Discrete - Time Signals and Systems

Spectrum Representation

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Spectrum

- **A compact representation of the frequency content of a signal that is composed of sinusoids**

- **Any complicated signal can be constructed out of sums of sinusoidal signals of *different* amplitudes, phases, and frequencies. *spectrum is simply the collection of these parameters***

Two-sided spectrum

The most general method for producing new signals from sinusoids is the '*additive linear combination*'

Mathematically the signal can be represented as,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k), \qquad \textbf{Eq.1}$$

Where each amplitude, phase, and frequency may be choosen independently

$From\ the\ properties\ of\ sinuosids,$

$$s(t) = A\cos(2\pi f_0 t + \phi) = \Re e\left\{ Xe^{j2\pi f_0 t} \right\}$$

$$where, X = Ae^{j\phi}$$

$$\because Xe^{j2\pi f_0 t} = Ae^{j\phi}e^{j2\pi f_0 t} = Ae^{j(2\pi f_0 t + \phi)}$$

$$\Re e\left\{ Xe^{j2\pi f_0 t} \right\} = \Re e\left\{ A\left[\cos(2\pi f_0 t + \phi) + j\sin(2\pi f_0 t + \phi)\right]\right\}$$

$$= A\cos(2\pi f_0 t + \phi) = s(t)$$

$eq.1,\ x(t),\ can\ now\ be\ written\ as,$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{ X_k e^{j2\pi f_k t} \right\}$$

$$where,\ X_0 = A_0,\ and\ X_k = A_k e^{j\phi_k}$$

Continue **Two-sided spectrum,…...**

Using inverse Euler formula,

$$\Re e\left\{X_k e^{j2\pi f_k t}\right\} = \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t}$$

x(t) can now be written as,

$$x(t) = X_0 + \sum_{k=1}^{N}\left\{\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t}\right\}$$

The above equation shows that each sinusoid in the sum, decomposes into two rotating phasors, one with positive frequency 'f_k' and '$-f_k$'

Two-sided spectrum: Definition

The signal *x(t)* as defined in *eq.1,* is composed of 2*N*+1 frequencies with 2*N*+1 complex amplitudes, this kind of representation is known as two-sided spectrum of a signal

The mathematical notation for two-sided spectrum is,

$$\left\{ \begin{array}{l} \left(0, X_0\right), \left(f_1, \frac{1}{2} X_1\right), \left(-f_1, \frac{1}{2} X_1^*\right), \ldots\ldots\ldots. \\ \qquad\qquad\qquad \left(f_k, \frac{1}{2} X_k\right), \left(-f_k, \frac{1}{2} X_k^*\right), \ldots\ldots\ldots. \end{array} \right\}$$

Continue Two-sided spectrum: Definition,……

Each pair $\left(f_k, \frac{1}{2}X_k\right)$, indicates the size and relative phase of the sinusoidal component contributing at frequency f_k

The above representation is also known as *frequency-domain* representation of the signal, the original signal *x(t)* is known as *time-domain* representation

Example1: Two-sided spectrum

$$x(t) = 4\cos(2\pi(5)t)$$

Using Euler's Identity,

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$4\cos(2\pi(5)t) = \tfrac{4}{2}e^{j2\pi(5)t} + \tfrac{4}{2}e^{-j2\pi(5)t}$$

There is no constant or DC term in the above equation, The two-sided spectrum of the signal is the sum of two rotating phasors represented by

$$\{(0,0),\ (5,2),\ (-5,2)\}$$

Comments & Graphical plot of the spectrum

$$\{(0,0),\ (5,2),\ (-5,2)\}$$

The DC term, with frequency '0', has no contribution in the signal

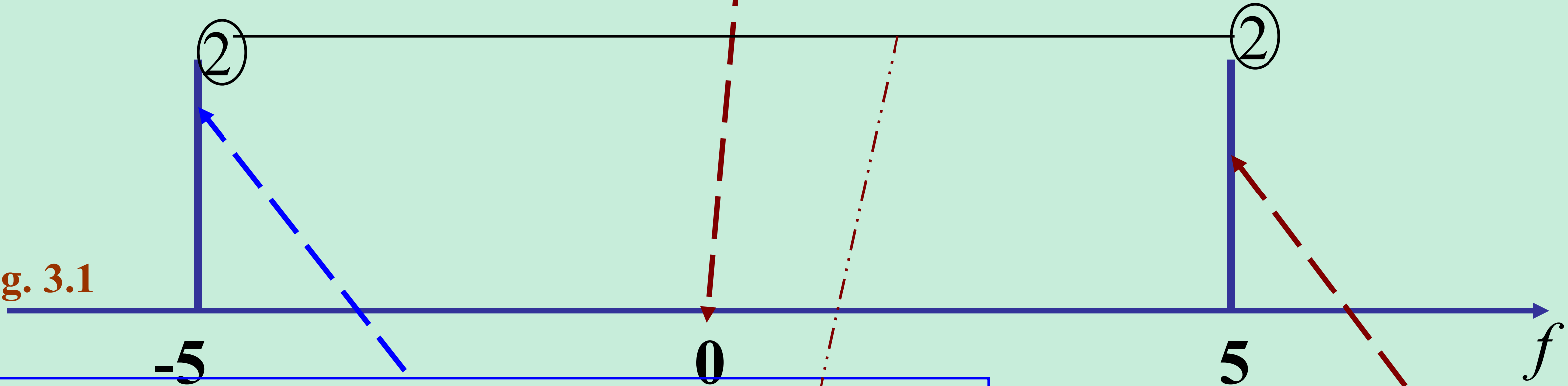


Fig. 3.1

Negative frequency '5' with amplitude '2'

Positive frequency '5' with amplitude '2'

Note that the amplitude is a real number as there is no phase term in sinusoidal signal

What is negative frequency?

- By definition, frequency (number of repetitions per second) is a positive quantity
- Mathematically, frequency is an absolute quantity

$$\cos(\omega_0 t) = \cos(-\omega_0 t), \quad also,$$

$$e^{\pm j\omega_0 t} = \cos(\omega_0 t) \pm j\sin(\omega_0 t)$$

which suggests that frequency of the sinusoid is an absolute quantity $|\omega_0|$

In conclusion, the presence of a spectral component at $-\omega_0$, has only *mathematical significance*, suggesting the presence of $e^{-j\omega_0 t}$, in the series, which is required to express a sinusoid in exponential form

In a practical sense the negative frequency can symbolize the motion. Towards you (positive), away from you (negative)

Example 2: With phase term

$$x(t) = 4\sin(2\pi(5)t),\ \textit{This can be written as}$$

$$x(t) = 4\cos(2\pi(5)t + \pi/2),\ \textit{Euler identity},$$

$$x(t) = \frac{4}{2}\left\{e^{j2\pi(5)t}e^{j\pi/2} + e^{-j2\pi(5)t}e^{-j\pi/2}\right\},\ \textit{the spectrum is},$$

$$\left\{(5, 2e^{j\pi/2}), (-5, 2e^{-j\pi/2})\right\}$$

The only differnce is the phase, associated with amplitude

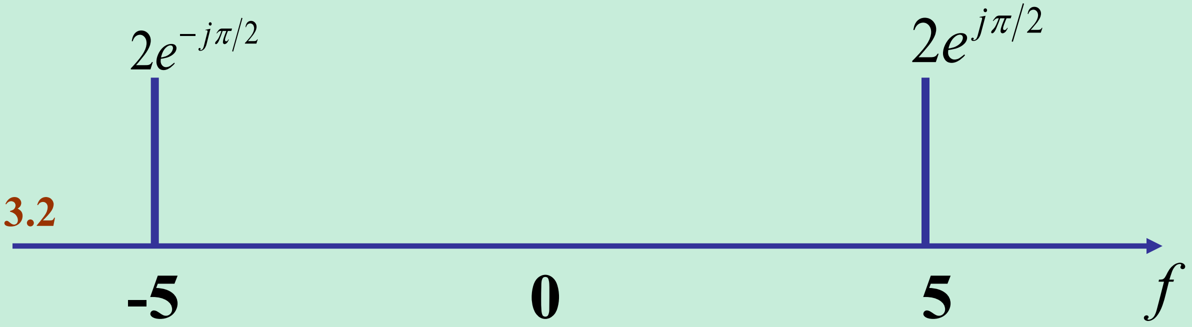


Fig. 3.2

Example 3: Sum of a constant and 2 sinusoids

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Applying Euler identity,

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$+ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

The DC component value corresponds to a frequency of '0'

Mathematically the spectrum,

$$\{(0,10),(100,7e^{-j\pi/3})(-100,7e^{j\pi/3}),(250,4e^{j\pi/2}),(-250,4e^{-j\pi/2})\}$$

Graphical plot of the spectrum

'DC' term

Amplitude along with phase

Notice that the phase for the same positive and negative frequency

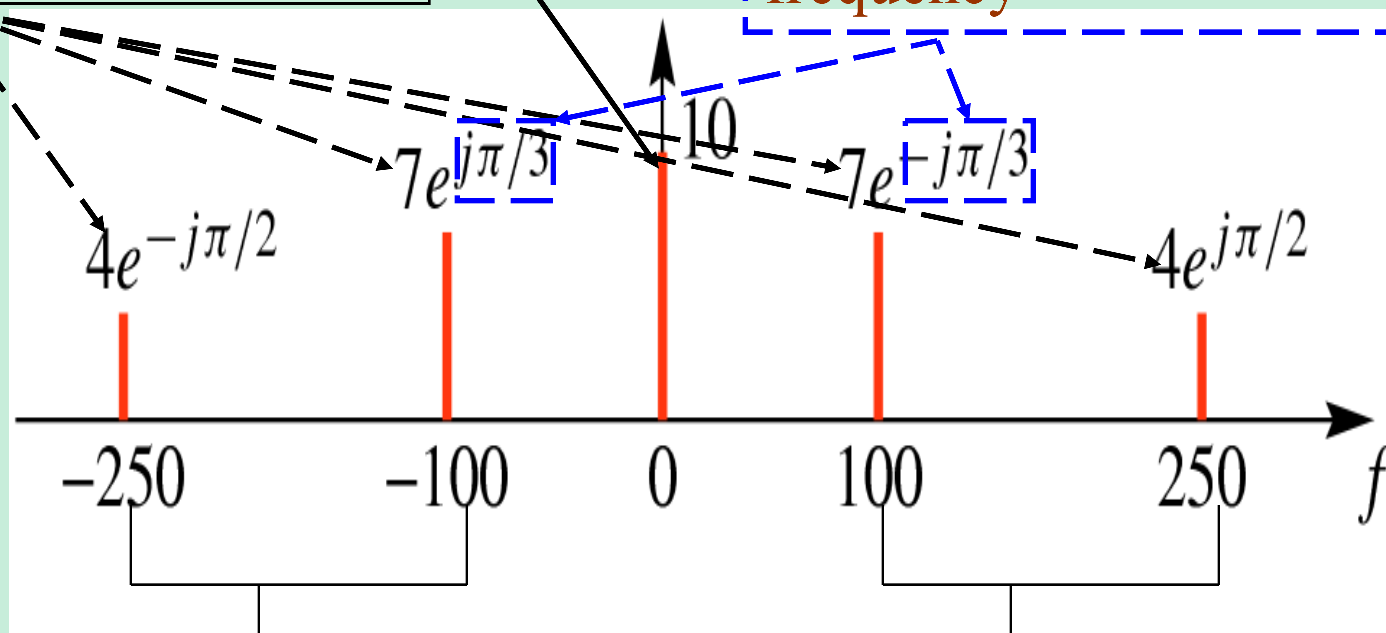


Fig. 3.3

Negative frequencies

Positive frequencies

A Change in Notation

The spectrum involves the multiplication of every X_k with ½, except for X_0 , this multiplication factor can be eliminated for mathematical simplicity, by introducing a new notation,

$$a_k = \begin{cases} A_0 & for\ \ k = 0 \\ \dfrac{1}{2} A_k e^{j\phi_k} & for\ \ k \neq 0 \end{cases}$$

The spectrum set will now denoted as a set of (f_k, a_k), subsequently The Fourier series follows the same notation,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 kt}$$

Multiplication of Sinusoids

When two sinusoids are multiplied, the resulting signal should also be represented as an additive linear combination of complex exponential signals.

When two sinusoids with different frequencies are multiplied, an interesting audio effect called *'beat note'* is observed

A sinusoid of 660 Hz sounds like, 

Graphical illustration of a sinusoid of 660Hz

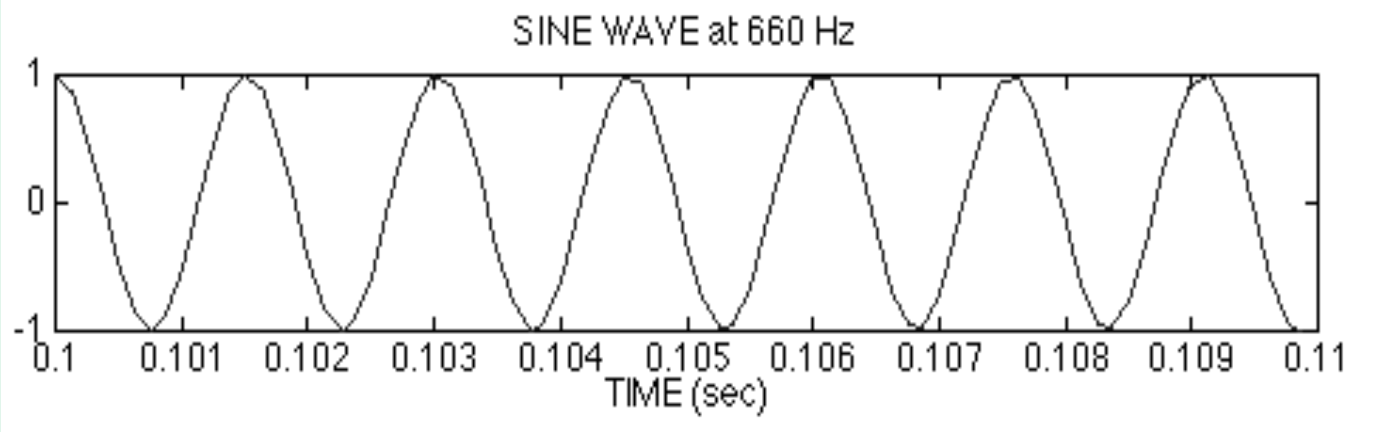


Fig. 3.4

The effect of multiplying the above sinusoid by another one with a frequency of 12 Hz

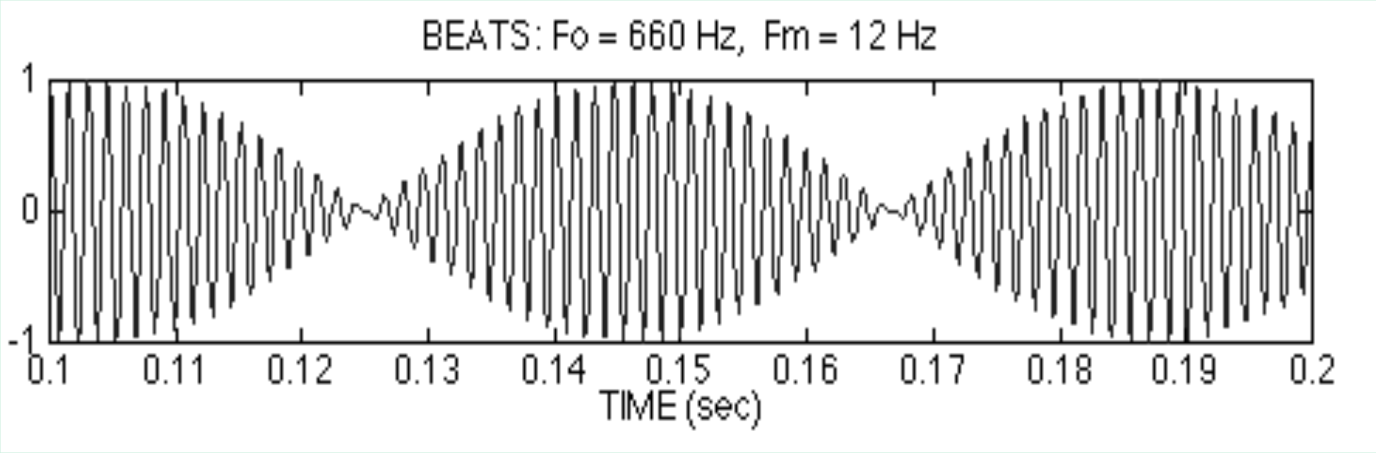


Fig. 3.5

Spectrum of Product

Consider the signal obtained by the product of two sinusoids At 5 Hz and ½ Hz

$$x(t) = \cos(2\pi(1/2)t)\cos(2\pi(5)t)$$

$$\cos(2\pi(1/2)t) = \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)$$

$$\cos(2\pi(5)t) = \left(\frac{e^{j\pi 10t} + e^{-j\pi 10t}}{2}\right)$$

$$x(t) = \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)\left(\frac{e^{j\pi 10t} + e^{-j\pi 10t}}{2}\right)$$

$$x(t) = \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2} \right) \left(\frac{e^{j\pi 10t} + e^{-j\pi 10t}}{2} \right)$$

$$= \frac{e^{j\pi 11t}}{4} + \frac{e^{-j\pi 9t}}{4} + \frac{e^{j\pi 9t}}{4} + \frac{e^{-j\pi 11t}}{4}$$

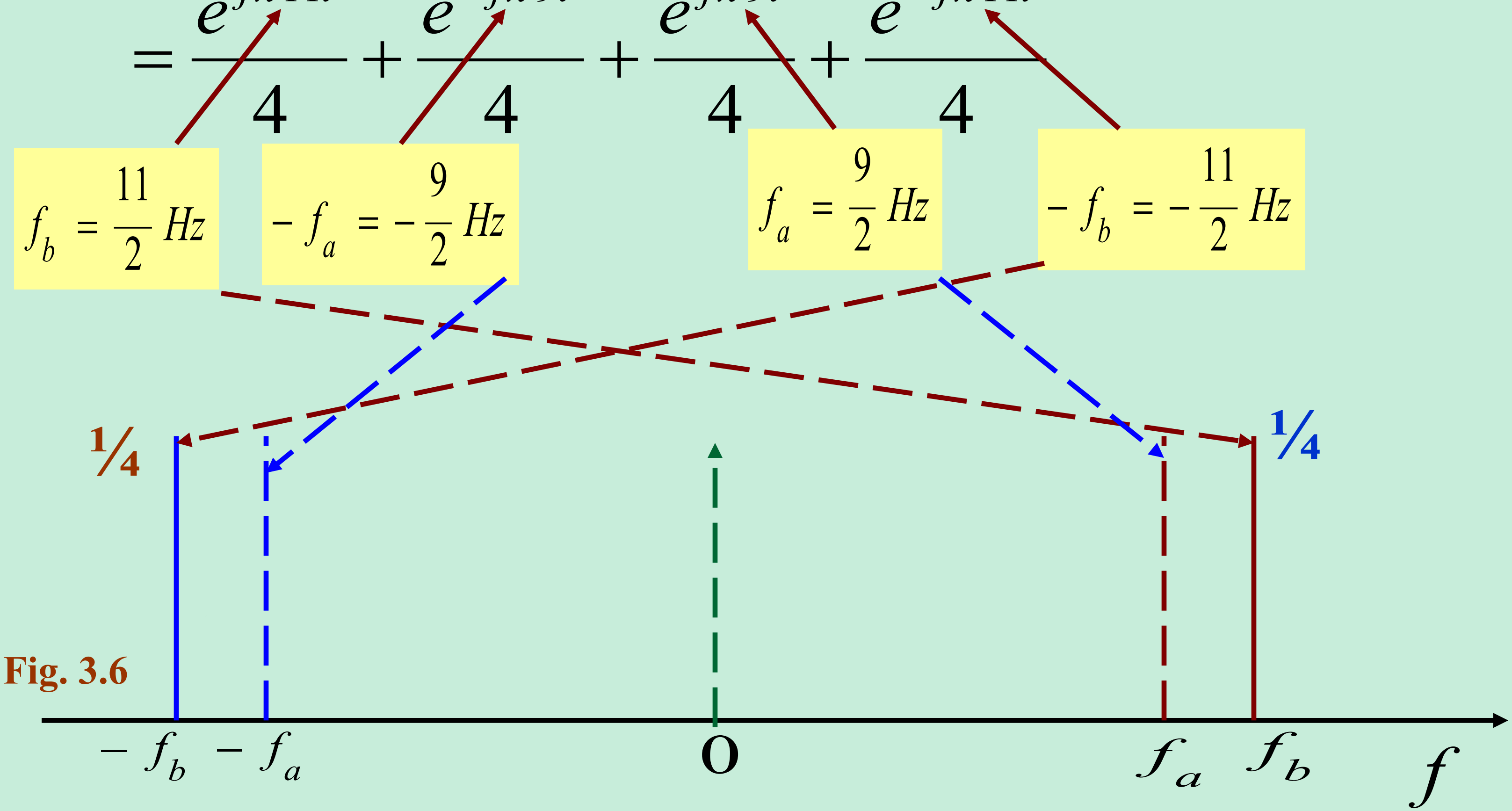


Fig. 3.6

Addition of Sinusoids

Beat notes can also be produced by adding two sinusoids whose frequencies are closely spaced, the central idea is that multiplying sinusoids is equivalent to addition, as shown in the phasors.

$$Let\ the\ frequencies\ of\ sinusoids\ be\ f_1\ and\ f_2,$$

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$$

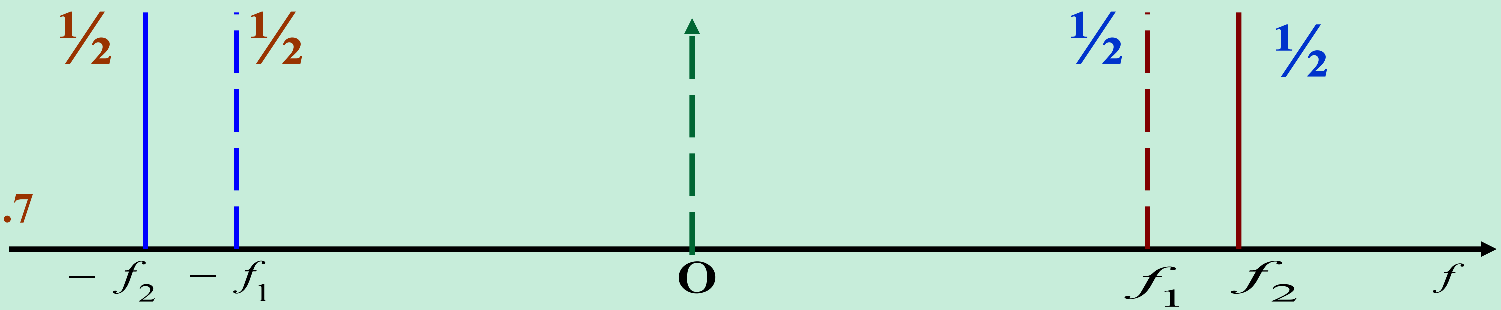


Fig. 3.7

As demonstrated earlier, each sinusoid has two components, one at Positive and another at negative frequency, totaling 4 components

Using the complex exponential representation, *x(t)* can be written as a product of two cosines

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t)$$

$$Let\ the\ center\ frequency\ be,\ f_c = \frac{(f_1 + f_2)}{2}$$

$$and\ deviation\ frequency\ be,\ f_\Delta = \frac{(f_2 - f_1)}{2}, \quad f_2 \geq f_1$$

$$\therefore f_1 = f_c - f_\Delta \ and \ f_2 = f_c + f_\Delta$$

$$x(t) = \Re e\left\{e^{j2\pi f_1 t}\right\} + \Re e\left\{e^{j2\pi f_2 t}\right\}$$

$$= \Re e\left\{e^{j2\pi(f_c - f_\Delta)t} + e^{j2\pi(f_c + f_\Delta)t}\right\}$$

$$= \Re e\left\{e^{j2\pi f_c t}\left(e^{-j2\pi f_\Delta t} + e^{j2\pi f_\Delta t}\right)\right\}$$

$$= \Re e\left\{e^{j2\pi f_c t}\left(2\cos(2\pi f_\Delta t)\right)\right\}$$

$$= 2\cos(2\pi f_c t)\cos(2\pi f_\Delta t)$$

Thus, the equivalence of product and sum is proved

To obtain beats with multiplication, the frequency difference between the sinusoids should be high

As seen from the above equation the frequencies in the sum should be closely spaced, so that the difference between the terms, f_c and f_Δ is high

$$f_c = 200Hz \qquad f_\Delta = 20Hz$$

$$\textit{The time interval between nulls is } \frac{1}{2}\left(1/f_\Delta\right) = 25m\sec$$

The variation of f_Δ causes the resulting Signal to fade in and out, this effect is called *'beating'*, in other words the envelope is formed By the f_Δ component

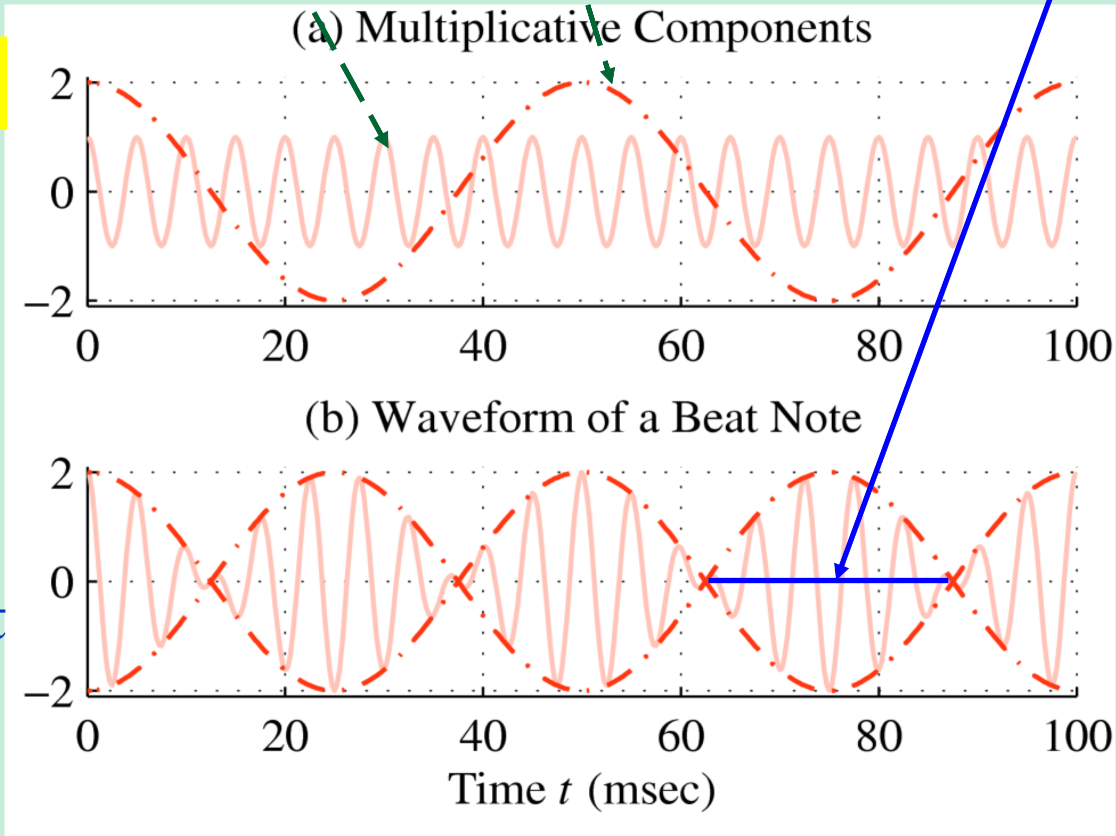


Fig. 3.8

$f_c = 200Hz \qquad f_\Delta = 9Hz$

The time interval between nulls is $\frac{1}{2}\left(1/f_\Delta\right) = 55.6m\sec$

Notice the effect of changing the deviation frequency f_Δ, the envelope changes more slowly, an increase in time difference between the nulls is also seen

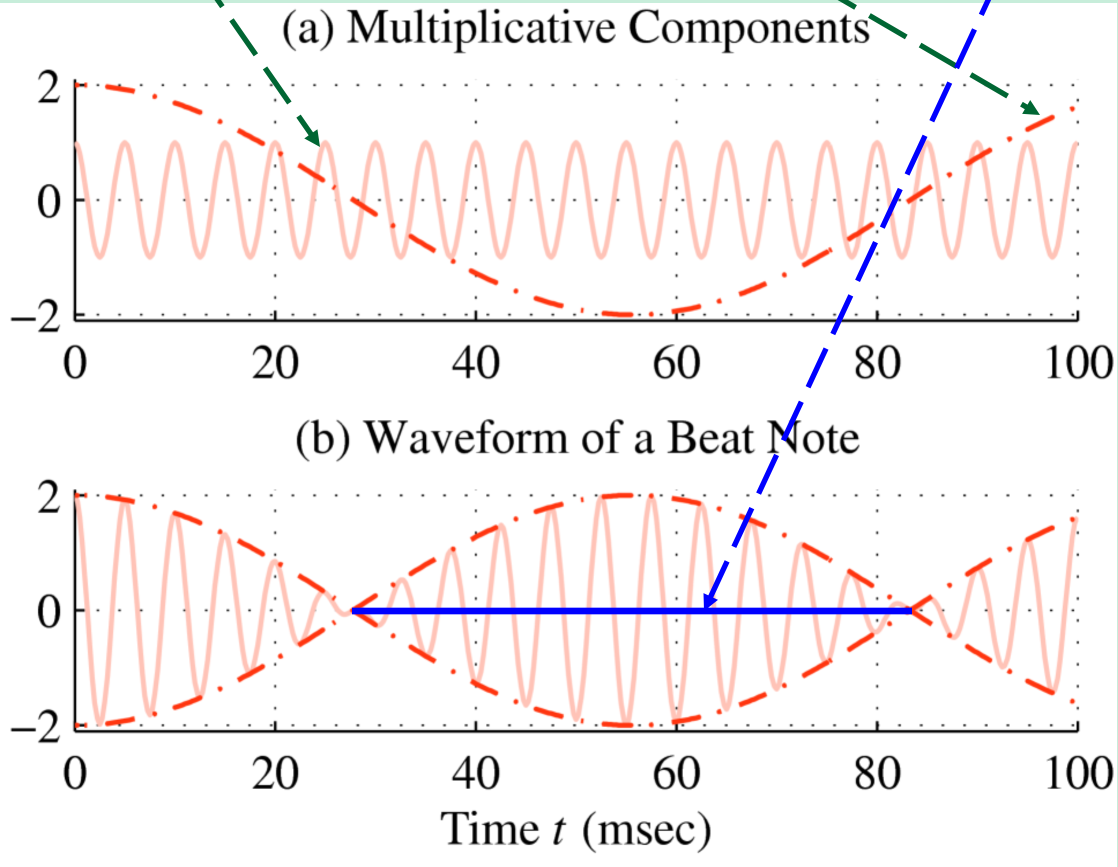


Fig. 3.9

Amplitude Modulation

Multiplying sinusoids results in a useful modulation scheme called '*amplitude modulation*', this technique is used to broadcast AM radio

The AM signal is a product of the form,

$$x(t) = v(t)\cos(2\pi f_c t)$$

f_c **, is called the carrier frequency, it is much higher than the signal frequency**

Spectrum helps in understanding how modulation works

Example: Amplitude Modulation

$$Let,\ v(t) = 5 + 4\cos(40\pi t)\, and,$$

$$carrier\ \ frequency\ f_c = 200Hz$$

$$x(t) = \left[5 + 4\cos(40\pi t)\right]\cos(400\pi t)$$

$$= 5\cos(400\pi t) + 4\cos(40\pi t)\cos(400\pi t)$$

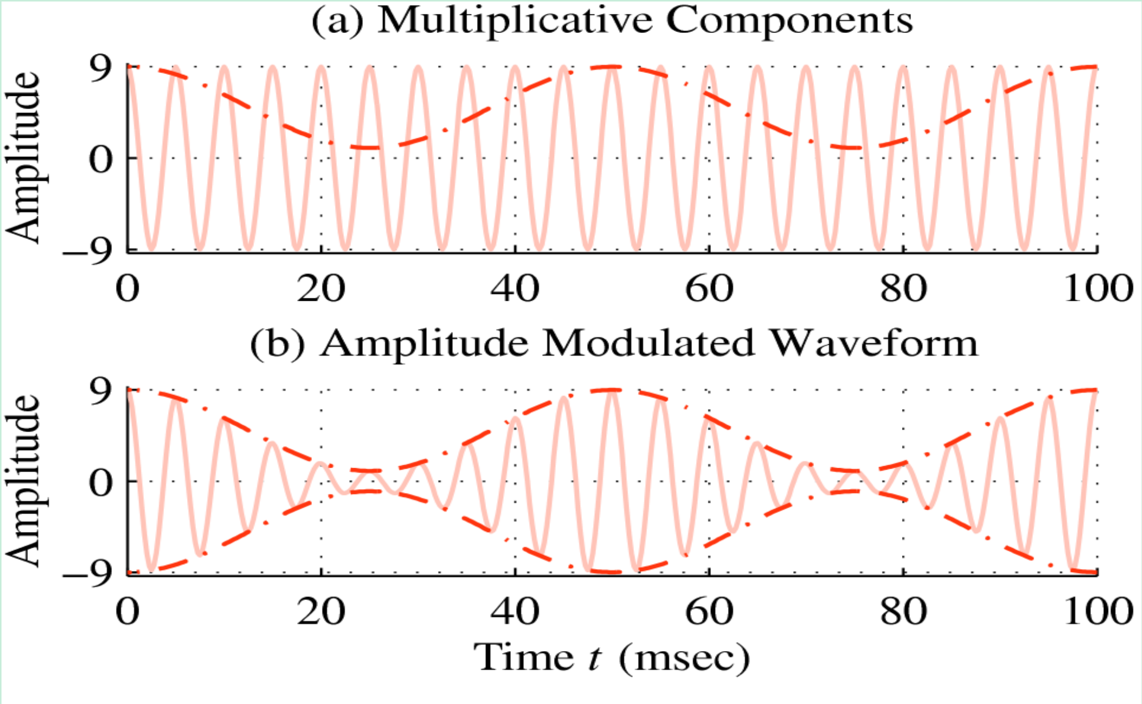
$$converting\ the\ above\ equation\ into\ exponentials,$$

$$x(t) = \frac{5}{2}e^{j400\pi t} + e^{j440\pi t} + e^{j360\pi t} + \frac{5}{2}e^{-j400\pi t}$$

$$+ e^{-j440\pi t} + e^{-j360\pi t}$$

Except for a 'DC' component the AM signal is nothing but a beat waveform

Time domain

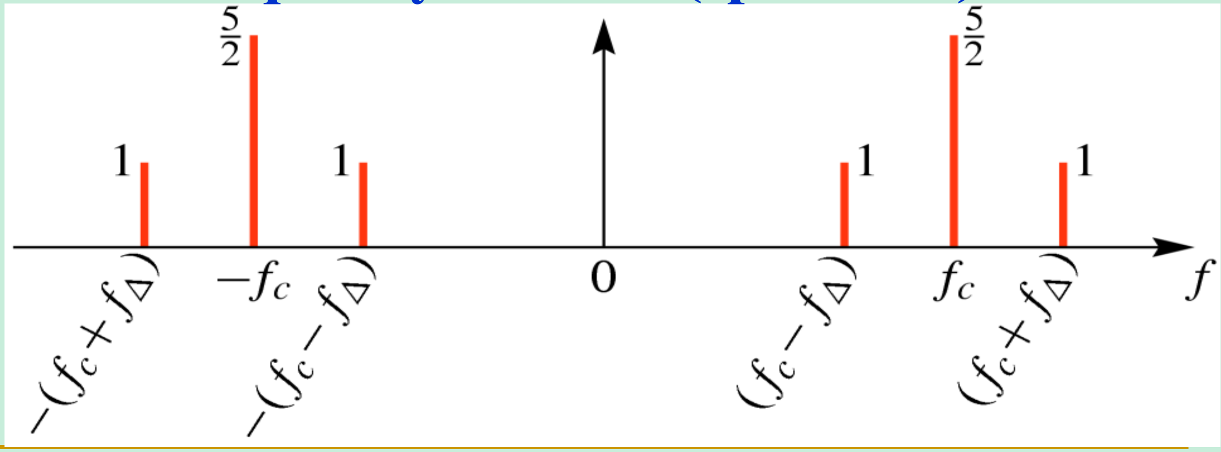


Notice that the only difference compared to beats is, there are no Nulls.

The formation of Envelope is important For the detection Process, this is the Reason for choosing A higher carrier frequency

Frequency domain (spectrum)

Fig. 3.10



The carrier frequency is 700 Hz, and the signal frequency is maintained at 20 Hz, the formation of envelope can be clearly seen, the receiver's job will be easy, which basically tracks the envelope

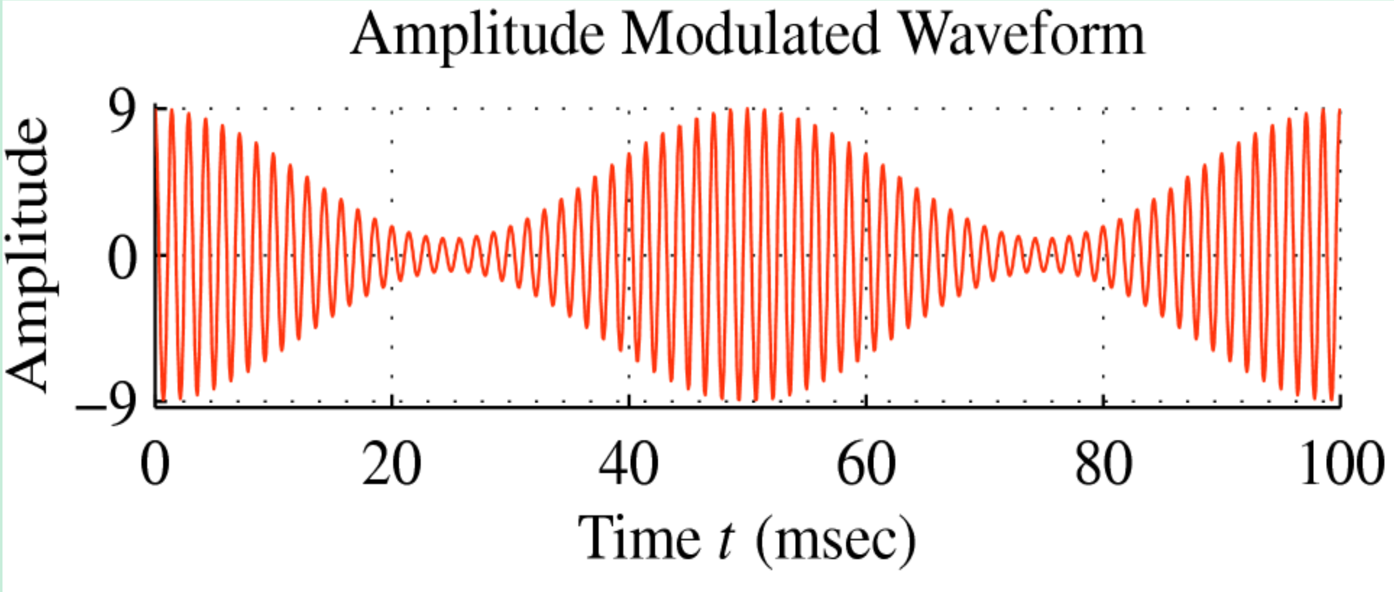


Fig. 3.12

The spectrum contains two identical subsets centered around f_c and $-f_c$

Periodic waveforms

The condition $x(t+T_0)=x(t)$ is satisfied by periodic signals

The time interval T_0 is called period of $x(t)$

If T_0 is the smallest such interval it is called the fundamental period

$\cos(2\pi t)$ is periodic for $T = 1,2,3....n$ sec,
where 'n' is an integer,
The fundamental period, $T_0 = 1\,\text{sec}$

Periodic waveforms *continued*….

Periodic signals can be synthesized by adding the harmonically related sinusoids

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \phi_k)$$

$$The\ frequency\ of\ k^{th}\ cosine\ component,\ f_k = kf_0$$

$$f_k\ is\ known\ as\ k^{th}\ harmonic\ of\ fundamental\ frequency\ f_0$$

Sum of two periodic signals

$$z(t) = ax(t) + by(t)$$

$$x(t)\ period = T_1$$

$$y(t)\ period = T_2$$

$z(t)$ *will be periodic with a period of* T_0,

$T_0 = kT_1 = lT_2$, *or*

$$\frac{T_1}{T_2} = \frac{l}{k},\ 'l'\ \text{and}\ 'k'\ \text{are integers}$$

$$= \text{a rational number}$$

Mathematically the procedure is called 'gcd',

gcd (*greatest common devisor*)

Example : $x(t) = 2\sin(2/3\,t) + 3\cos(1/2\,t)$

$T_1 = 3\pi, T_2 = 4\pi$

$T_0 = 4(3\pi) = 3(4\pi) = 12\pi$

Signal	*Period*
$\cos(t)$	2π
$\sin(t)$	2π
$\cos(2t)$	π
$\sin(2t)$	π
$\cos(\pi t)$	2
$\sin(\pi t)$	π
$\cos(2\pi t)$	2

Signal	*Period*
$\sin(2\pi t)$	2
$\cos(nt)$	$2\pi/n$
$\sin(nt)$	$2\pi/n$
$\cos(2\pi t/k)$	k
$\sin(2\pi t/k)$	k
$\cos(2\pi nt/k)$	k/n
$\sin(2\pi nt/k)$	k/n

Example: Synthesizing of a periodic signal

- **The signal approximating the waveform produced by a man speaking the vowel sound "*Ah*", is synthesized**
- **The fundamental frequency is 100Hz**
- **The signal is obtained by summing 5 harmonics, with different amplitudes and phases**
- **The period of the waveform obtained at each stage depends on the 'gcd' of the harmonics added**

Complex amplitudes and harmonic frequencies for the periodic signal approximating the vowel "ah", note that the fundamental frequency is 100Hz, the coefficient values for the negative frequencies will be complex conjugates

k	f_k (Hz)	a_k	Mag	Phase
1	100	0	0	0
2	200	$386 + j6101$	6 113	1.508
3	300	0	0	0
4	400	$-4433 + j14024$	14 708	1.877
5	500	$24000 - j4498$	24 418	−0.185
6	600	0	0	0
⋮	⋮	⋮	⋮	⋮
15	1500	0	0	0
16	1600	$828 - j6760$	6 811	−1.449
17	1700	$2362 + j0$	2 362	0

Spectrum of the signal synthesized, for clarity phase and magnitude are plotted separately

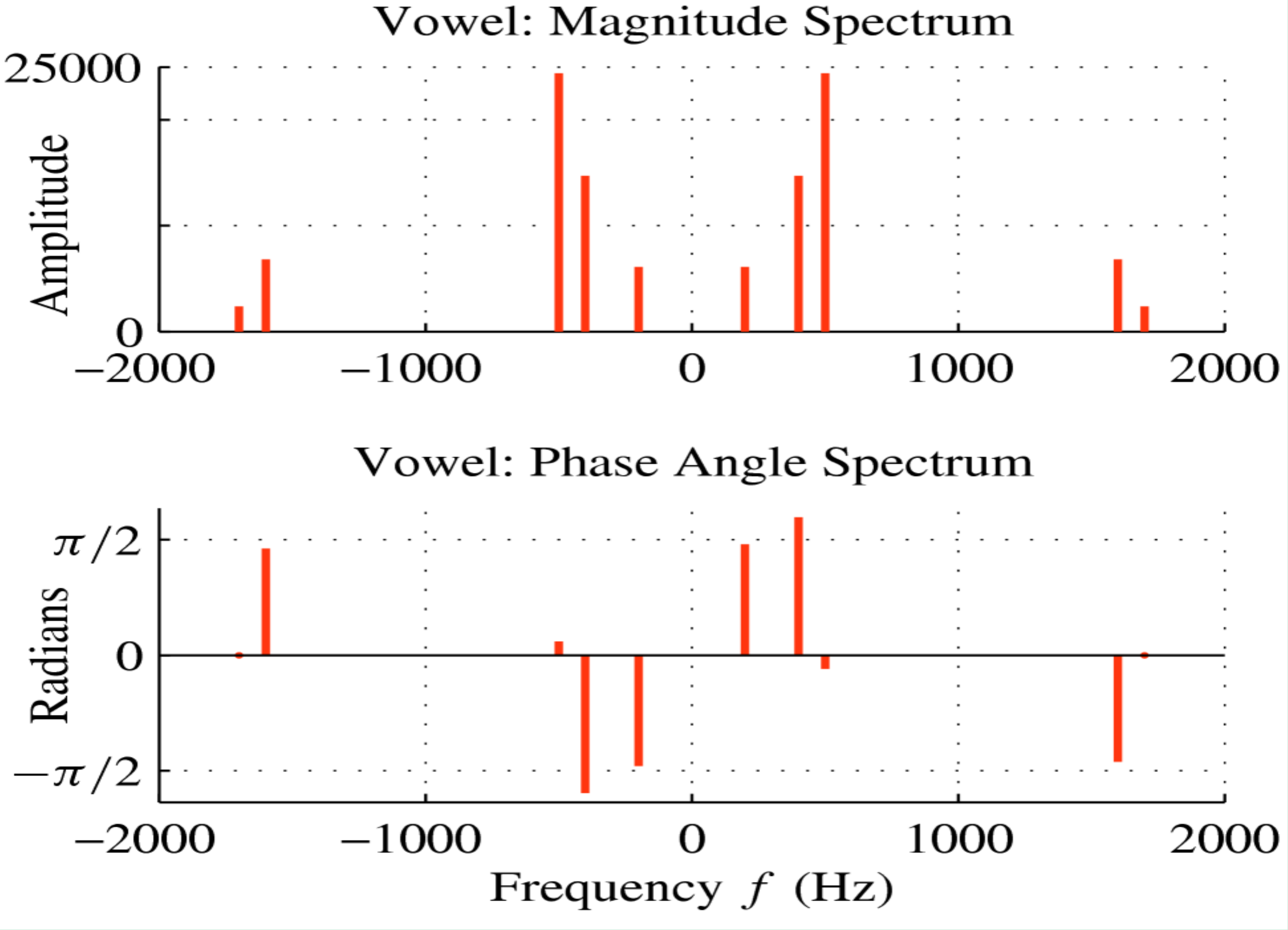
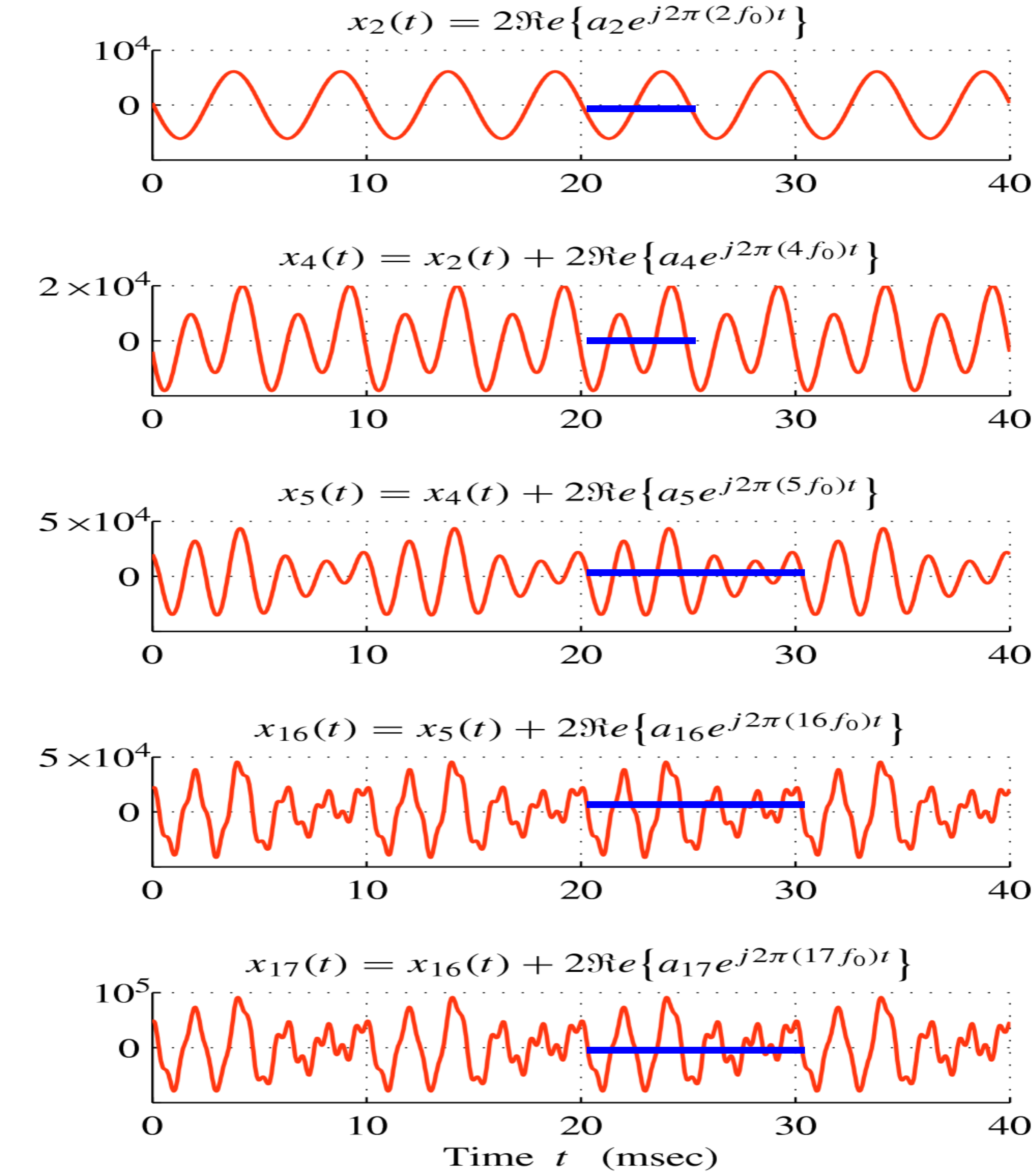


Fig. 3.14



fundamental frequency $= 200Hz$
Time period $= 5ms$

fundamental frequency $= 100Hz$
Time period $= 10ms$

Fig. 3.15

Periodic waveforms can only be synthesized by summing the harmonically related sinusoids

Synthesized periodic waveform

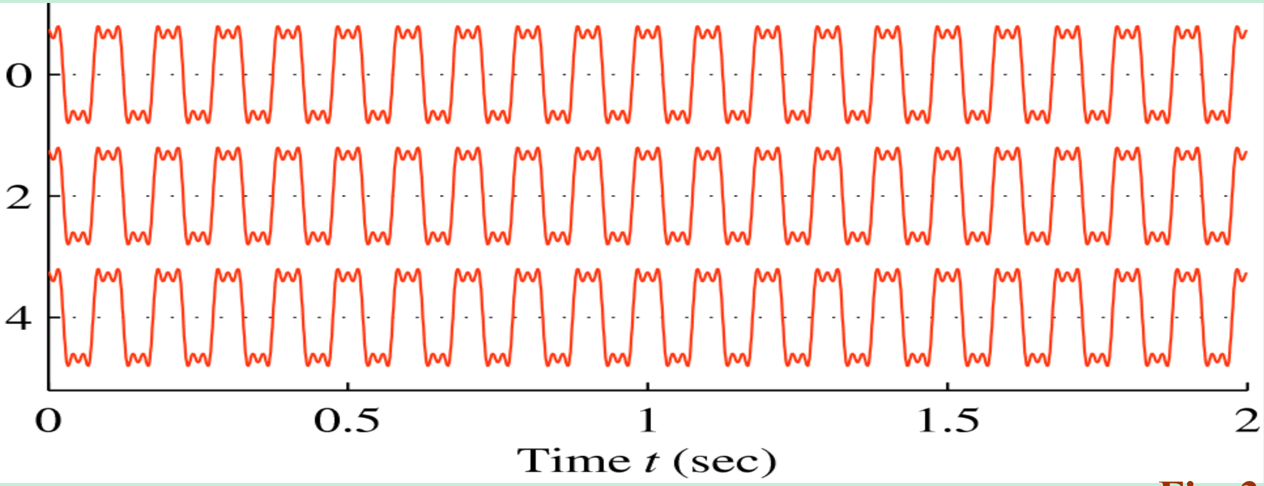


Fig. 3.16

Spectrum of the signal, it can be seen that all the frequencies are multiples of 10

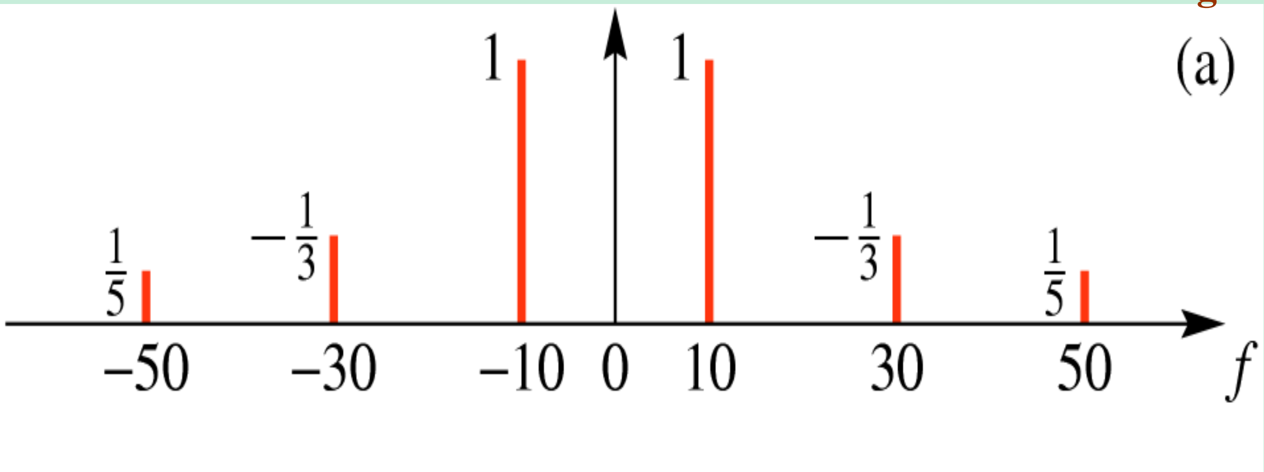


Fig. 3.17

Synthesized Waveform, *'non periodic'*

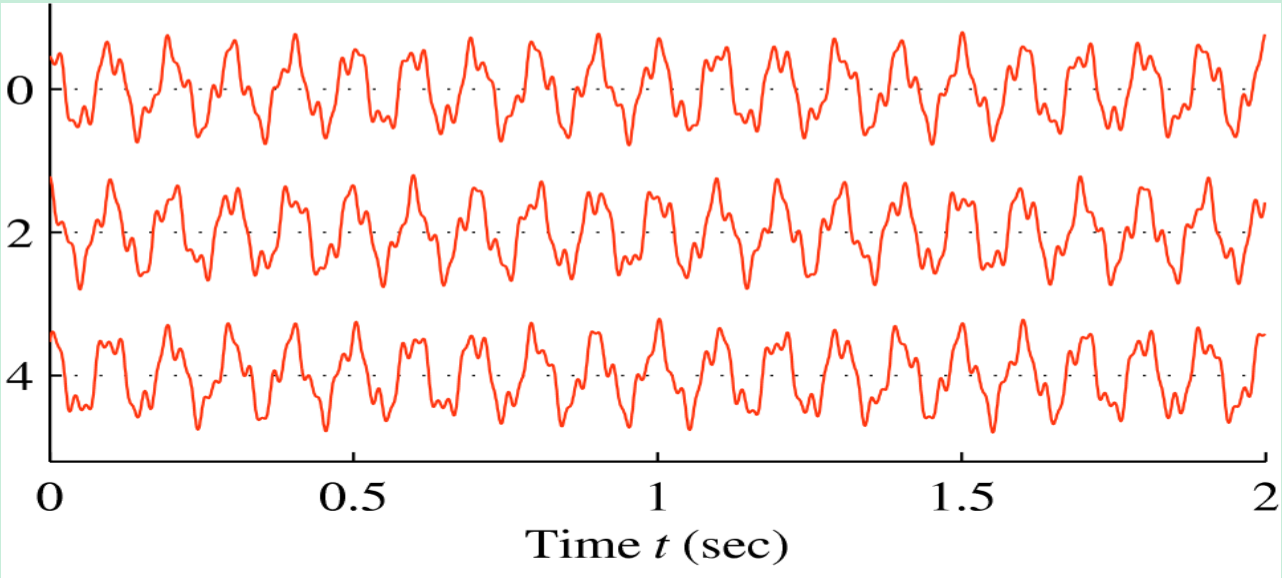


Fig. 3.18

Spectrum of the signal, it can be seen that the frequencies do not have a common multiple, so they can not produce a periodic waveform

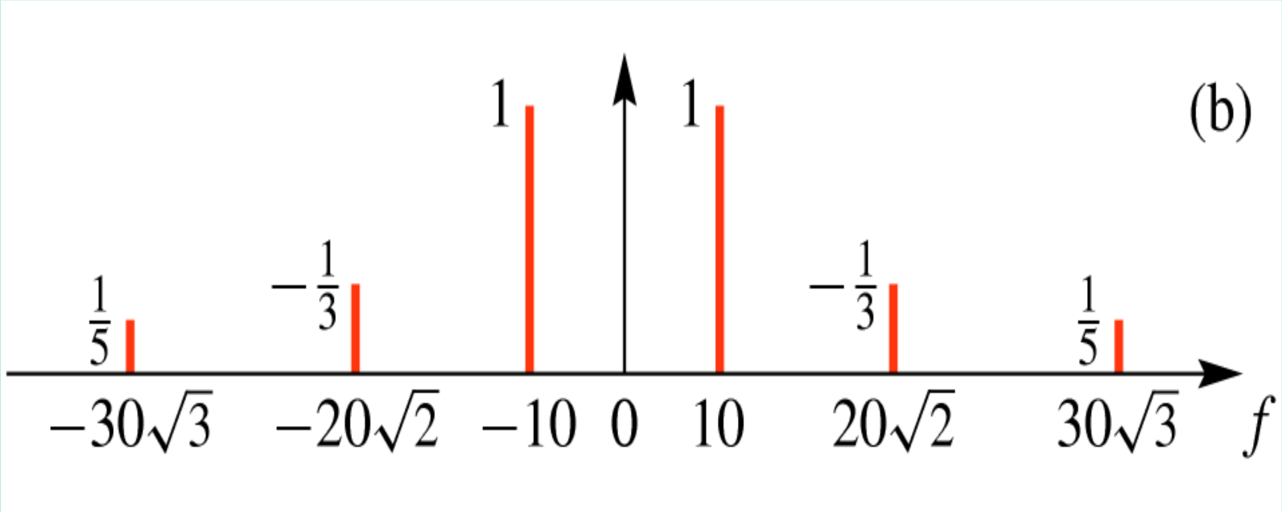


Fig. 3.19

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, "Signal Processing First", Prentice Hall, 2003