Discrete - Time Signals and Systems

Sinusoids

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Sinusoids

What are sinusoids?

- Both sine and cosine trigonometric functions are collectively known as sinusoids

What is the most used among the two?

- Cosine

Why?

- Cosine is an even function (Provides considerable mathematical advantage), moreover sine is a 90^{o} phase shifted version of cosine

Why are Sinusoids so important?

- Every possible signal can be obtained by summing up an infinite number of orthogonal basis signals with different amplitudes

Mathematically,

$$g(t) = c_1 \phi_1(t) + c_2 \phi_2(t) + c_3 \phi_3(t) + \ldots\ldots + c_\infty \phi_\infty(t) \quad = \sum_{n=1}^{\infty} c_n \phi_n(t)$$

The set of continuous signals $\{\phi_n(t)\}$, *are orthogonal to each other*, they are called basis functions or signals

- The above decomposition of signals is called a **Generalized Fourier series**
- More about why to decompose a signal, in later part

How are sinusoids related with Generalized Fourier series?

- There exists a large number of orthogonal signal sets which can be used as basis signals for generalized Fourier series, Like,
  - Exponential functions
  - Walsh functions
  - Bessel functions
  - Legendre polynomial functions
  - Laguerre functions
  - Jacobi Polynomials
  - Hermite Polynomials
  - Chebyshev Polynomials

  And, **sinusoids**

Sinusoids are the most popular basis signals

$$\{1, \cos(\omega_0 t), \cos(2\omega_0 t), ..., \cos(n\omega_0 t), \sin(\omega_0 t),$$
$$\sin(2\omega_0 t), ... \sin(n\omega_0 t), ...\}$$

- The above signal set, known as trigonometric Fourier series, satisfies the orthogonality requirement of a generalized Fourier series
- As it can be seen the signal set is completely sinusoids
- **They are easy to handle mathematically**
- Simple visualisation of the concept of frequency
- Some physically generated signals are sinusoids, like sounds from a tuning fork

Definitions of ‘sine’ and ‘cosine’ functions

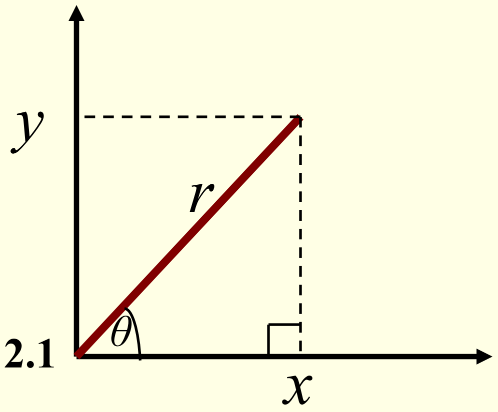


Fig 2.1

$$sin\theta = \frac{y}{r} \qquad y = r\, sin\theta$$

$$cos\theta = \frac{x}{r} \qquad x = r\cos\theta$$

- ‘***sine’*** and ***‘cosine’*** functions are defined through a right-angled triangle
- Both the trigonometric functions take an ***‘angle’*** as their argument, the units are ***‘radians’***
- The algebraic signs of *x* and *y*, come into play after the 1st quadrant, both the functions have a period of 2π

Graphical illustration to explain the mathematical properties of *'sine'* and *'cosine'* functions

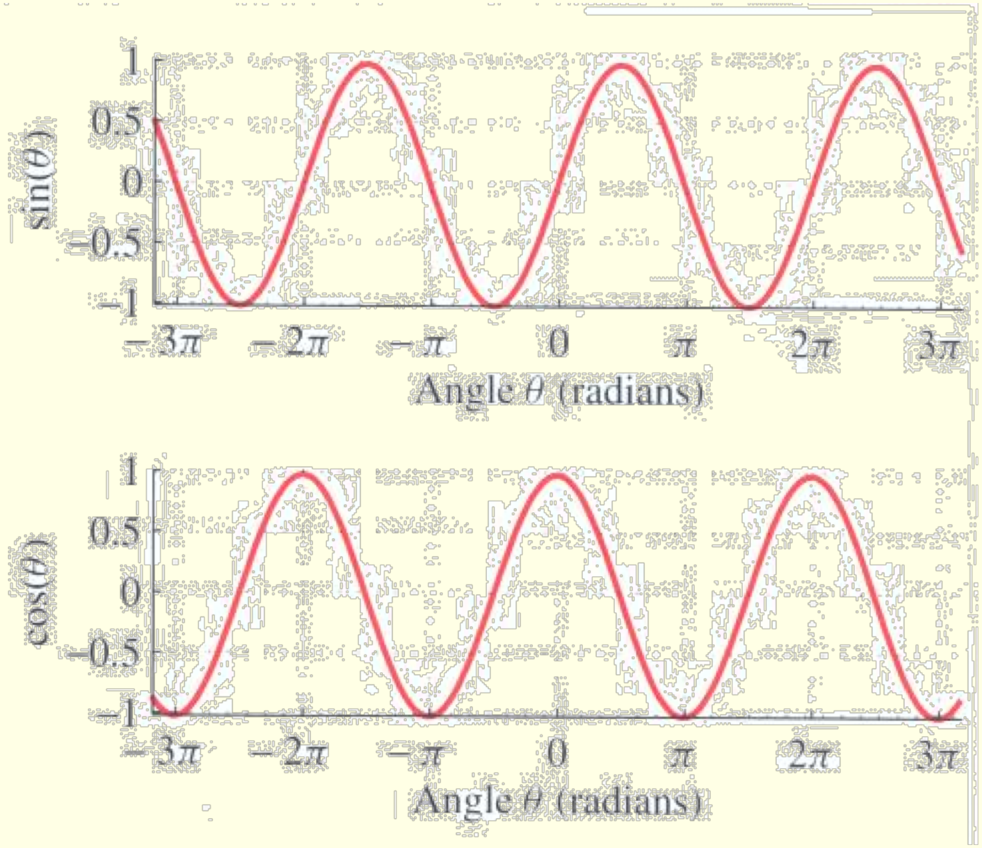


Fig 2.2

Both functions oscillate between +1 and -1, table 2.1 explains More properties based on the above plot

Basic properties of sine and cosine functions

Property	Equation
Equivalence	$sin\theta = cos(\theta - \pi/2)$ or $cos\theta = sin(\theta + \pi/2)$
Periodicity	$sin\theta = sin(\theta + 2\pi k)$ and $cos\theta = cos(\theta + 2\pi k)$ k, an integer
Evenness of cosine	$cos(-\theta) = cos(\theta)$
Oddness of sine	$sin(-\theta) = -sin(\theta)$
Zeros of sine	$sin(\pi k) = 0$, when k is an integer
Ones of cosine	$cos(2\pi k) = 1$, when k is an integer
Minus ones of cosine	$cos(2\pi(k + 0.5)) = -1$, when k is an integer

Table 2.1

Some Basic Trigonometric identities

Number	Equation
1	$sin^2\theta + cos^2\theta = 1$
2	$cos2\theta = cos^2\theta - sin^2\theta$
3	$sin2\theta = 2sin\theta\cos\theta$
4	$sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Table 2.2

Combining identities 1 and 2,

$$cos^2\theta = \frac{1}{2}(1 + \cos 2\theta), \quad sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

From calculus,

$$\frac{d\sin\theta}{d\theta} = \cos\theta, \qquad \frac{d\cos\theta}{d\theta} = -\sin\theta$$

The most general Mathematical formula for a cosine (sinusoidal) signal

'cosine' Trigonometric function, $\frac{\pi}{2}$ *phase-shift produces 'sine'*

Amplitude

Phase-shift, *units* ***'rad'***

$$x(t) = A\cos(\omega_0 t + \phi)$$

Radian Frequency, *units* ***'rad/sec'***

cyclic frequency, $f_0 = \frac{\omega_0}{2\pi}$ *units* ***'sec^{-1}'***

Continuous-Time, one-dimensional signal

Sinusoidal signal generated from formula1,

Frequency determines the periodicity

$$x(t) = 10\cos(2\pi(440)t - 0.4\pi)$$

Phase determines the Initial position of sinusoid

Maximum and minimum Values of cos(.) is +1 and -1 so, the maximum and minimum values of x(t) are '+amplitude' and '–amplitude', in this case it is +10 and -10

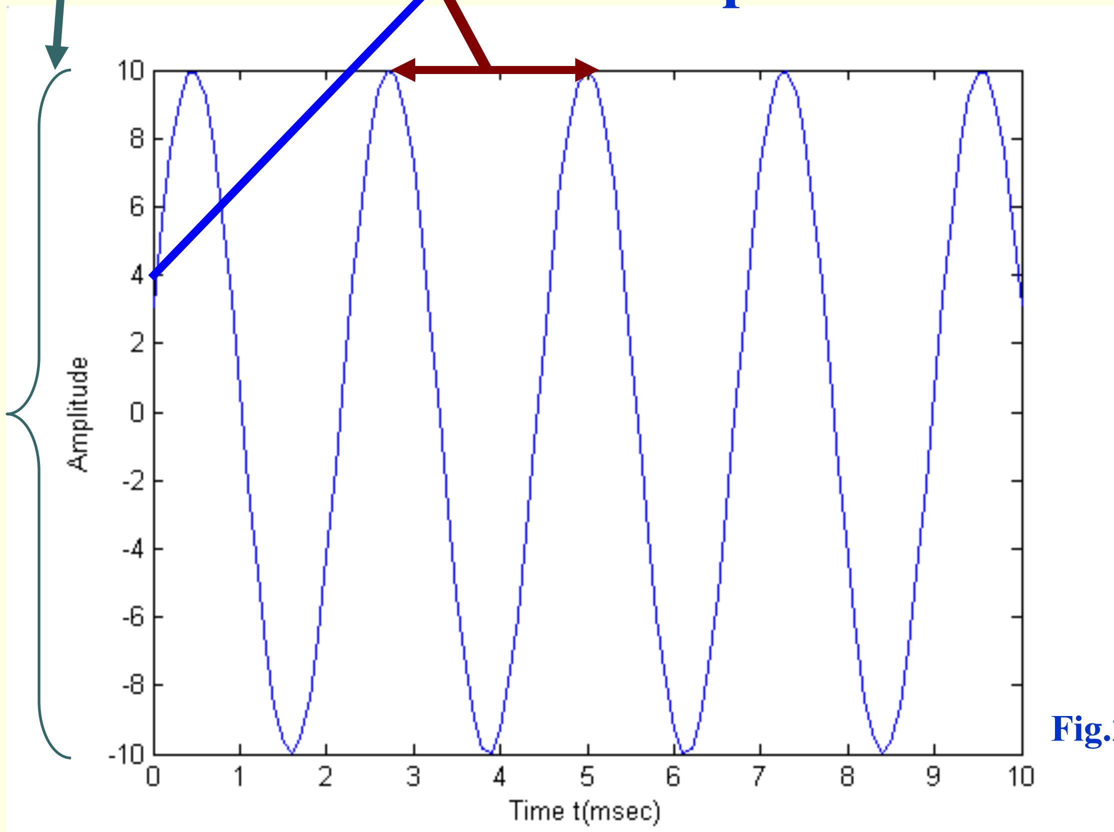


Fig.2.3

Sinusoidal signal generated from formula2,

The time gap between cycles increased

$$x(t) = 5\cos(2\pi(40)t - \pi)$$

Notice the change in Beginning position of Sinusoid, result of phase

Notice the changes in Maximum and minimum values. As the amplitude changes to 5, maximum and minimum values of x(t) also changed to '+5' and '-5'

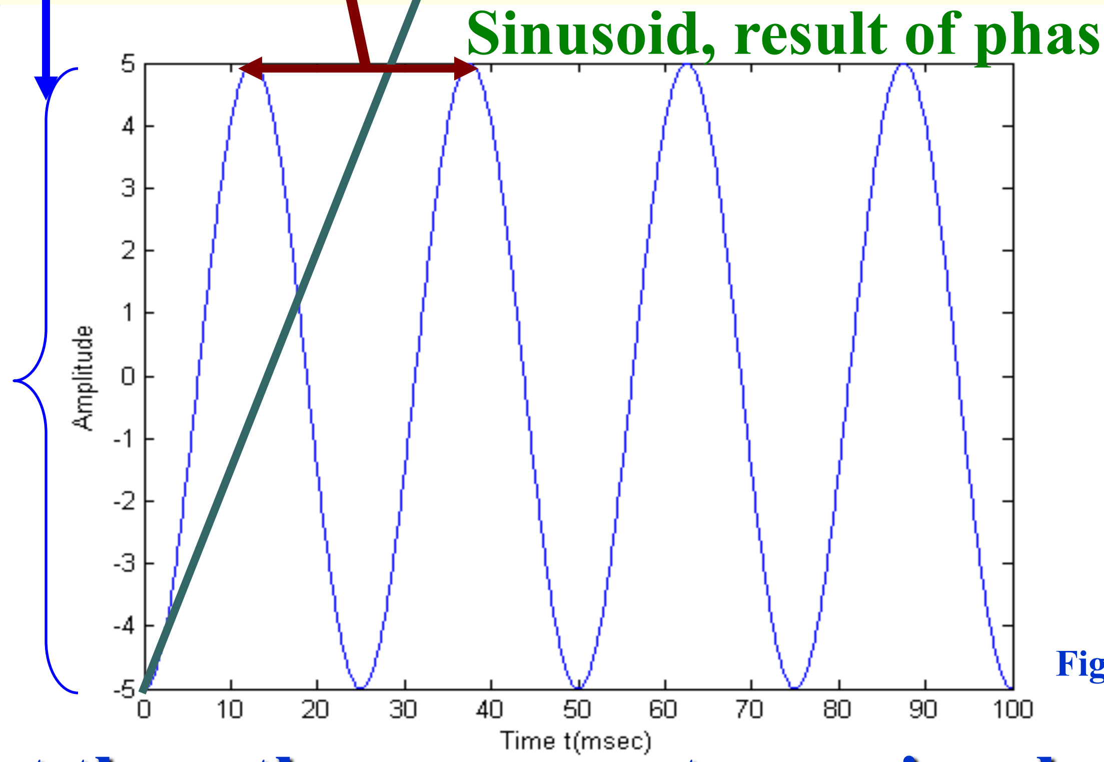


Fig.2.4

It can be concluded that these three parameters uniquely determine the sinusoidal waveform

Can sinusoids be generated physically?

- Yes. In fact, any kind of sound signal can be decomposed into sinusoids of different frequencies with different amplitudes

Tuning fork can generate a pure sinusoidal signal of, as presented by the mathematical formula $x(t) = A\cos(\omega_0 t + \phi)$

Fig.2.5

How can we know that this sound is a sinusoidal?

- **A plot of the sound signal can help us in visualization**

How can we get the plot of the sound signal we just heard?

- The digital recording of sound signal can be done with a microphone and computer equipped with A-to-D converter
- Microphone converts the sound into an electrical signal
- After sampling, quantization, and digitization a sequence of numbers are stored in the computer
- Appropriate plotting scheme is selected
- Plotting the sound helps us in clearly seeing that the sound signal produced by tuning fork is indeed a typical sinusoidal signal

The frequency of the above sound signal is 440 Hz, Fig 2.1 shows the plot

$$x(t) = A\cos(\omega_0 t + \phi)$$

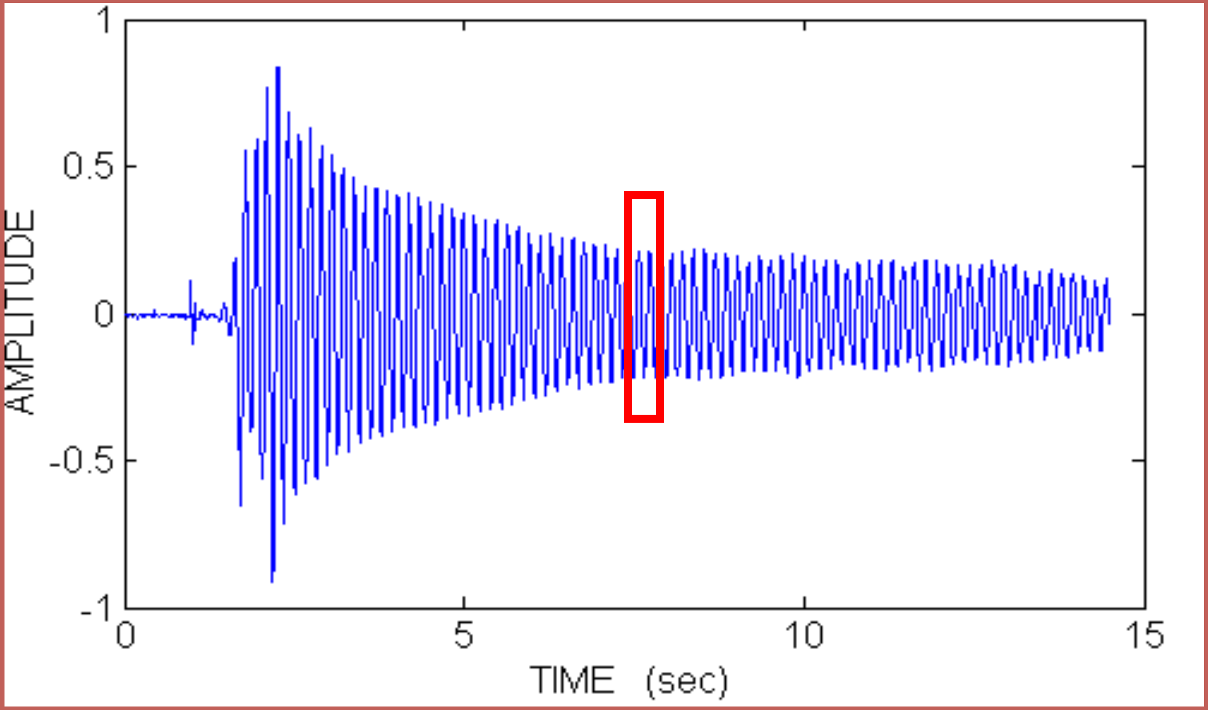


Fig.2.6

- The amplitude decays slowly, Ideally we should be able to see a constant sinusoidal signal, with a frequency of 440 Hz
- The amplified version, shows a signal that resembles a sinusoidal function

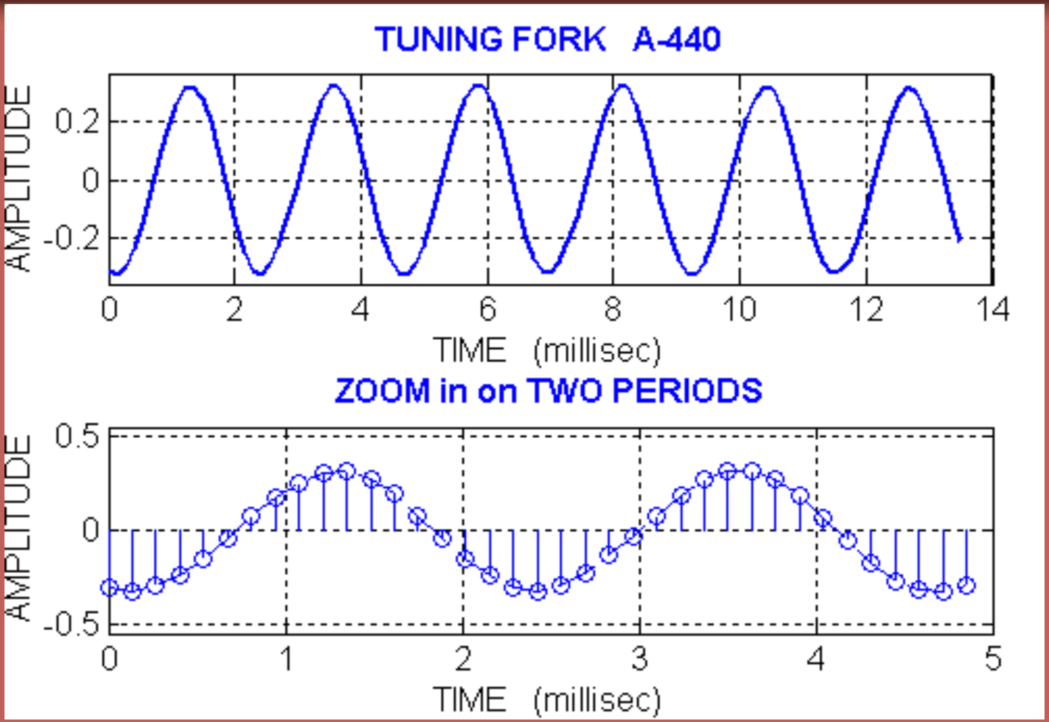


Fig.2.7

How can we tell that the frequency is 440, if we didn't see the marks on tuning fork ?

Frequency and period of signal are related as,

$$\omega_0 = \frac{2\pi}{T_0}$$

***T₀*=2.27*msec*, as seen from the zoom plot, so the frequency Is close to 440 Hz.**

There are some sounds with only one frequency component, Like, Bluebird, whose whistle produces a sinusoid of frequency **‘801.4Hz’**

Fig.2.8

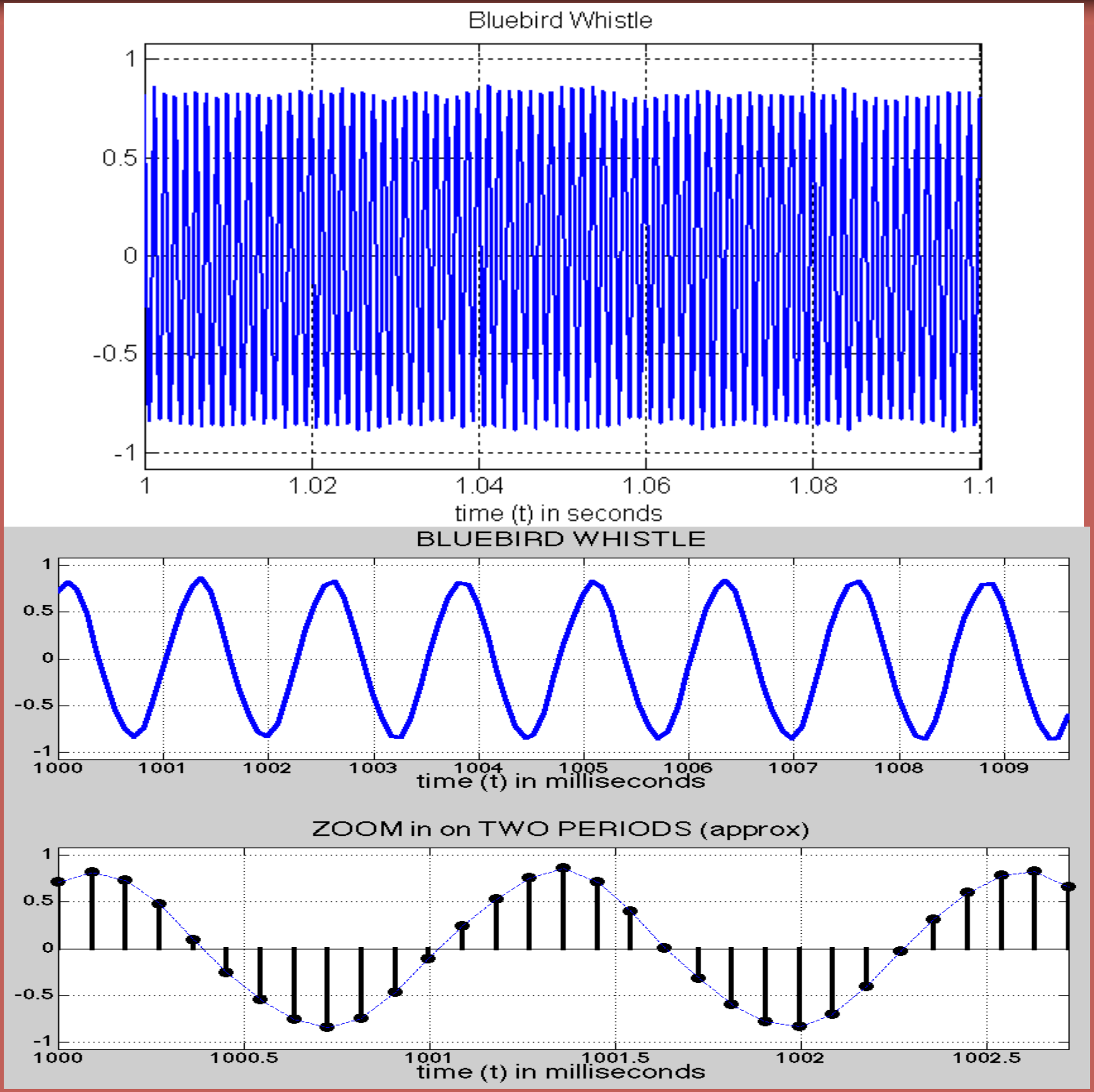
Blue Bird

Fig.2.9

Kitty-cat whistle

Frequency, 961.9Hz

Notice the difference in sound



Bluebird Whistle
1
0.5
0
-0.5
-1
1
1.02
1.04
1.06
1.08
1.1
time (t) in seconds
BLUEBIRD WHISTLE
1000
1001
1002
1003
1004
1005
1006
1007
1008
1009
time (t) in milliseconds
ZOOM in on TWO PERIODS (approx)
1000
1000.5
1001
1001.5
1002
1002.5
time (t) in milliseconds

Fig.2.10

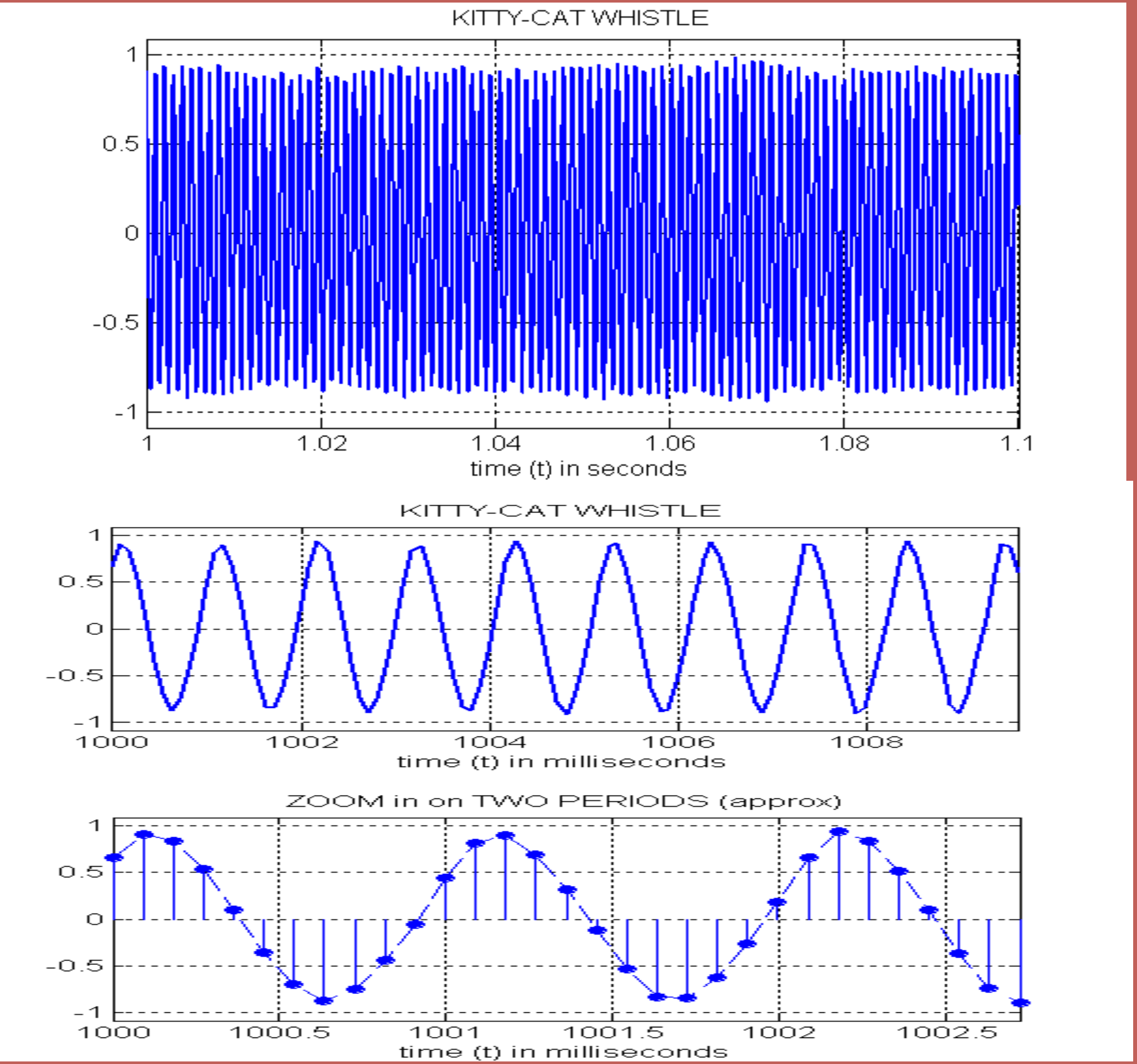

KITTY-CAT WHISTLE
1
0.5
0
-0.5
-1
1
1.02
1.04
1.06
1.08
1.1
time (t) in seconds
KITTY-CAT WHISTLE
1000
1002
1004
1006
1008
time (t) in milliseconds
ZOOM in on TWO PERIODS (approx)
1000
1000.5
1001
1001.5
1002
1002.5
time (t) in milliseconds

Fig.2.11

Relation between Frequency and Period

- **Sinusoids are periodic signals,** $x(t) = x(t + T_0), T_0$ is the period
- **The period of sinusoid, denoted by is the length of one cycle of sinusoid**

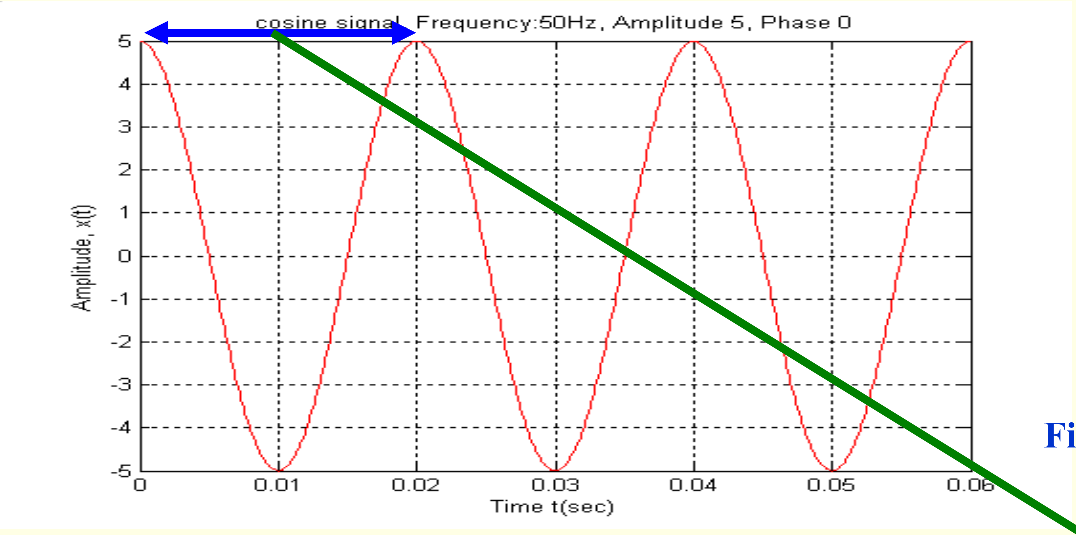


Fig.2.12

For the above signal, $\omega_0 = 2\pi(50),\ T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\pi(50)} = 0.02\,\text{sec}$

What is the most obvious effect of Frequency?

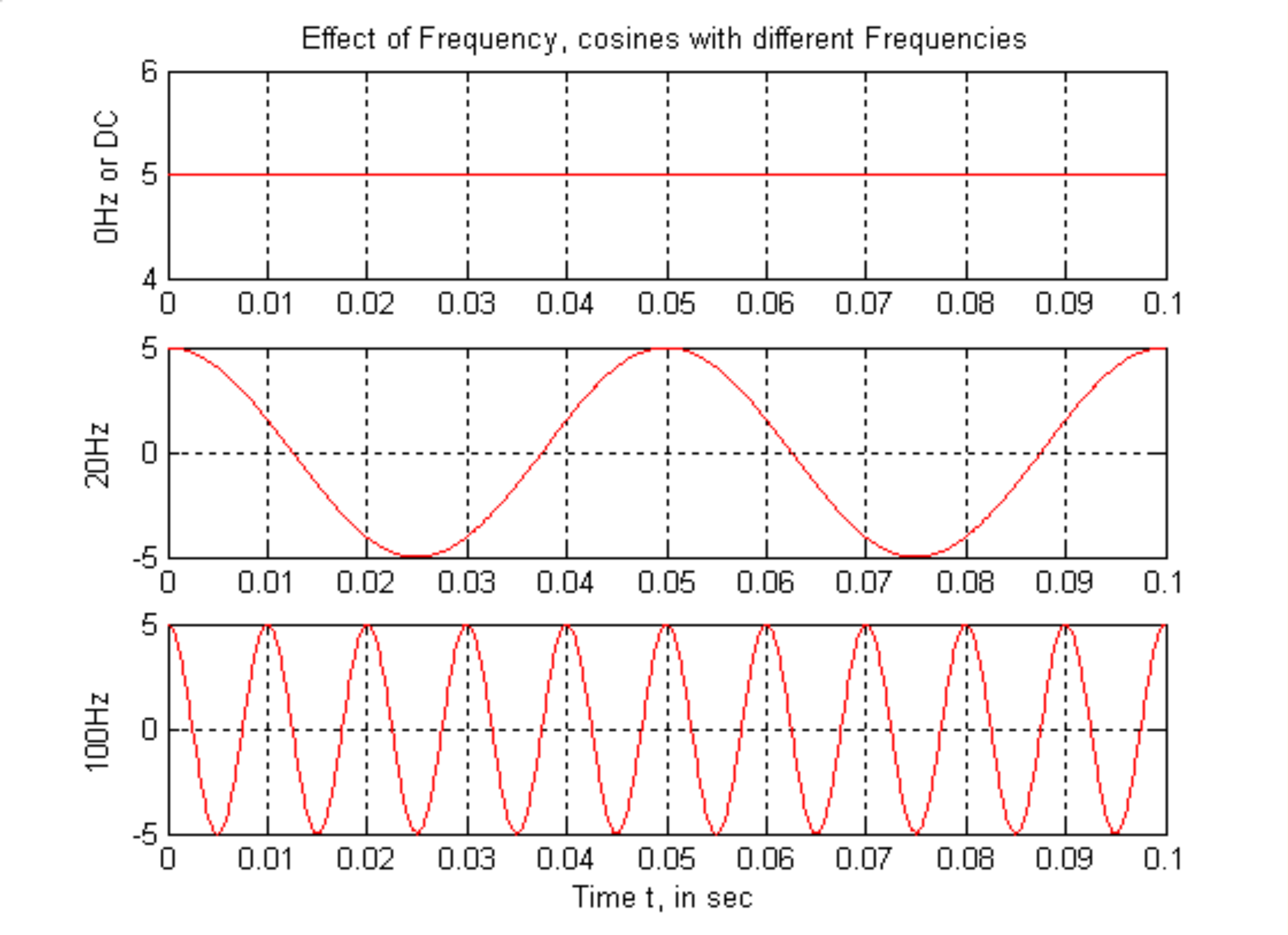


Fig.2.13

The higher the frequency, the more the signal varies rapidly with time, Similarly less time period means more frequency

Concept of Time-shift

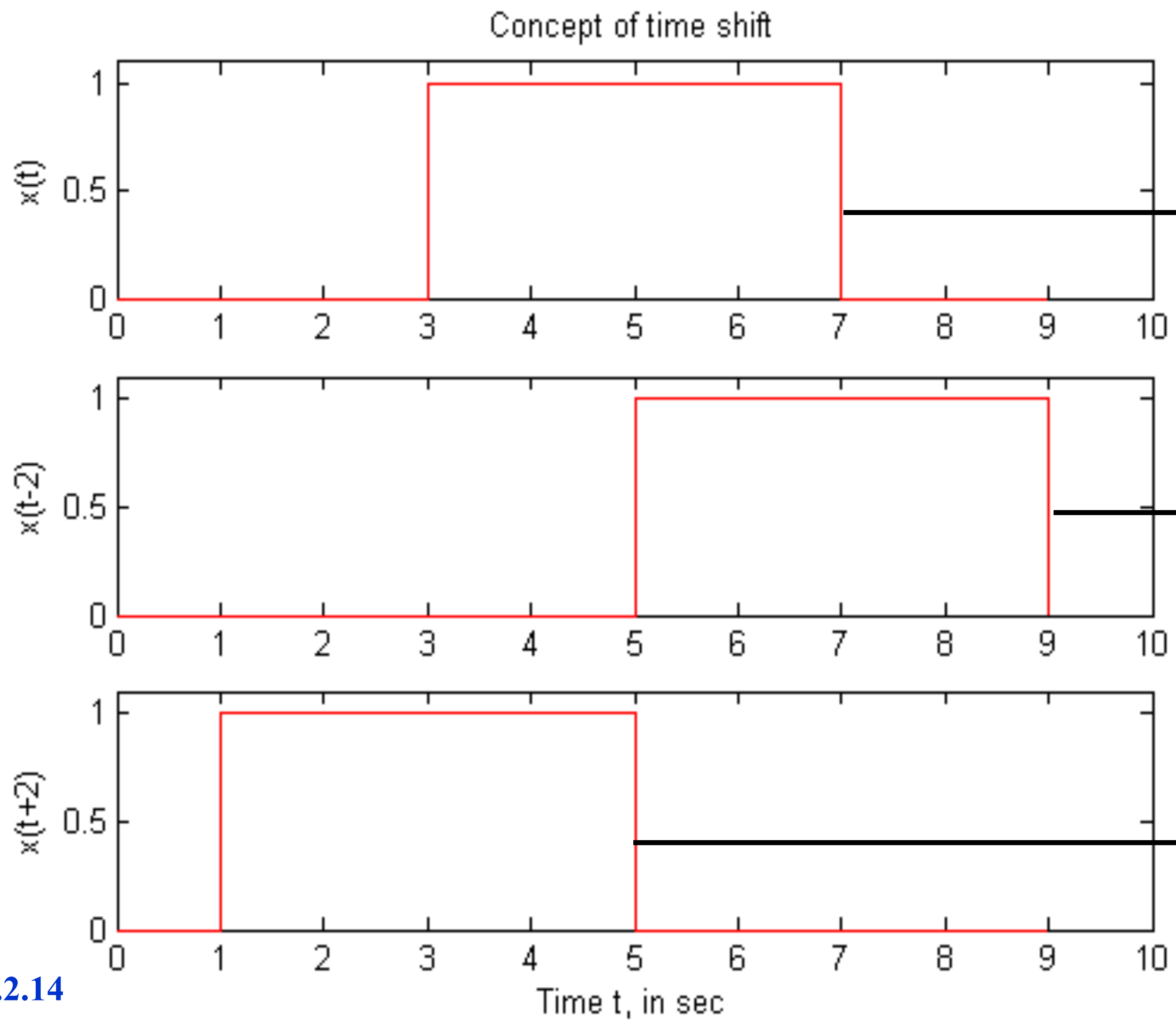


Fig.2.14

Original Signal,

$$x(t) = 1 \quad 3 \le t \le 7$$
$$= 0 \quad elsewhere$$

Right-shifted signal, notice the argument, '***t-2***'

Left-shifted signal, notice the argument, '***t+2***'

Concept of Phase-shift

$x(t) = \cos(2\pi(20)t + \phi)$,**cosine of frequency 20Hz and amplitude '1'**

Phase is varied

Notice the Shifts in maxima and minima of cosine wave, also notice the sine waveform for a phase shift of

$\phi = 0$

$\phi = \frac{\pi}{4}$

$\phi = \frac{\pi}{2}$

$\phi = \pi$

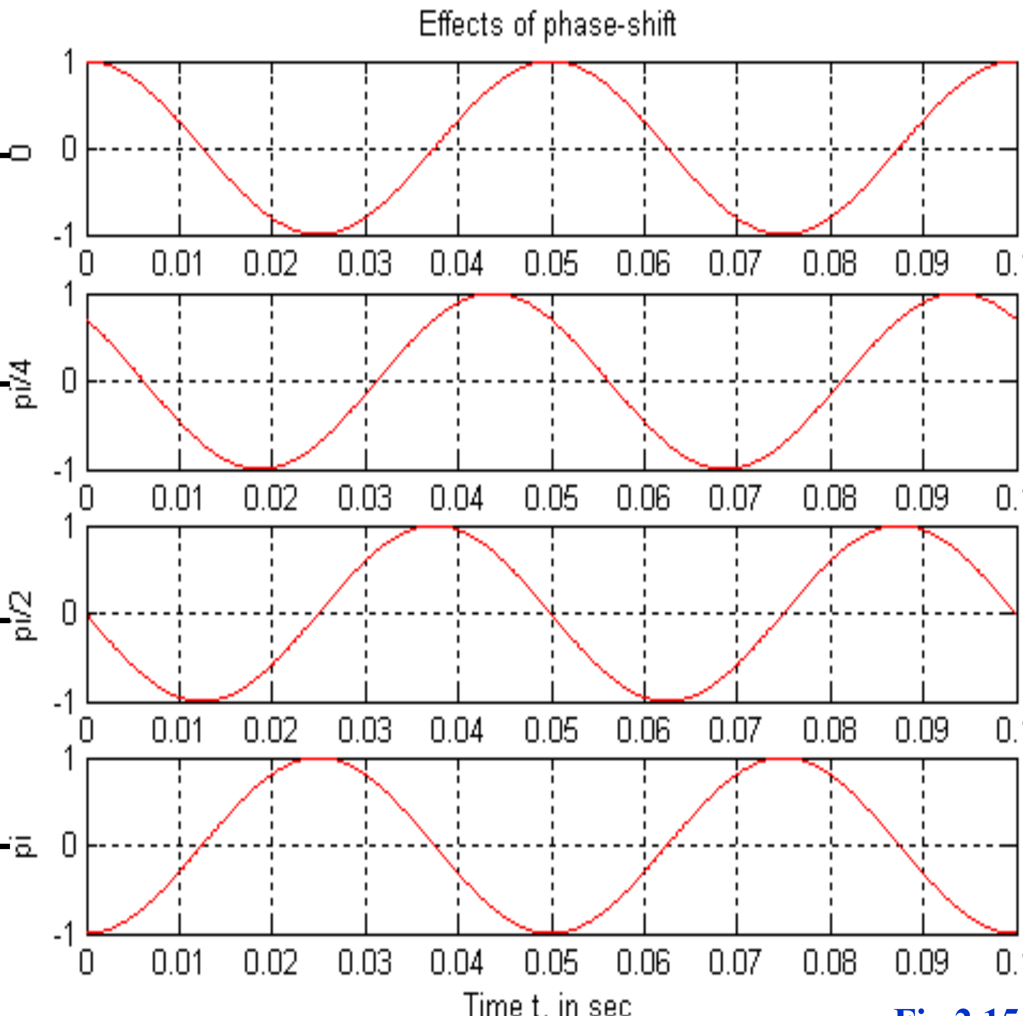


Fig.2.15

$$x(t) = \cos(2\pi(20)t + \frac{\pi}{2})$$
$$= \sin(2\pi(20)t)$$

Relation between Phase-shift and Time-shift

- **As it was demonstrated in the previous page, because of periodicity, one could get the same effect through time shift**
- **There exists mathematical relation between time and phase shifts. For a standard sinusoidal signal,**

$$x(t) = Acos(\omega_0 t + \phi)$$

$$Let\ x_0(t)\ be\ the\ signal\ with\ zero\ phase-shift$$

$$x_0(t) = A\cos(\omega_0 t), time-shifted\ x_0(t)$$

$$x_0(t - t_1) = A\cos(\omega_0(t - t_1)) = A\cos(\omega_0 t - \omega_0 t_1)$$

$$Compare\ this\ with\ standard\ equation,$$

$$\phi = -\omega_0 t_1, t_1 = \frac{-\phi}{\omega_0} = \frac{-\phi}{2\pi f_0}$$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, "Signal Processing First", Prentice Hall, 2003