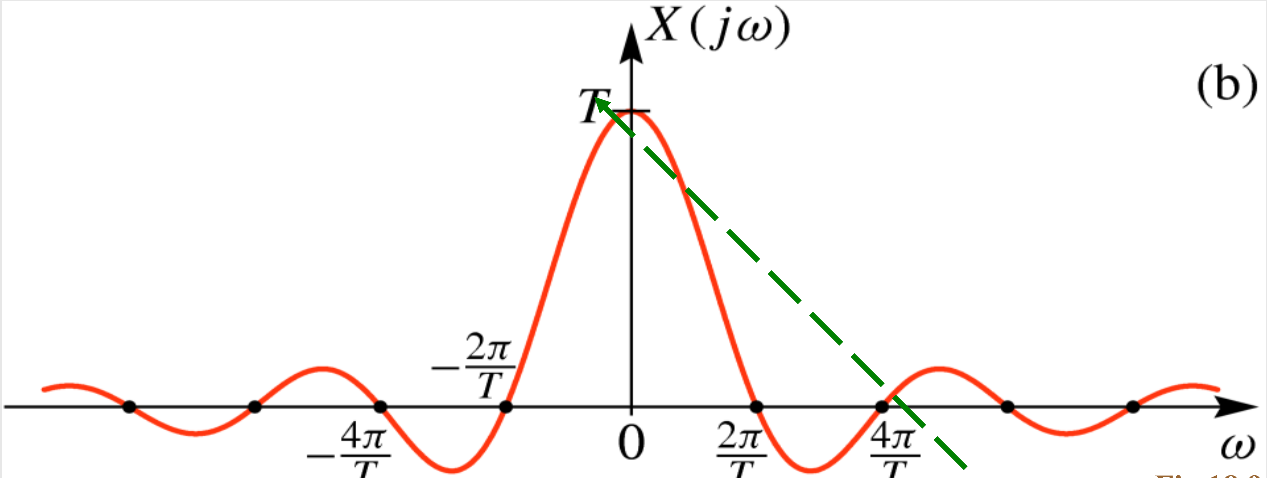


Fig.18.9

$$X(j\omega)=\frac{\sin(\omega T/2)}{\omega/2}$$

$$Note: X(j0)=\lim_{\omega\to 0}\frac{\sin(\omega T/2)}{\omega/2},$$

$$using\ L'Hospitals\ rule;\ X(j0)=\lim_{\omega\to 0}\frac{T/2\cos(\omega T/2)}{1/2}=T$$

Fourier Transform Pair 2 -- Synthesis

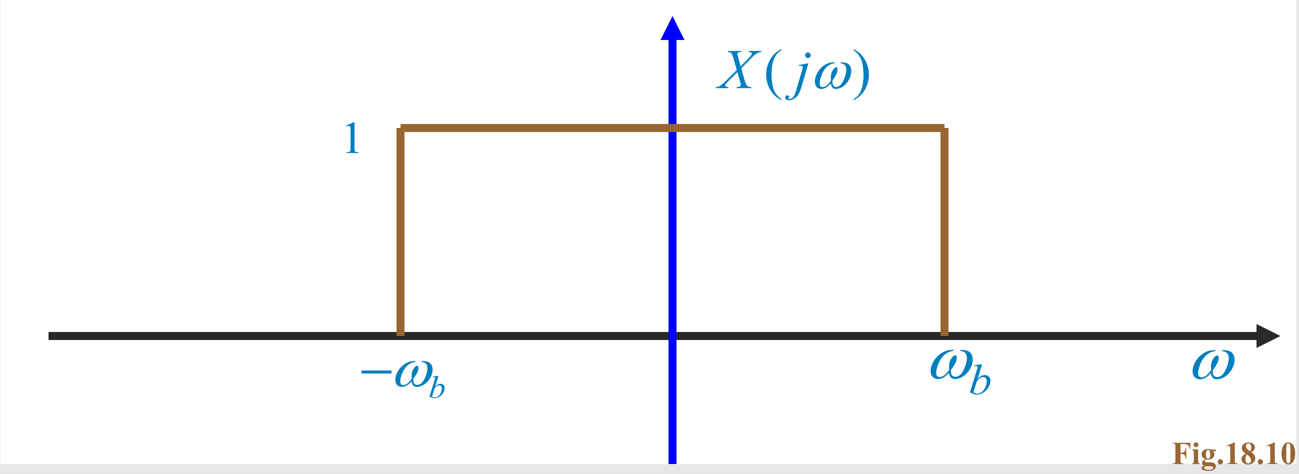


Fig.18.10

The frequency domain information is given above,

Find the corresponding time-domain signal

$$X(j\omega)=\left[u(\omega+\omega_b)-u(\omega-\omega_b)\right]$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega \quad : \quad \textit{Synthesis equation}$$

Fourier Transform Pair 2 – **Synthesis** *contd…*

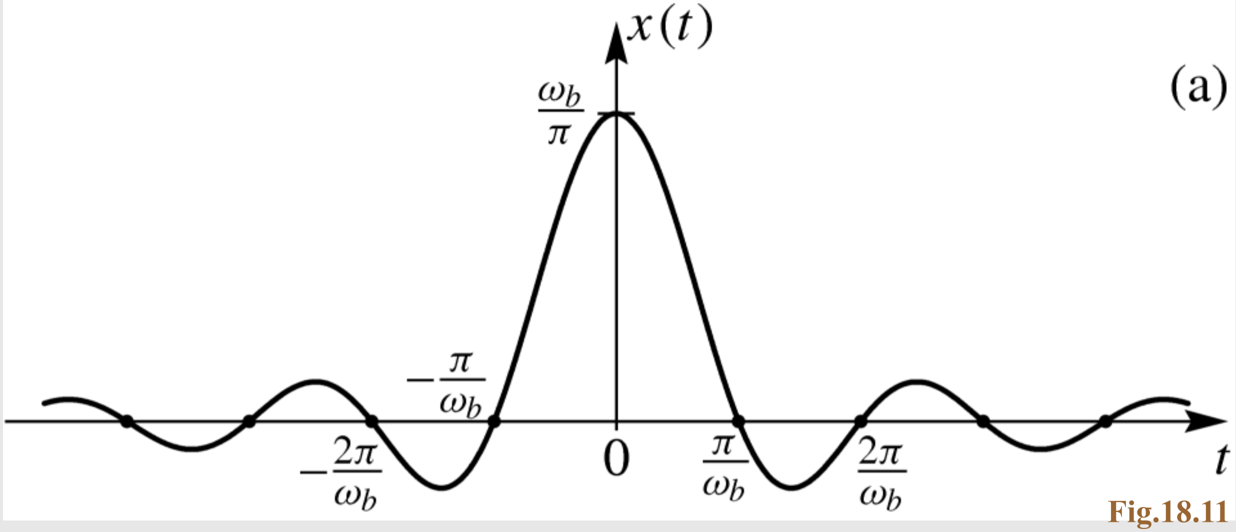
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[u(\omega+\omega_b)-u(\omega-\omega_b)\right]e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} e^{j\omega t}\,d\omega \quad (\textit{notice the change in limits})$$

$$= \left.\frac{e^{(j\omega t)}}{(2\pi jt)}\right|_{-\omega_b}^{\omega_b} = \frac{\left(e^{(j\omega_b t)} - e^{-(j\omega_b t)}\right)}{(j2\pi t)}$$

$$= \frac{2j\sin(\omega_b t)}{(j2\pi t)} = \frac{\sin(\omega_b t)}{(\pi t)}$$

Fourier Transform Pair 2 – Synthesis *contd…*



Frequency domain *Time domain*

$$\left[u(\omega+\omega_b)-u(\omega-\omega_b)\right] \overset{F}{\longleftrightarrow} \frac{\sin(\omega_b t)}{(\pi t)}$$

Note : finite - time signals are infinite in frequeny domain

finite - frequency signals are infinite in time domain

Fourier Transform Pair 3

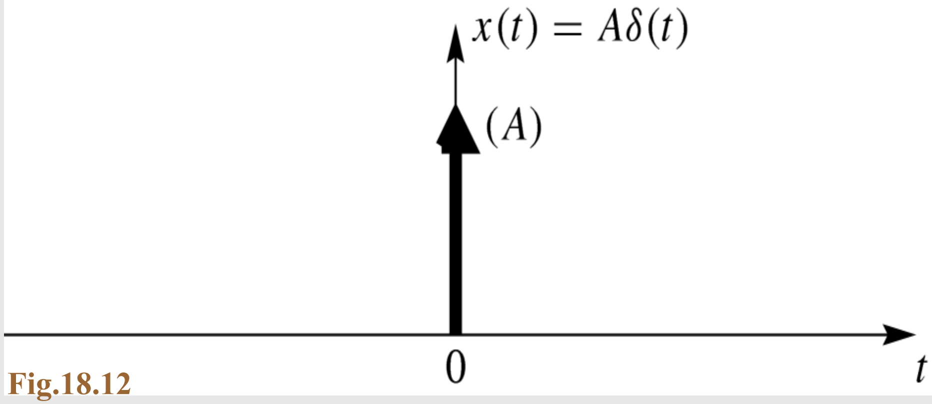


Fig.18.12

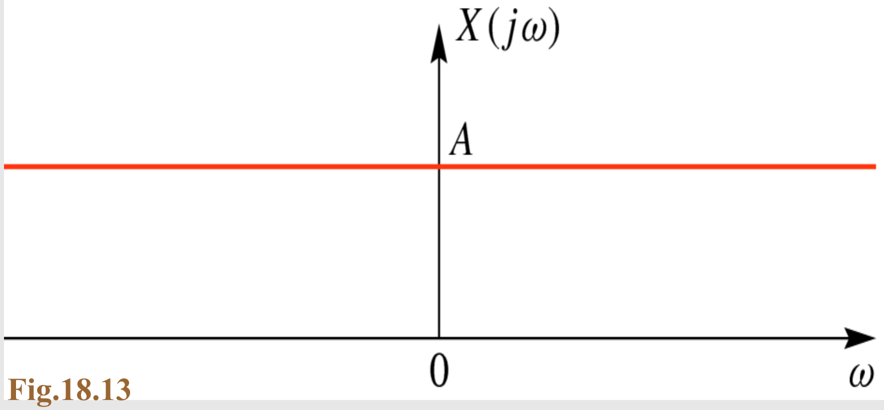


Fig.18.13

$$x(t) = A\delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} A\delta(t)e^{-j\omega t}dt$$

$$= Ae^{-j\omega t}\Big|_{t=0} = A$$

$$A\delta(t) \overset{F}{\longleftrightarrow} A$$

Fourier Transform Pair 3 -- Synthesis

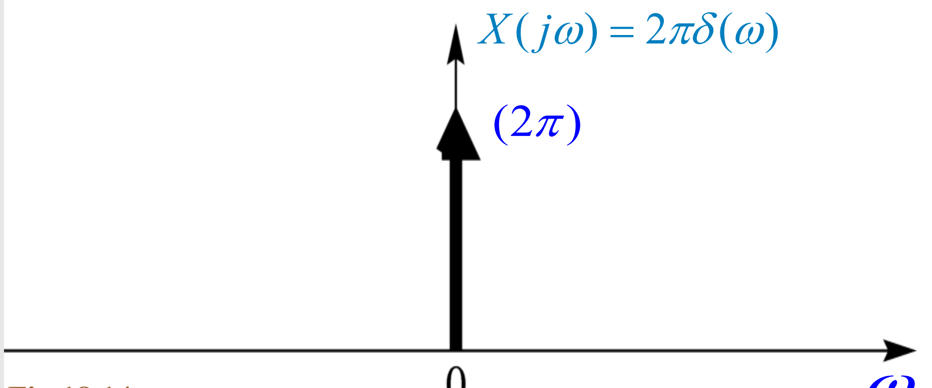


Fig.18.14

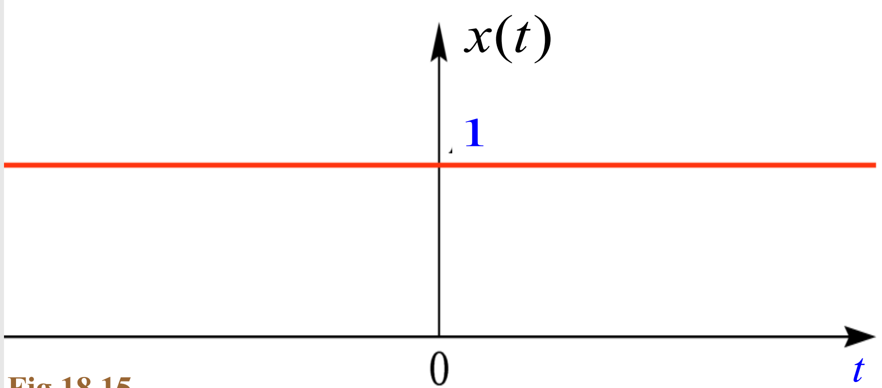


Fig.18.15

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad : \quad \textit{Synthesis equation}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega$$

$$= e^{j\omega t}\Big|_{\omega=0} = 1$$

Fourier Transform Pair 4

$$Let\ X(j\omega) = 2\pi\delta(\omega-\omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad : \quad Synthesis\ equation$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

$$= e^{j\omega t}\Big|_{\omega=\omega_0} = e^{j\omega_0 t}$$

Frequency domain — *Time domain*

$$2\pi\delta(\omega-\omega_0) \overset{F}{\longleftrightarrow} e^{j\omega_0 t}$$

$$2\pi\delta(\omega+\omega_0) \overset{F}{\longleftrightarrow} e^{-j\omega_0 t}$$

Fourier Transform Pair 5

Find the Fourier Transform for $x(t) = \sin(\omega_0 t)$

$$x(t) = \sin(\omega_0 t) = \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \right)$$

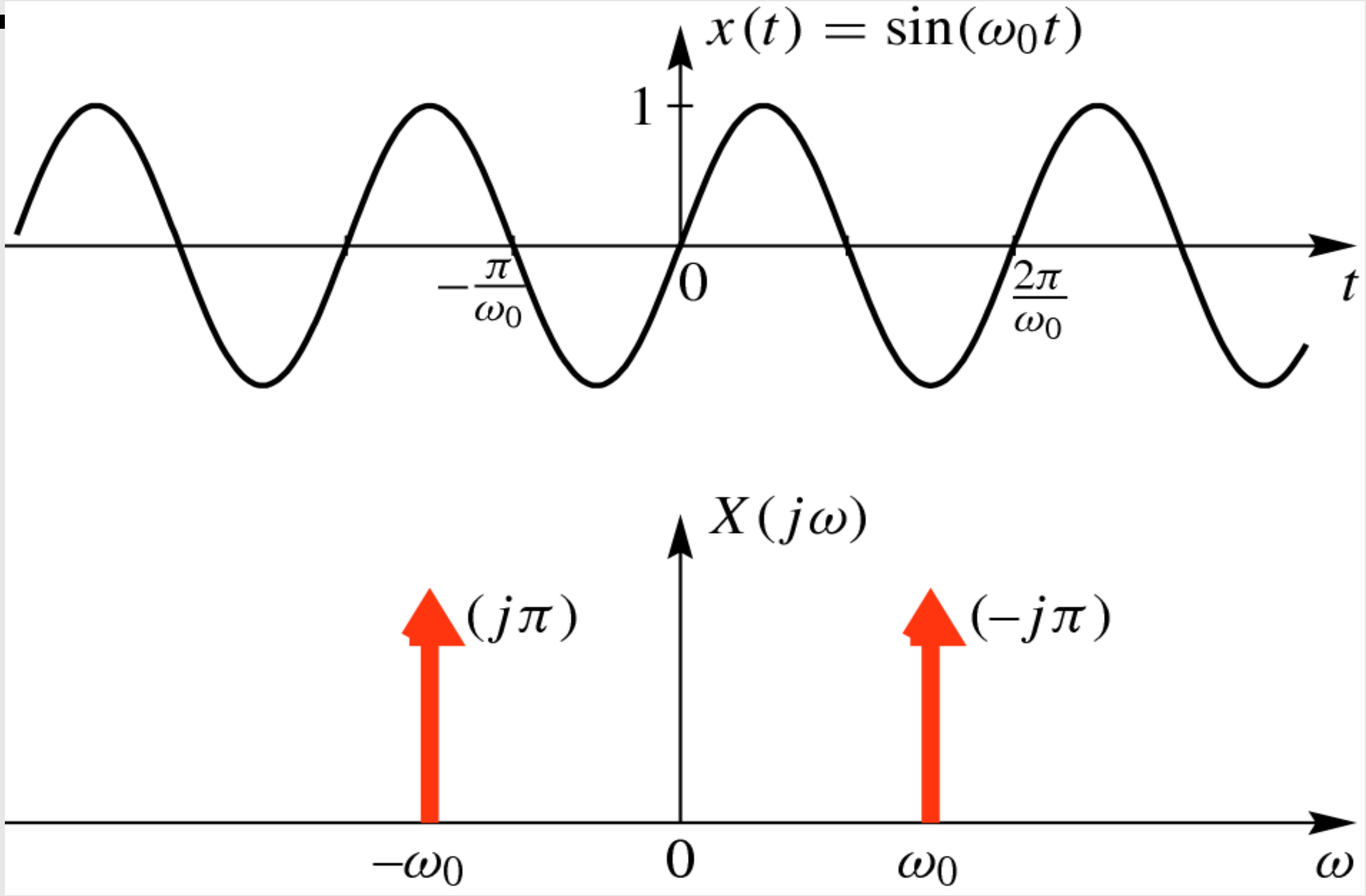
$x(t) \xleftrightarrow{F} X(j\omega)$, *use the identities from previous example*

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega + \omega_0)$$

Time domain		*Frequency domain*
$ax_1(t) + bx_2(t)$	$\xleftrightarrow{F}$	$aX_1(j\omega) + bX_2(j\omega)$
$\sin(\omega_0 t)$	$\xleftrightarrow{F}$	$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$

Fourier Transform Pair 5 *contd….*



Fourier Transform Pair 6

$$Find\ the\ Fourier\ Transform\ for\ x(t) = \cos(\omega_0 t)$$

$$x(t) = \cos(\omega_0 t) = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right)$$

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega),\ use\ the\ identities\ from\ previous\ example$$

$$e^{j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega + \omega_0)$$

Time domain *Frequency domain*

$$a x_1(t) + b x_2(t) \overset{F}{\longleftrightarrow} a X_1(j\omega) + b X_2(j\omega)$$

$$\cos(\omega_0 t) \overset{F}{\longleftrightarrow} \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}$$

$$Where: \quad x(t) = x(t + T_0)$$

$$\omega_k = k\omega_0 = \frac{k2\pi}{T_0}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt,$$

$$x(t) \quad \stackrel{F}{\longleftrightarrow} \quad X(j\omega)$$

$$F\{x(t)\} = F\left\{ \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right\}$$

Periodic Signals *contd…*

$$e^{j\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - \omega_0)$$

$$\therefore e^{jk\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Time domain *Frequency domain*

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t} \xleftrightarrow{F} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example: Square Wave

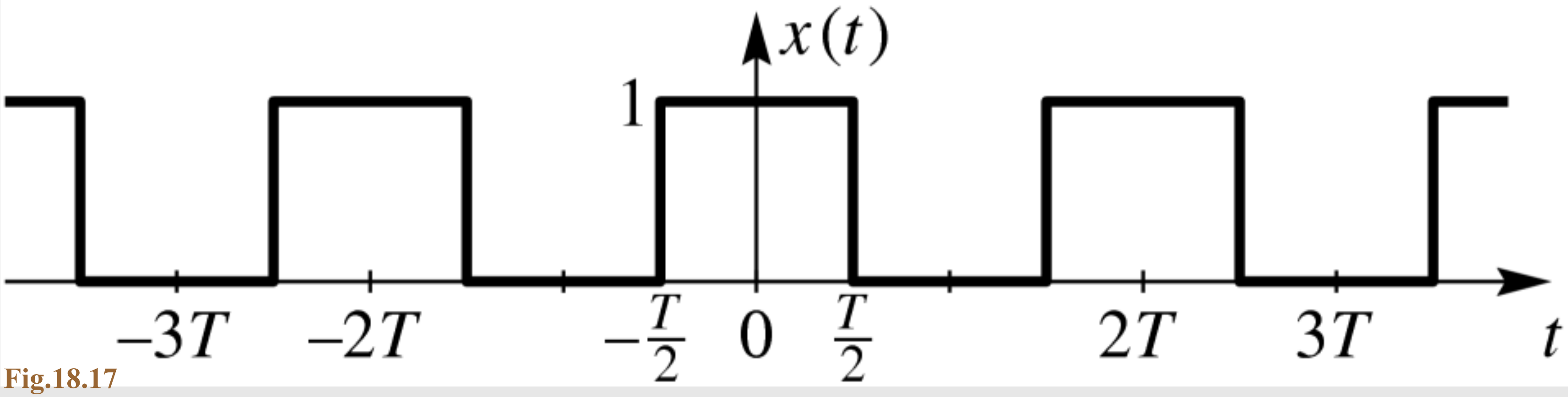


Fig.18.17

$$a_k = \frac{1}{T_0}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt,$$

$$= \frac{1}{T_0}\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)\Bigg|_{-T/2}^{T/2} = \left(\frac{e^{-jk\omega_0 T/2} - e^{jk\omega_0 T/2}}{-jk\omega_0 T_0}\right)$$

Example: Square wave *contd…*

Assume a 50% duty cycle, which means,

$$T = T_0/2; \quad \therefore \ \omega_0 T = \pi \ \ and \ \ \omega_0 T_0 = \frac{2\pi}{T_0} T_0 = 2\pi$$

$$\left(\frac{2\sin(k\omega_0 T/2)}{k\omega_0 T_0} \right) = \left(\frac{\sin(\pi k/2)}{k\pi} \right)$$

$$a_0 = \frac{1}{T_0} \int_{-T/2}^{T/2} dt = \frac{T}{T_0} = \frac{1}{2}$$

$$a_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} & k \neq 0 \\ \dfrac{1}{2} & k = 0 \end{cases}$$

Example: Square wave *contd…*

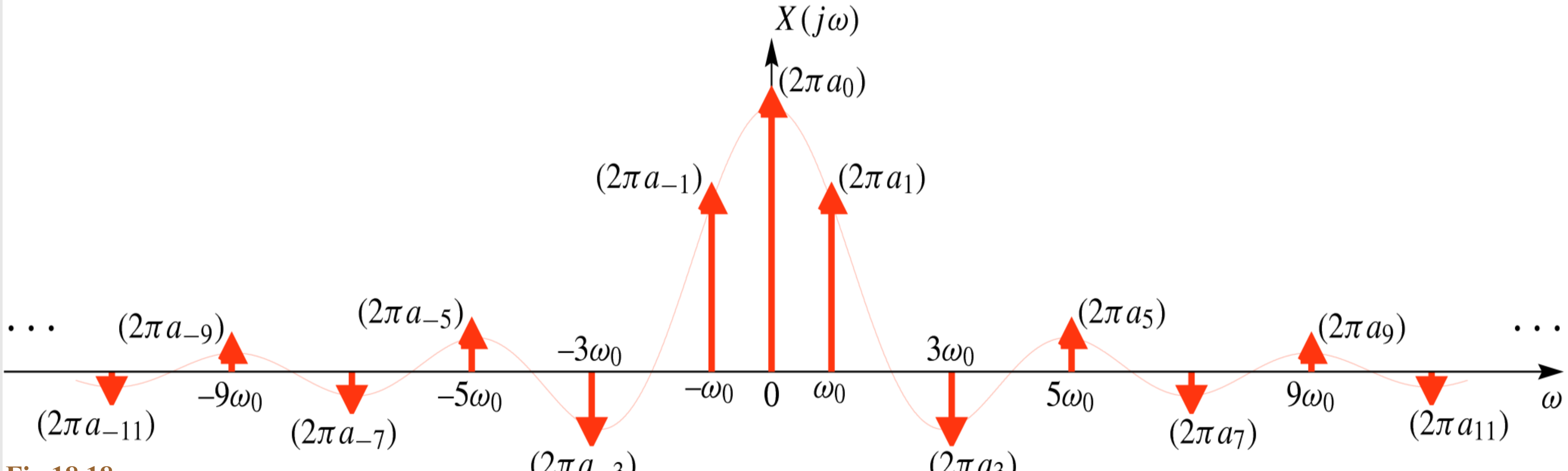


Fig.18.18

$$\sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t} \overset{F}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$X(j\omega) = \pi\delta(\omega) + \sum_{k=-\infty}^{\infty} \left(\frac{2\sin(\pi k/2)}{k} \right) \delta(\omega - k\omega_0)$$

Example: Impulse Train

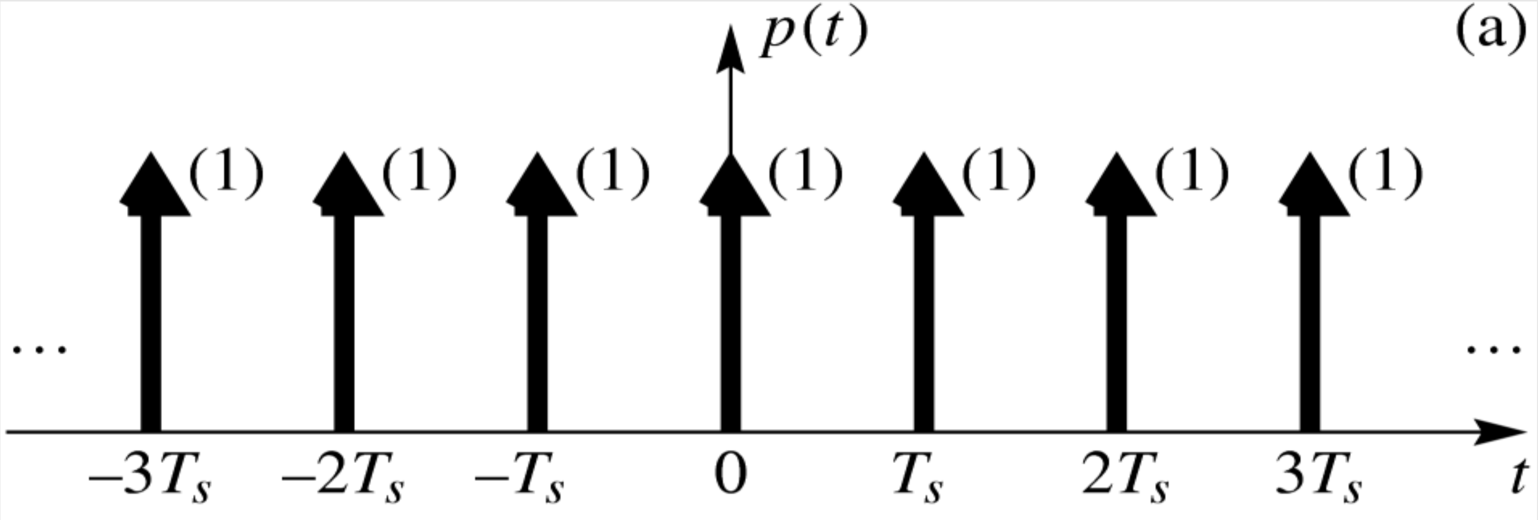


Fig.18.19

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$a_k = \frac{1}{T_0} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt$$

Example: Impulse Train *contd…*

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-jk\omega_s t}\,dt = \frac{1}{T_s}$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(\omega - k\omega_s)$$

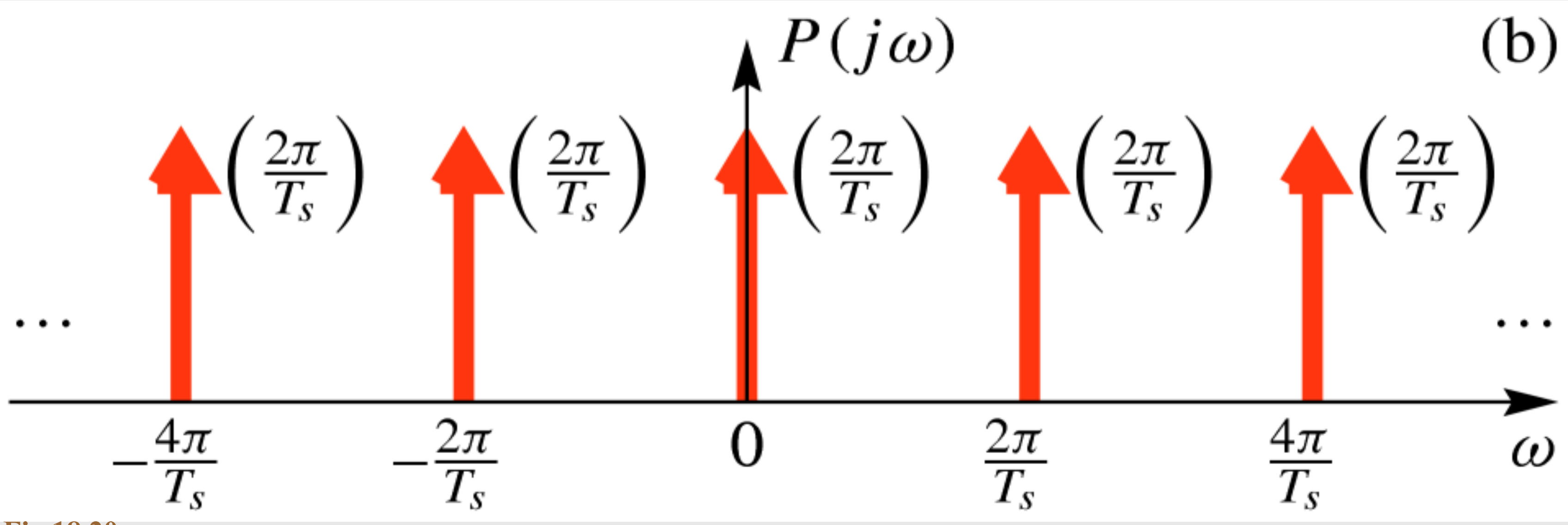


Fig.18.20

Table 1: Fourier Transform Pairs

$Time\ domain: x(t)$	$Frequency\ domain: X(j\omega)$
$e^{-at}u(t), \quad (a>0)$	$\dfrac{1}{(a+j\omega)}$
$e^{bt}u(-t), \quad (\mathrm{b}>0)$	$\dfrac{1}{(b-j\omega)}$
$u(t+T/2)-u(t-T/2)$	$\dfrac{\sin(\omega T/2)}{\omega/2}$
$\dfrac{\sin(\omega_b t)}{(\pi t)}$	$u(\omega+\omega_b)-u(\omega-\omega_b)$
$\delta(t)$	1
$\delta(t-t_d)$	$e^{-j\omega t_d}$

Table 2: Fourier Transform Pairs

$Time\ domain: x(t)$	$Frequency\ domain: X(j\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$A\cos(\omega_0 t+\phi)$	$\pi Ae^{j\phi}\delta(\omega-\omega_0)+\pi Ae^{-j\phi}\delta(\omega+\omega_0)$
$\cos(\omega_0 t)$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$
$\sin(\omega_0 t)$	$-j\pi\delta(\omega-\omega_0)+j\pi\delta(\omega+\omega_0)$
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$\sum_{k=-\infty}^{\infty} 2\pi a_k\delta(\omega-k\omega_0)$
$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\omega-k\omega)$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 11.1- 11.4 “Signal Processing First”, Prentice Hall, 2003