Discrete - Time Signals and Systems

Fourier Series
Analysis and Synthesis 1

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General definition of orthogonal signals

A set $\{\phi_n(t)\}, \quad n = 0, \pm 1, \pm 2, \ldots\ldots \pm M, \ldots\ldots$ **, is orthogonal ove some interval [*a*, *b*] if;**

$$\int_a^b \phi_m(t)\phi_n^*(t)dt = \begin{cases} \lambda_n & m = n \\ 0 & m \neq n \end{cases}$$

'*' denotes conjugate for complex functions

In other words, each signal in the set is orthogonal to every other signal in the set

Example

Family of Harmonically related complex exponentials

$$\phi_n(t) = \left\{ e^{jn\omega_0 t} \right\} \qquad n = 0, \pm 1, \pm 2, \ldots\ldots \pm M, \ldots.$$

$$\omega_0 = \frac{2\pi}{T_0},$$

$$T_0 \ldots is\ the\ fundamental\ period$$

$$\phi_n(t)\ are\ periodic\ signals,$$

$$\phi_n(t + T_0) = \phi_n(t)$$

$$Which\ means,$$

$$\phi_n(t+T_0) = e^{jn\omega_0(t+T_0)}$$

$$= e^{(jn\omega_0 t + jn\omega_0 T_0)}$$

$$= e^{jn\omega_0 t} e^{jn\omega_0 T_0}$$

$$= e^{jn\omega_0 t} e^{jn\frac{2\pi}{T_0}T_0} = e^{jn\omega_0 t} e^{jn2\pi}$$

$$e^{jn2\pi} = 1,\ for\ all\ n\ ,'n'\ is\ an\ integer$$

$$\therefore \phi_n(t+T_0) = e^{jn\omega_0 t} = \phi_n(t)$$

Continue example….

The set $\{\phi_n(t)\}$ *is orthogonal over any period* $(t_1, t_1 + T_0)$

Proof:

$$\int_{t_1}^{t_1+T_0} \phi_n(t)\phi_m^*(t)dt = \int_{t_1}^{t_1+T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= \int_{t_1}^{t_1+T_0} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$= \int_{t_1}^{t_1+T_0} e^{j(n-m)\omega_0 t} dt$$

$$= \frac{1}{j(n-m)} \left(e^{j(n-m)\frac{2\pi}{T_0}t} \right) \Bigg|_{t_1}^{t_1+T_0}$$

$$= \frac{1}{j(n-m)} \left(e^{j(n-m)\frac{2\pi}{T_0}(t_1+T_0)} - e^{j(n-m)\frac{2\pi}{T_0}t_1} \right)$$

1

$$= \frac{1}{j(n-m)} \left(e^{j(n-m)\frac{2\pi}{T_0}t_1} e^{j(n-m)\frac{2\pi}{T_0}T_0} - e^{j(n-m)\frac{2\pi}{T_0}t_1} \right)$$

$$= \frac{1}{j(n-m)} \left(e^{j(n-m)\frac{2\pi}{T_0}t_1} - e^{j(n-m)\frac{2\pi}{T_0}t_1} \right)$$

$$= 0, \ if \ m \neq n$$

$for \quad m = n$

$$\int_{t_1}^{t_1+T_0} \phi_n(t)\phi_m^*(t)dt = \int_{t_1}^{t_1+T_0} e^{jn\omega_0 t} e^{-jn\omega_0 t} dt$$

$$= \int_{t_1}^{t_1+T_0} e^{0} dt = (t)\Big|_{t_1}^{t_1+T_0}$$

$$= t_1 + T_0 - t_1 = T_0 = \lambda_n$$

Thus we have proved for the family of harmonically related complex exponentials $\phi_n(t) = \left\{ e^{jn\omega_0 t} \right\} \qquad n = 0, \pm 1, \pm 2, \ldots.. \pm M, \ldots.$

$$\int_a^b \phi_m(t)\phi_n^*(t)dt = \begin{matrix} [\lambda_n = T_0] \\ 0 \end{matrix} \Bigg|_{m \neq n}^{m=n}$$

Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

Fig. 4.1

The Fourier series and integral is a most beautiful and fruitful development, which is central to the areas of communications, signal processing and antennas. Taken by the beauty of Fourier series , Maxwell called it a great *'mathematical poem'*

It is Fourier's investigation into the propagation of heat in solid bodies that led to the powerful insight called Fourier series and Fourier integral

The result, any signal can be expressed as a sum of sinusoids, was announced in a paper on the theory of heat. However, due to the lack of mathematical rigor and generality, the paper was not published, the impact was felt decades later

Fourier Series: Definition

Approximation of periodic signals [$x(t \pm T_0) = x(t)$], by harmonically related periodic exponential functions, $\left\{e^{jn\omega_0 t}\right\}$ leads to Fourier series theory

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \qquad \textbf{Eq.1}$$

$$Where,\ \omega_0 = \frac{2\pi}{T_0}$$

The frequency of k^{th} *complex exponential,*

$f_k = k/T_0$ *or* $\omega_k = k\omega_0$, *All the frequencies are integer multiples of fundamental frequency,* $f_0 = 1/T_0\ Hz$

Calculation of Fourier coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

$$\int_0^{T_0} x(t) e^{-j(2\pi/T_0)lt} dt$$

$$= \int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \right) . \left(e^{-j(2\pi/T_0)lt} dt \right)$$

Integration and summation are inter changeble,

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j\omega_0 (k-l)t} dt$$

From orthogonality property,

$$\int_0^{T_0} e^{j\omega_0(k-l)t}\,dt = \begin{array}{c|l} T_0 & k = l \\ 0 & k \neq l \end{array}$$

$$\therefore \sum_{k=-\infty}^{\infty} a_k \int_0^{T_0} e^{j\omega_0(k-l)t}\,dt = a_l T_0$$

From above proof, Fourier analysis equation,

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt \qquad \textbf{Eq.1}$$

And the Fourier synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt} \qquad \textbf{Eq.2}$$

Eq.1 and 2 play the fundamental role of signal analysis and synthesis

General example

x(t), periodic signal, with period T_0

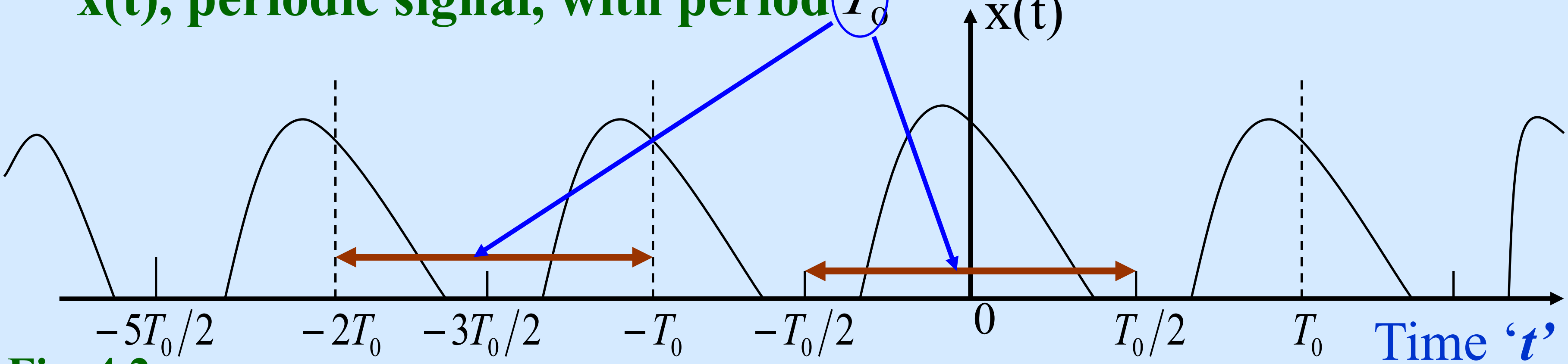


Fig. 4.2

a_k: Fourier coefficients, contains spectral information, often complex numbers, so absolute value is shown on the plot

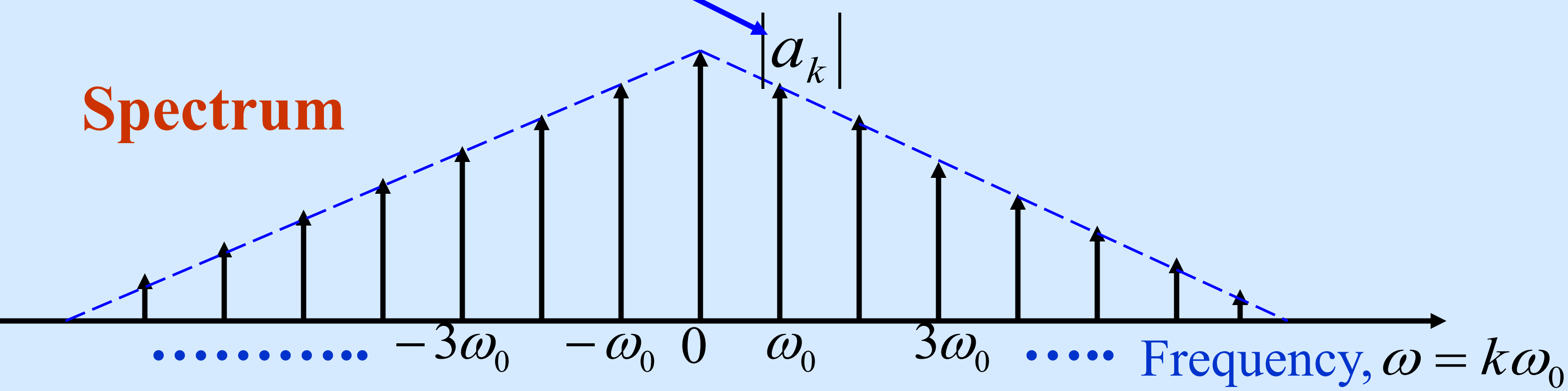


Fig. 4.3

Example 1

$$x(t) = \cos\omega_0 t + \sin^2\omega_0 t$$
$$= \cos\omega_0 t + 1/2(1 - \cos 2\omega_0 t)$$
$$= 1/2 + \cos\omega_0 t - 1/2\cos 2\omega_0 t$$

using the Euler identity, $\cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$,

$$x(t) = 1/2 + \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} - \frac{1}{2}\left(\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2}\right)$$
$$= \frac{1}{2} + \frac{e^{j\omega_0 t}}{2} + \frac{e^{-j\omega_0 t}}{2} - \frac{e^{j2\omega_0 t}}{4} - \frac{e^{-j2\omega_0 t}}{4}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

$$= \ldots\ldots + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots\ldots$$

Through comparison,

$$a_{-1} = a_1 = \frac{1}{2}$$

$$a_{-2} = a_2 = -\frac{1}{4}$$

$$a_0 = \frac{1}{2}$$

$$a_k = 0, \quad |k| > 2$$

In cases as above, where one can reduce the function into complex exponentials, there is no need to integrate to get the Fourier coefficients

Example 2

$$x(t) = 8\cos 3\omega_0 t$$

$$\textit{using the Euler identity}, \cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2},$$

$$x(t) = 8\left(\frac{e^{j3\omega_0 t} + e^{-j3\omega_0 t}}{2}\right)$$

$$= 4e^{j3\omega_0 t} + 4e^{-j3\omega_0 t}, \textit{comparing again with definition},$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

$$= \ldots.. + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots..$$

$$a_{-3} = a_3 = 4$$

$$a_k = 0, \ \textit{otherwise}$$

Example 3: Periodic train of rectangular pulse of width(τ) and period(T_0)

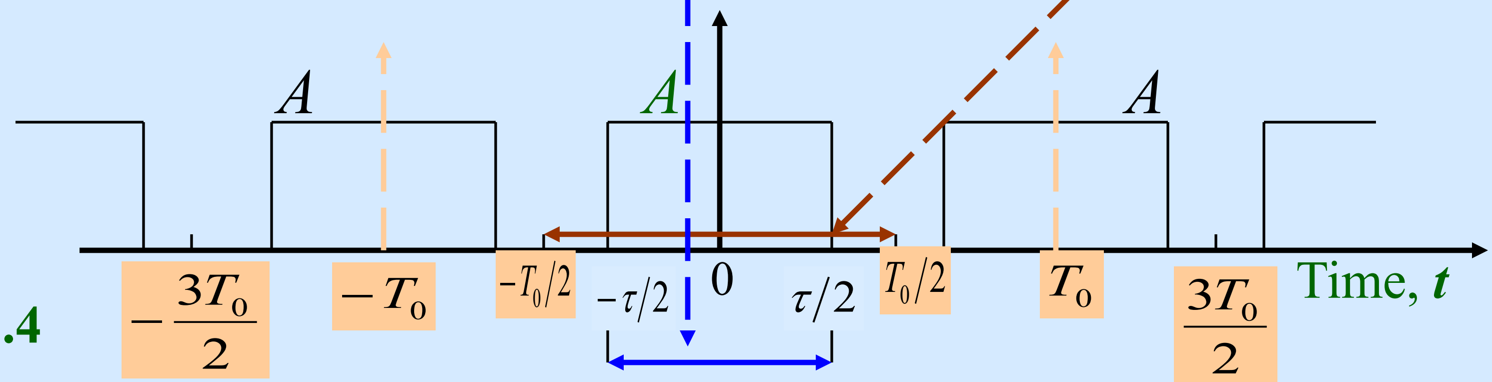


Fig. 4.4

Calculation of Fourier coefficients, a_k

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$The\ limits\ of\ the\ period\ \{-T_0/2, T_0/2\}, with\ amplitude\ 'A',$

$$= \frac{A}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 kt}dt$$

$The\ function\ is\ non-zero\ in\ the\ width\ '\tau',$

$$= \frac{A}{T_0}\int_{-\tau/2}^{\tau/2} e^{-j\omega_0 kt}dt$$

$$= \left(\frac{A}{T_0}.\frac{1}{-j\omega_0 k}\right)\left(e^{-j\omega_0 kt}\Big|_{-\tau/2}^{\tau/2}\right)$$

$$= -\frac{A}{jT_0\omega_0 k}\left(e^{-j\omega_0 k\tau/2} - e^{j\omega_0 k\tau/2}\right)$$

$$= -\frac{A}{jT_0\omega_0 k}\left(-2j\sin(\omega_0 k\,\tau/2)\right)$$

$$= \frac{2A}{T_0\left(\frac{2\pi}{T_0}\right)k}\sin\left(\left(\frac{2\pi}{T_0}\right)k\,\tau/2\right) \qquad \because \omega_0 = \frac{2\pi}{T_0}$$

$$a_k = \frac{A}{\pi k}\sin\left(\frac{\pi k\tau}{T_0}\right), \qquad k \neq 0$$

$For\ \ k = 0,$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)0t}dt$$

$$a_0 = \frac{A}{T_0}\int_{-\tau/2}^{\tau/2} 1.dt = \frac{A}{T_0}\left[\tau/2 - (-\tau/2)\right]$$

$$a_0 = \frac{A\tau}{T_0},\ 'a_0'\ is\ also\ known\ as\ 'DC'\ term$$

Example 4: The Square wave

A square wave is defined as,

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{T_0}{2} \\ 0 & \text{for } \frac{T_0}{2} \le t \le T_0 \end{cases}$$

The period of square wave T_0

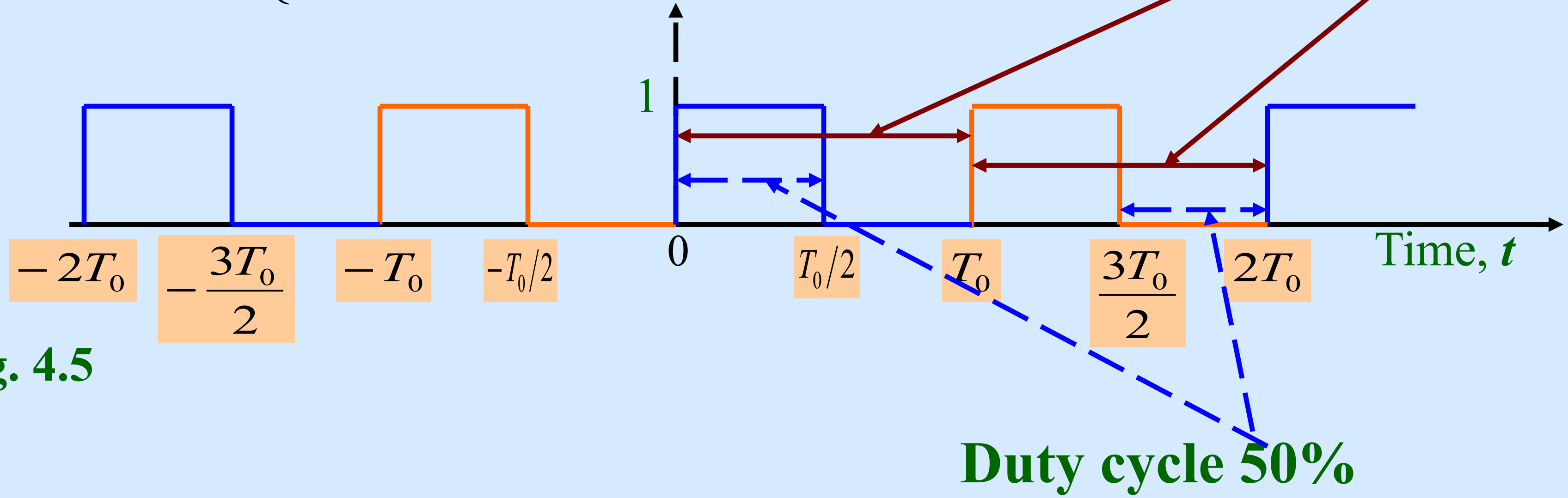


Fig. 4.5

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$The\ limits\ of\ the\ period\ are\ \{0, T_0\},\ with\ amplitude\ '1',$

$$= \frac{1}{T_0}\int_0^{T_0} e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt, \quad The\ function\ is\ non-zero\ in\ \{0, T_0/2\},$$

$$= \frac{1}{T_0}\int_0^{T_0/2} e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$$= \left(\frac{1}{T_0}\cdot\frac{1}{-j\left(\frac{2\pi}{T_0}\right)k}\right)\left(e^{-j\left(\frac{2\pi}{T_0}\right)kt}\Big|_0^{T_0/2}\right)$$

$$=\left(\frac{-1}{j2\pi k}\right)\left(e^{-j\left(\frac{2\pi}{T_0}\right)k\frac{T_0}{2}}-e^{-j\left(\frac{2\pi}{T_0}\right)k0}\right)$$

$$=\left(\frac{-1}{j2\pi k}\right)\left(e^{-j\pi k}-1\right)$$

$$\because e^{-j\pi k}=\left(e^{-j\pi}\right)^k \ and \ e^{-j\pi}=-1 \ \ e^{-j\pi k}=(-1)^k,$$

$$a_k=\frac{1-(-1)^k}{j2\pi k}, \ \ for \ \ k\neq 0$$

$$For \ \ k=0, \qquad a_0=\frac{1}{T_0}\int_0^{T_0/2} e^{-j\left(\frac{2\pi}{T_0}\right)0t}dt$$

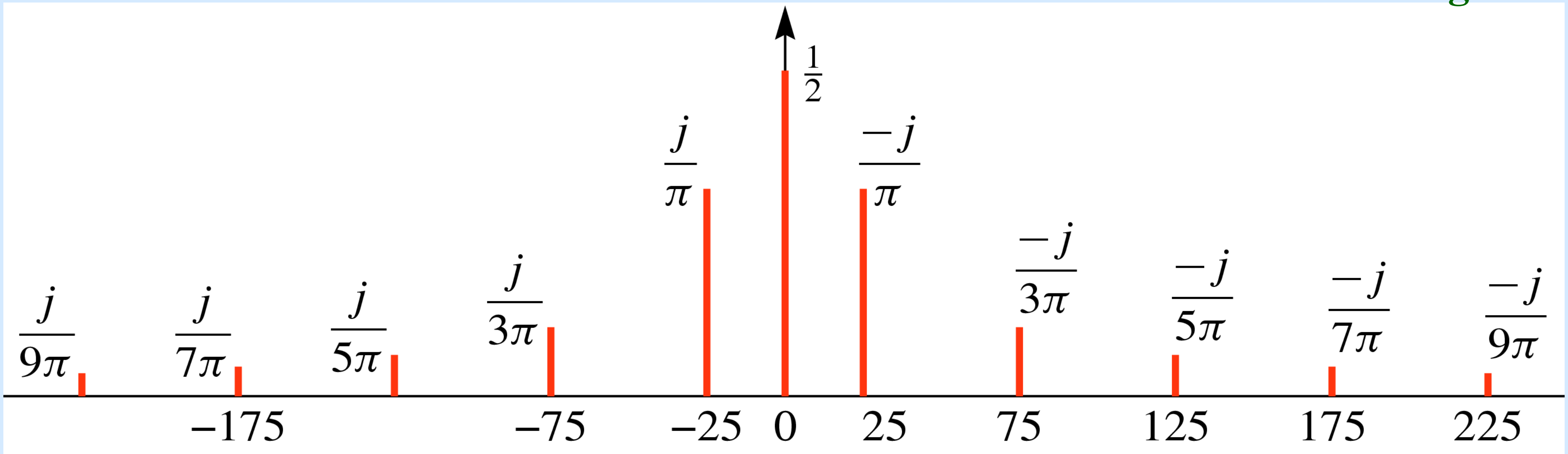
$$a_0=\frac{1}{T_0}\int_0^{T_0/2}1.dt=\frac{1}{T_0}\left[T_0/2\right]=\frac{1}{2}$$

$$a_0=\frac{1}{2}, \ \ a_0 \ is \ also \ known \ as \ 'DC' \ term$$

The final answer for the Fourier series coefficients of the square wave can be summarized in three different cases,

$$a_k = \begin{cases} \dfrac{1}{j\pi k} & \text{for } k = \pm 1,\ \pm 3,\ \pm 5,\ldots. \\ 0 & \text{for } k = \pm 2,\ \pm 4,\ \pm 6,\ldots. \\ \dfrac{1}{2} & \text{for } k = 0 \end{cases}$$

Fig. 4.6



Spectrum of a square wave from Fourier series coefficients, fundamental frequency of 25Hz, frequency range [-225 - 225] is shown, k=-9 to +9

Synthesis of Square wave

Given the Fourier series coefficients, using the synthesis Equation the signal can be reconstructed

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt}$$

The quality of signal synthesized improves with the no. of Fourier coefficients used

The summation of terms in the above equation can sometimes result in simple cosine or sine functions

Synthesis using the 1st three harmonics

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt}, \textit{using the first 3 harmonics},$$

$$x(t)=\sum_{k=-3}^{3} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt}$$

$$= a_{-3}e^{-j3\omega_0 t} + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t}$$

$$\because a_{-3}=\frac{-1}{j3\pi},\ a_3=\frac{1}{j3\pi}$$

$$a_{-2}=a_2=0$$

$$a_{-1}=\frac{-1}{j\pi},\ a_1=\frac{1}{j\pi}$$

$$a_0=\frac{1}{2}$$

$$x_3(t) = a_{-3}e^{-j3\omega_0 t} + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t}$$

$$= a_0 + a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t} + a_{-3}e^{-j3\omega_0 t} + a_3 e^{j3\omega_0 t}$$

$$= \frac{1}{2} + \frac{-1}{j\pi}e^{-j\omega_0 t} + \frac{1}{j\pi}e^{j\omega_0 t} + \frac{-1}{j3\pi}e^{-j3\omega_0 t} + \frac{1}{j3\pi}e^{j3\omega_0 t}$$

$$= \frac{1}{2} + \frac{j}{\pi}e^{-j\omega_0 t} + \frac{-j}{\pi}e^{j\omega_0 t} + \frac{j}{3\pi}e^{-j3\omega_0 t} + \frac{-j}{3\pi}e^{j3\omega_0 t}$$

$$\because -j = e^{-j\pi/2}, \, and \;\; j = e^{j\pi/2}$$

$$= \frac{1}{2} + \frac{1}{\pi}\left(e^{j\pi/2}e^{-j\omega_0 t} + e^{-j\pi/2}e^{j\omega_0 t}\right) + \frac{1}{3\pi}\left(e^{j\pi/2}e^{-j3\omega_0 t} + e^{-j\pi/2}e^{j3\omega_0 t}\right)$$

$$= \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t - \pi/2) + \frac{2}{3\pi}\cos(3\omega_0 t - \pi/2), \qquad or$$

$$x_3(t) \; = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t)$$

Synthesis, fundamental frequency of 25Hz, 3 harmonics

$$x_3(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

Fourier series coefficients
For 1st three harmonics

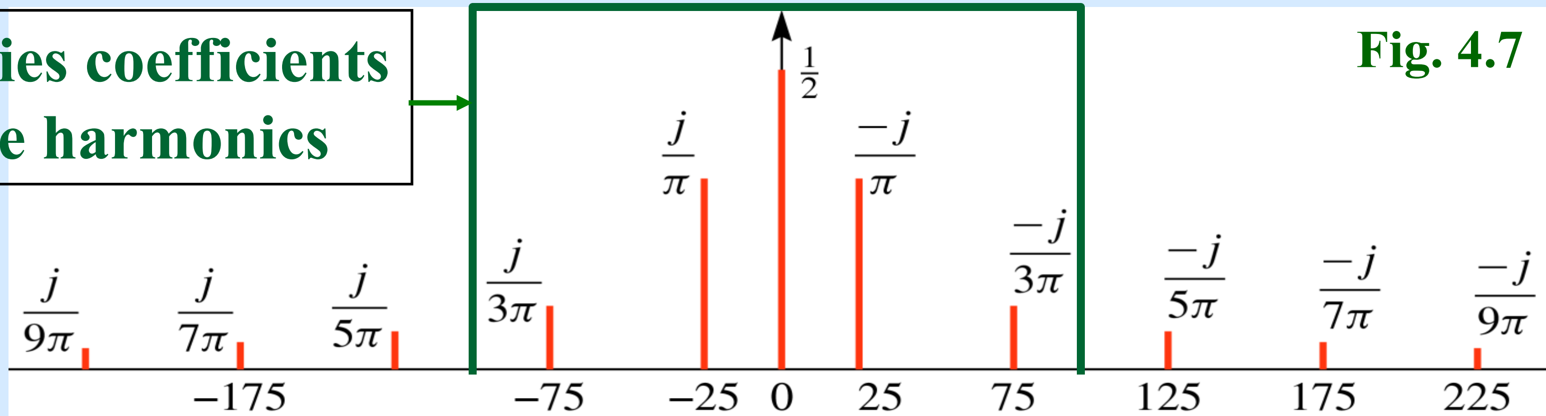


Fig. 4.7

***x(t)*, with a fundamental frequency of 25 Hz**

***$x_3(t)$*, with a fundamental frequency of 25 Hz**

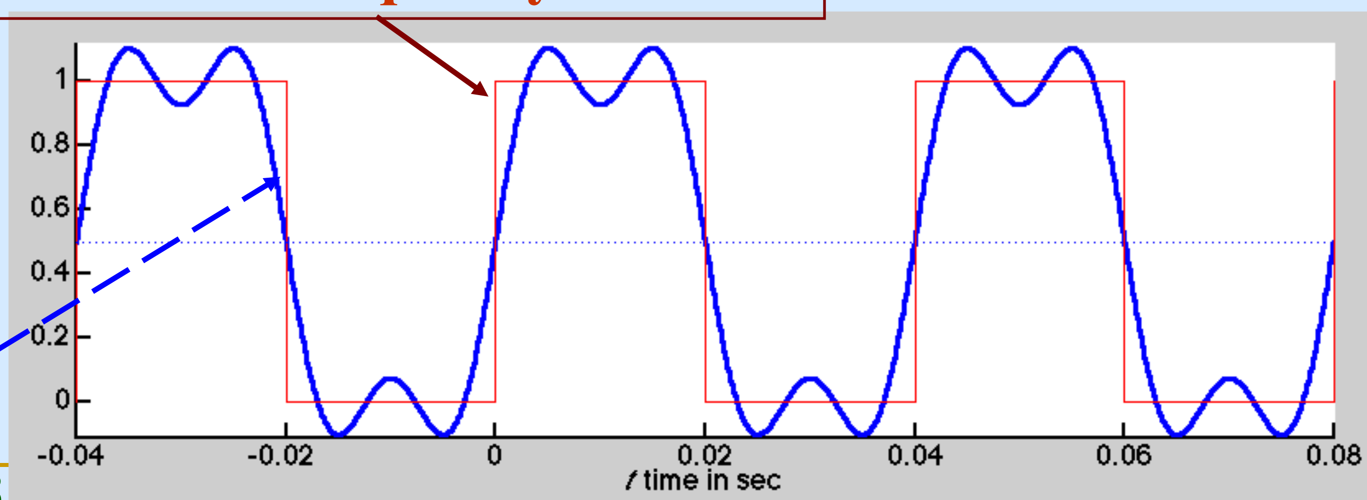


Fig. 4.8

$$x_7(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$

Fourier series coefficients for 1st seven harmonics

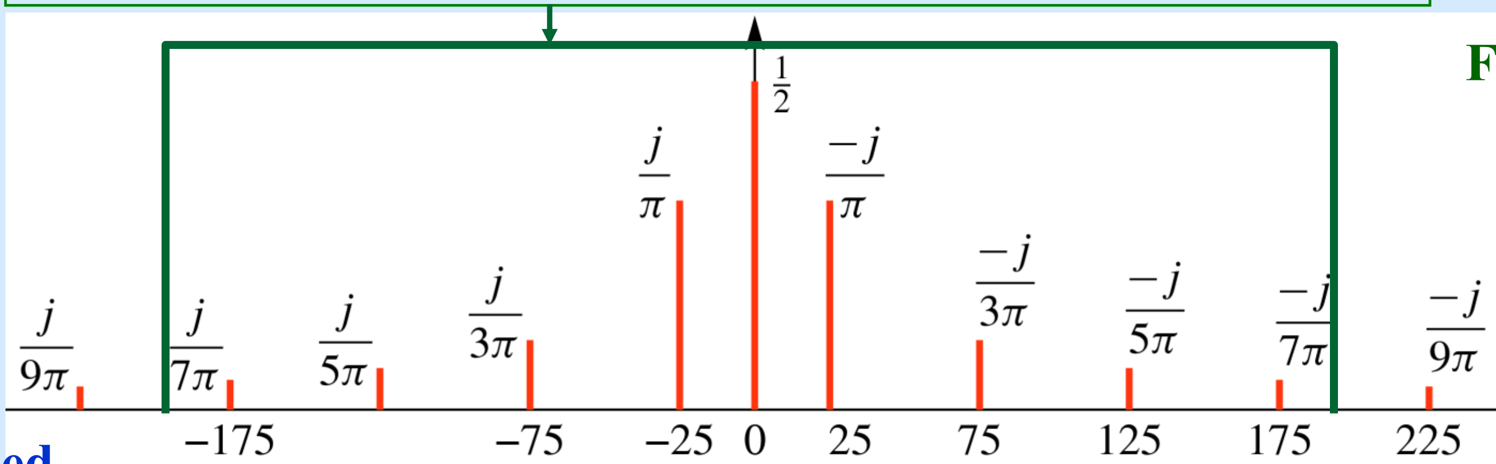


Fig. 4.9

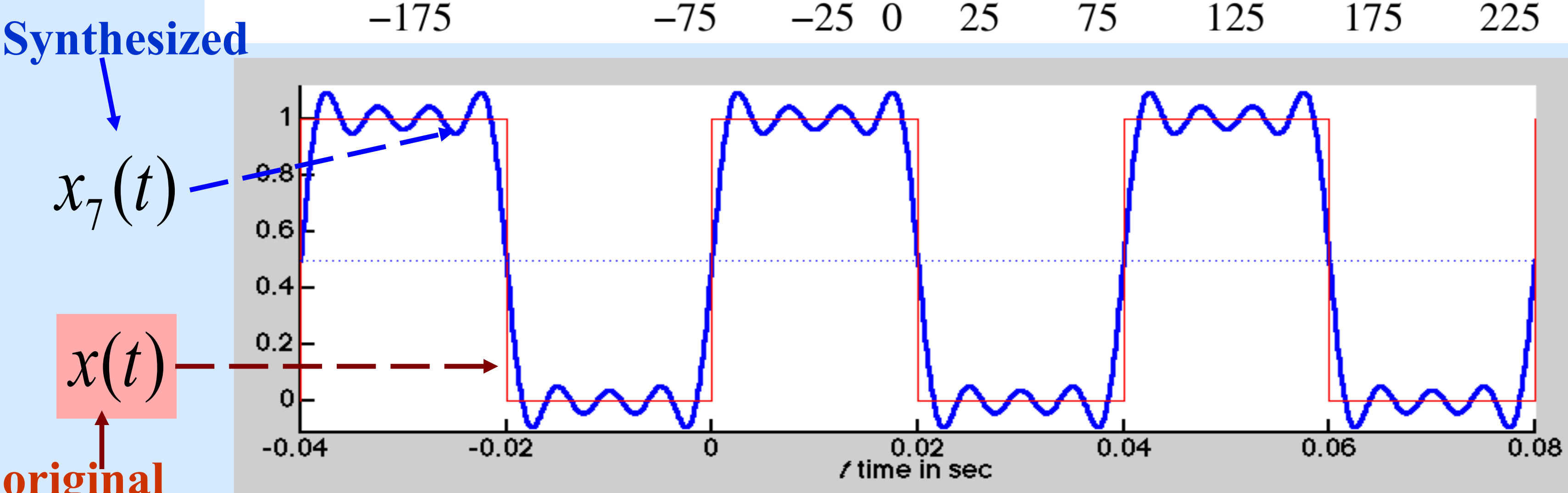
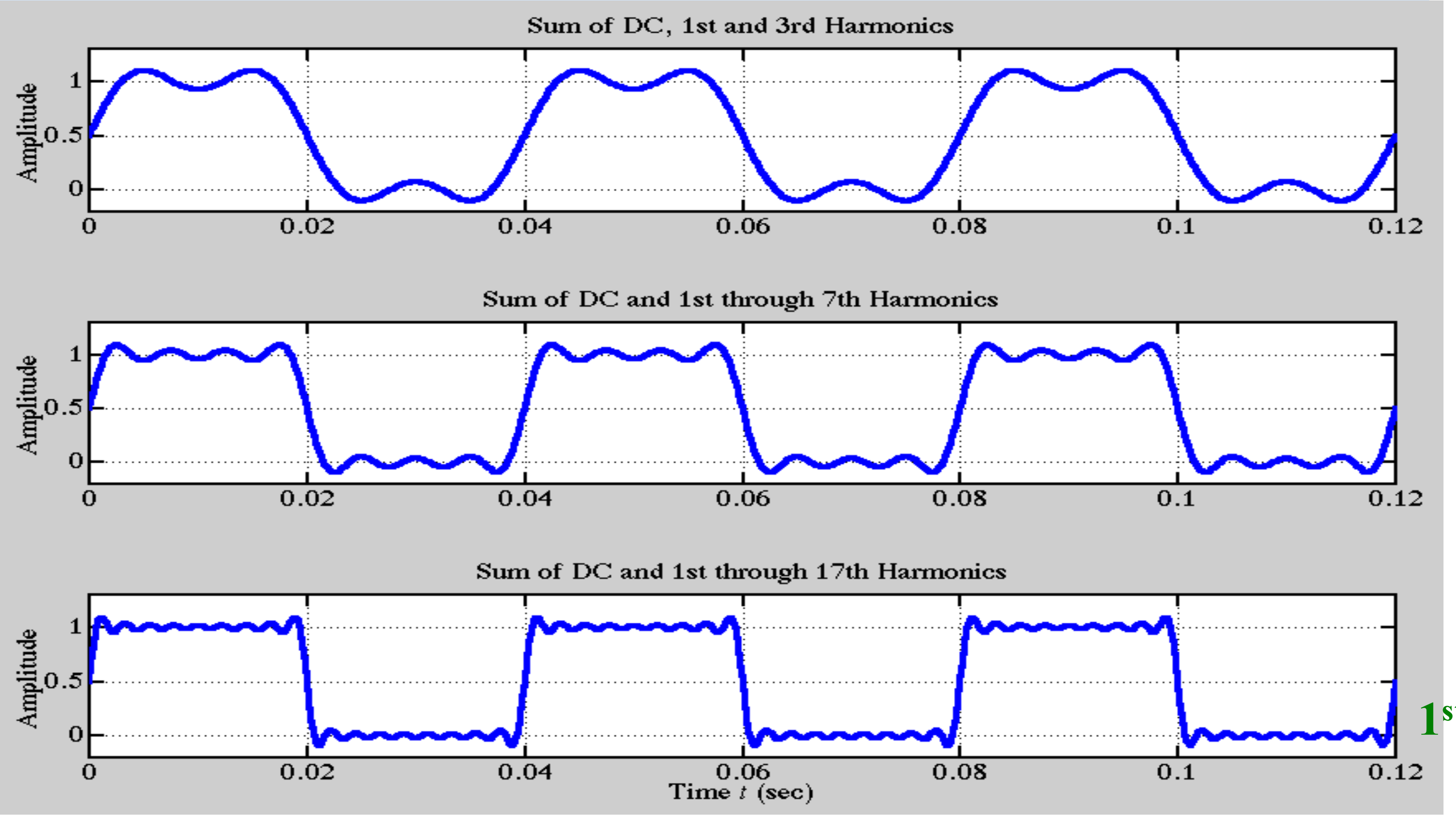


Fig. 4.10

General formula for synthesized signal with 'N' harmonics

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



1st three

1st seven

1st seventeen

Fig. 4.11

Notice the Gibbs phenomenon at discontinuities

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, "Signal Processing First", Prentice Hall, 2003