Discrete - Time Signals and Systems

Fourier Series
Analysis and Synthesis 2

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Example 5: The Triangular wave

Fundamental period of periodic wave is T_0

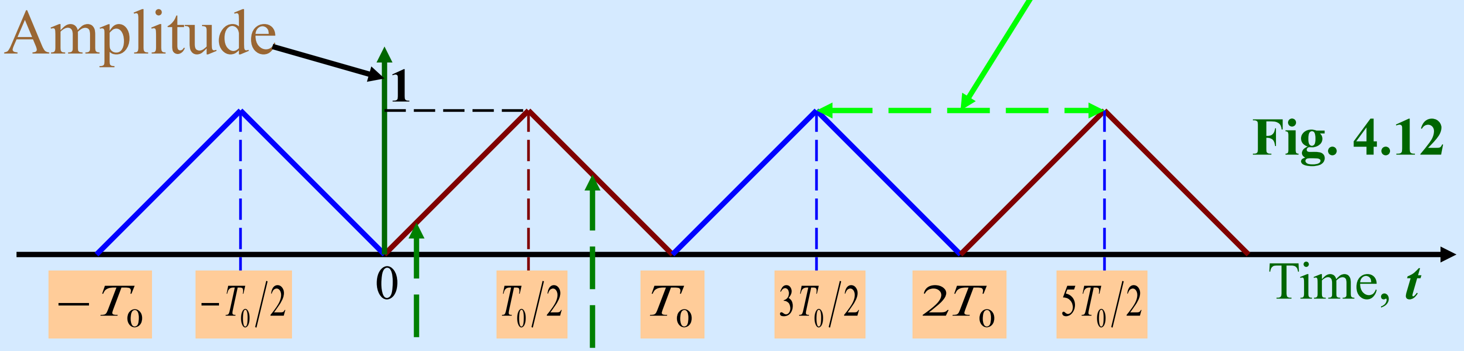


Fig. 4.12

$$x(t) = \begin{cases} 2t/T_0 & \text{for } 0 \le t \le \dfrac{T_0}{2} \\ 2(T_0 - t)/T_0 & \text{for } \dfrac{T_0}{2} \le t \le T_0 \end{cases}$$

The procedure to calculate the Fourier series coefficients is the same

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

The function has to be split into $\{0, T_0/2\}$ *and* $\{T_0/2, T_0\}$

$$= \frac{1}{T_0}\int_0^{T_0/2} (2t/T_0)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (2(T_0 - t)/T_0)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} (2t/T_0)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt - \frac{1}{T_0}\int_{T_0/2}^{T_0} (2t/T_0)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt$$

$$+ \frac{2}{T_0}\int_{T_0/2}^{T_0} e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt,$$

Part1 **Part2** **Part3**

$$= \underbrace{\frac{2}{T_0^2}\int_0^{T_0/2} te^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt}_{\text{Part1}} - \underbrace{\frac{2}{T_0^2}\int_{T_0/2}^{T_0} te^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt}_{\text{Part2}} + \underbrace{\frac{2}{T_0}\int_{T_0/2}^{T_0} e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt}_{\text{Part3}}$$

Integration by parts for part 1:

$$\int uv = u\int v - \int\left(du\int v\right),$$

$$\frac{2}{T_0^2}\int_0^{T_0/2} t e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt =$$

$$= \frac{2}{T_0^2}\frac{1}{-j\left(\frac{2\pi}{T_0}\right)k}\left(t e^{-j\left(\frac{2\pi}{T_0}\right)kt} \right)\Bigg|_0^{T_0/2} - \frac{2}{T_0^2}\frac{1}{\left(-j\left(\frac{2\pi}{T_0}\right)k\right)^2}\left(e^{-j\left(\frac{2\pi}{T_0}\right)kt} \right)\Bigg|_0^{T_0/2}$$

$$= \frac{e^{-j\pi k}}{-j2\pi k} + \frac{1}{2\pi^2 k^2}\left(e^{-j\pi k} - 1\right)$$

$Integration\ \ by\ parts\ for\ part\ 2:$

$$\int uv = u\int v - \int\left(du\int v\right),$$

$$\frac{2}{T_0^2}\int_{T_0/2}^{T_0} te^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt =$$

$$= \frac{2}{T_0^2}\frac{1}{-j\left(\frac{2\pi}{T_0}\right)k}\left(te^{-j\left(\frac{2\pi}{T_0}\right)kt}\right)\Bigg|_{T_0/2}^{T_0} - \frac{2}{T_0^2}\frac{1}{\left(-j\left(\frac{2\pi}{T_0}\right)k\right)^2}\left(e^{-j\left(\frac{2\pi}{T_0}\right)kt}\right)\Bigg|_{T_0/2}^{T_0}$$

$$= \frac{1}{-j\pi k} + \frac{e^{-j\pi k}}{j2\pi k} - \frac{1}{2\pi^2 k^2}\left(1 - e^{-j\pi k}\right)$$

For part 3 :

$$\frac{2}{T_0}\int_{T_0/2}^{T_0} e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt = \frac{2}{T_0}\,\frac{1}{-j\left(\frac{2\pi}{T_0}\right)k}\left(e^{-j\left(\frac{2\pi}{T_0}\right)kt}\right)\Bigg|_{T_0/2}^{T_0}$$

$$= \frac{-1}{j\pi k}\left(1-e^{-j\pi k}\right)$$

Adding parts 1 *and* 2,

$$= \frac{e^{-j\pi k}}{-j2\pi k} + \frac{1}{2\pi^2 k^2}\left(e^{-j\pi k}-1\right) - \left(\frac{1}{-j\pi k} + \frac{e^{-j\pi k}}{j2\pi k} - \frac{1}{2\pi^2 k^2}\left(1-e^{-j\pi k}\right)\right)$$

$$= \frac{e^{-j\pi k}}{-j\pi k} + \frac{1}{\pi^2 k^2}\left(e^{-j\pi k}-1\right) + \frac{1}{j\pi k}$$

$Adding\ the\ part3\ to\ the\ above\ result,$

$$= \frac{e^{-j\pi k}}{-j\pi k} + \frac{1}{\pi^2 k^2}\left(e^{-j\pi k} - 1\right) + \frac{1}{j\pi k} - \frac{1}{j\pi k} + \frac{e^{-j\pi k}}{j\pi k}$$

$$= \frac{1}{\pi^2 k^2}\left(e^{-j\pi k} - 1\right)$$

$The\ 'DC'\ component,$

$$a_0 = \ area\ of\ under\ triangle\ in\ one\ cycle = \frac{1}{2}$$

$Fourier\ series\ coefficients,$

$$a_k = \begin{cases} \dfrac{-2}{\pi^2 k^2} & k = \ \pm 1,\ \pm 3,\ \pm 5,\ ...... \\ 0 & k = \ \pm 2,\ \pm 4,\ \pm 6,\ ...... \\ \dfrac{1}{2} & k = \ 0 \end{cases}$$

Synthesis of Triangular wave

$$x_3(t) = \sum_{k=-3}^{3} a_k e^{j\left(\frac{2\pi}{T_0}\right)kt}, \text{ using the first 3 harmonics,}$$

$$= a_{-3}e^{-j3\omega_0 t} + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_3 e^{j3\omega_0 t}$$

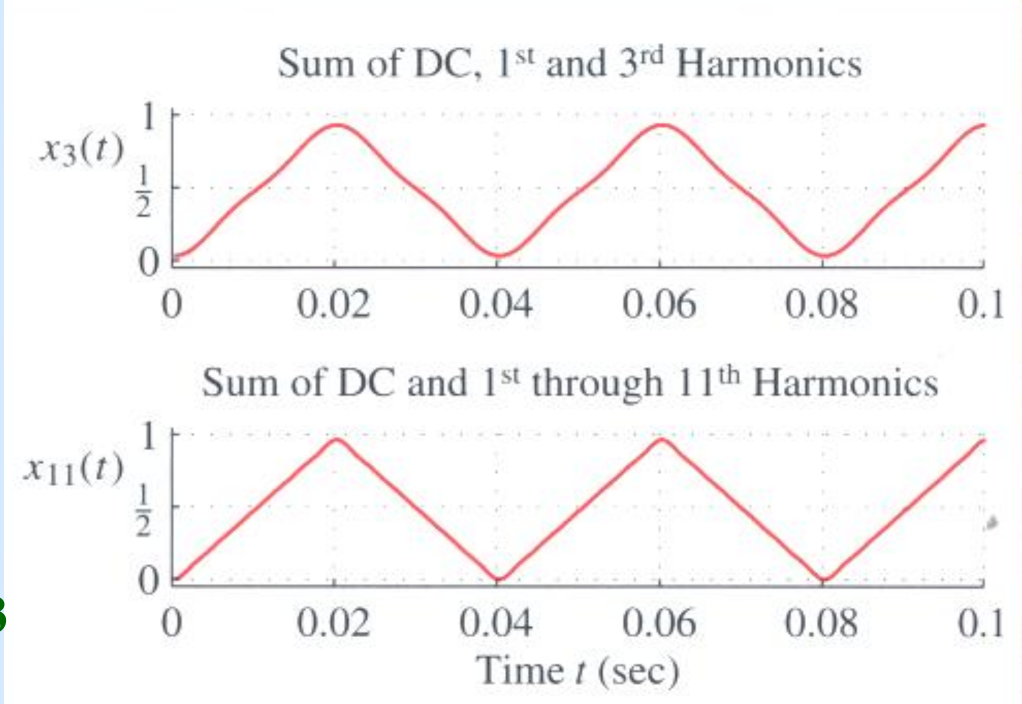
$$\because a_{-3} = a_3 = \frac{-2}{9\pi^2}, \quad a_{-2} = a_2 = 0,$$

$$a_{-1} = a_1 = \frac{-2}{\pi^2}, \quad a_0 = \frac{1}{2},$$

$$\therefore \quad x_3(t) = \frac{1}{2} - \frac{4}{9\pi^2}\cos(3\omega_0 t) - \frac{4}{\pi^2}\cos(\omega_0 t)$$

With a fundamental frequency 25 hz

$$x_3(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(50\pi t) - \frac{4}{9\pi^2}\cos(150\pi t)$$



3 harmonics

11 harmonics

Fig. 4.13

The triangular signal doesn't have discontinues, 11harmonic synthesized waveform resembles the original one closely, also unlike a square wave no Gibbs phenomenon is seen as there is no discontinuity

Review of Fourier Series: Analysis

$Fourier\ analysis\ equation,$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

$Given\ the\ periodic\ signal\ x(t)\ such\ that$

$x(t + mT_0) = x(t),$

$Analysis\ gives\ \{a_k, f_k\},$

$Where\ f_k = kf_0,$

$f_0\ being\ the\ fundamental\ frequency$

Review of Fourier Series: Synthesis

The Fourier synthesis equation,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_0 kt}$$

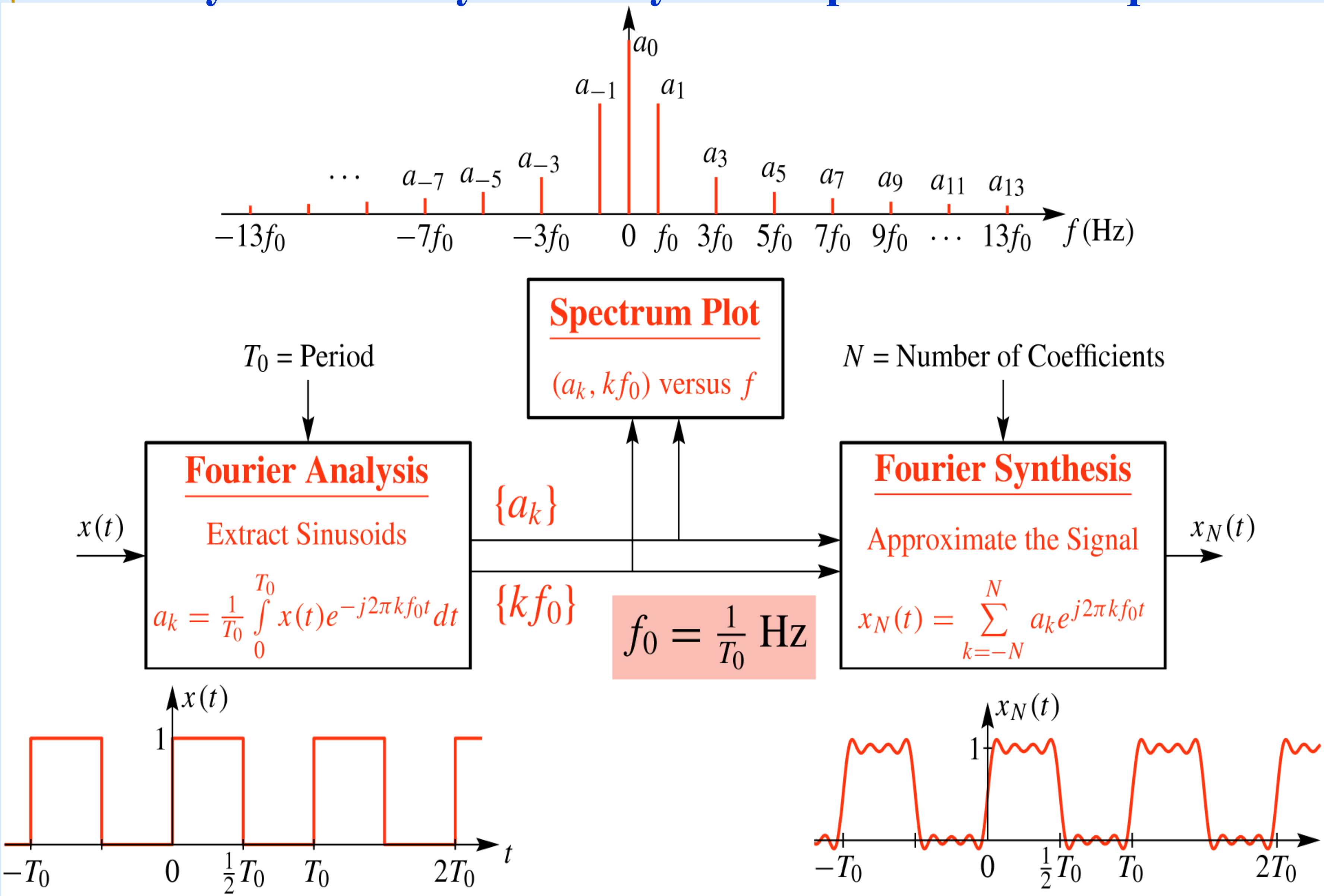
Given the spectrum plot $\{a_k, f_k\}$ *or* $\{a_k, kf_0\}$

and the 'N' number of coefficients,

synthesis can be done by $x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_0 kt}$

since $x_N(t)$*, is only an approximation,*

Summary of the analysis and synthesis process for a square wave



Review of Fourier Series: Synthesis, *continued*…..

There will be some error, Mean square error is a popular measurement

$$MSE = E_n - \sum_{k=-N}^{N} |a_k|^2$$

$$E_n = \frac{1}{T_0}\int_{<T_0>} |x(t)|^2\, dt \;=\; \textit{average signal power}$$

$$\text{or,}\quad \%\ \text{of error} = 1 - \frac{\sum_{k=-N}^{N} |a_k|^2}{E_n}$$

Another popular definition of error is,

$e_N(t) = x(t) - x_N(t),$

absolute value of $e_N(t)$, $|e_N(t)|$ *is defined as error function*

Gibbs phenomenon **can be observed in these error plots, notice the jump in error value at discontinuities, also notice that the error value didn't change at the discontinuity point, for 7 and 17th harmonics, this 9% overshoot is unavoidable in any discontinuous function**

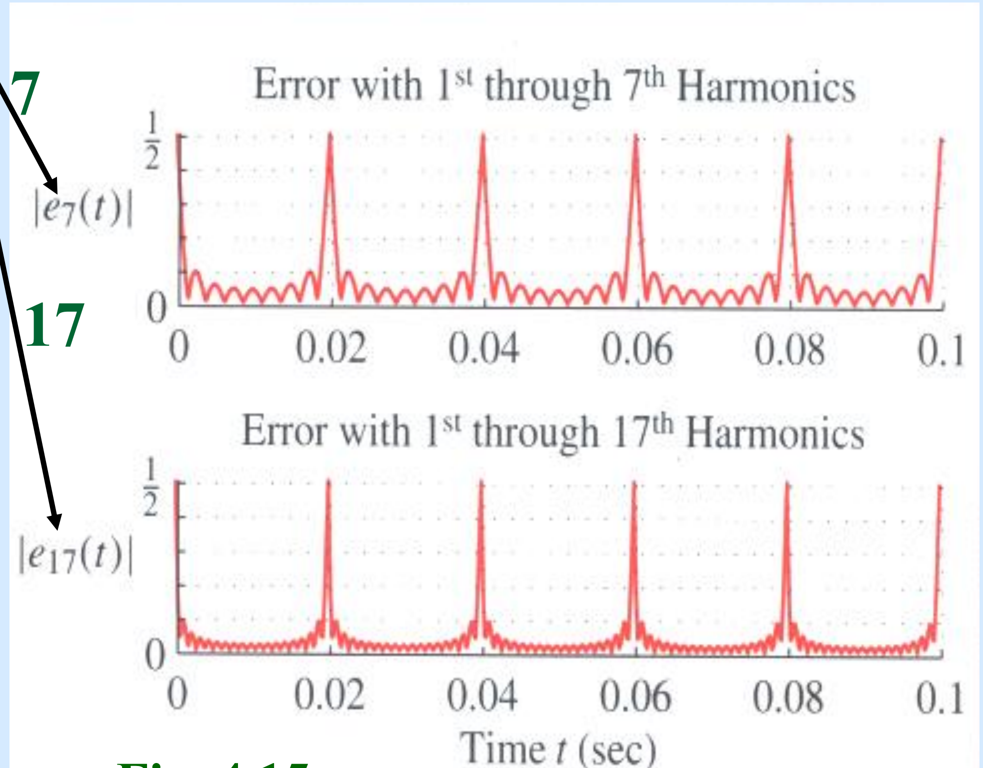


Fig. 4.15

There is considerable reduction with 17 harmonics in the error value at other points

Fourier series: Trigonometric form

Example, $x_3(t) = \sum_{k=-2}^{2} a_k e^{jk\omega_0 t}$, *using the first 3 harmonics,*

$$= a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}$$

$$= a_0 + \left(a_{-1}e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}\right) + \left(a_{-2}e^{-j2\omega_0 t} + a_2 e^{j2\omega_0 t}\right)$$

If x(t) is real,

$$a_k^* = \left[\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt\right]^* = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j\omega_0 kt}dt$$

$$a_{-k} = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0(-k)t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{j\omega_0 kt}dt$$

$$\therefore a_{-k} = a_k^*, \quad |a_{-k}| = |a_k|, \quad \langle a_{-k} = -\langle a_k$$

$$= a_0 + \left(a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t}\right) + \left(a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t}\right)$$

$$= a_0 + \left(a_1\left(\cos(\omega_0 t) + j\sin(\omega_0 t)\right) + a_1^*\left(\cos(\omega_0 t) - j\sin(\omega_0 t)\right)\right)$$

$$+ \left(a_2\left(\cos(2\omega_0 t) + j\sin(2\omega_0 t)\right) + a_2^*\left(\cos(2\omega_0 t) - j\sin(2\omega_0 t)\right)\right)$$

$$= a_0 + \left(a_1 + a_1^*\right)\cos(\omega_0 t) + j\left(a_1 - a_1^*\right)\sin(\omega_0 t)$$

$$+ \left(a_1 + a_1^*\right)\cos(2\omega_0 t) + j\left(a_1 - a_1^*\right)\sin(2\omega_0 t)$$

$$= A_0 + A_1 \cos(\omega_0 t) + A_2 \cos(2\omega_0 t)$$

$$+ B_1 \sin(\omega_0 t) + B_2 \sin(2\omega_0 t)$$

$$= A_0 + \sum_{k=1}^{2} A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)$$

In conclusion, A periodic signal x(t) can be approximated by a linear combination of sinusoids (both sine and cosines)

Fourier series: Trigonometric *general* form

$$\text{x(t)} = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t)$$

Where,

$$A_0 = a_0 = \frac{1}{T_0}\int_{<T_0>} x(t)dt$$

$$A_k = 2\Re e\{a_k\} = \frac{2}{T_0}\int_{<T_0>} x(t)\cos(k\omega_0 t)dt$$

$$B_k = -2\,\text{Im}\{a_k\} = \frac{2}{T_0}\int_{<T_0>} x(t)\sin(k\omega_0 t)dt$$

Example1 : The Square wave

Fourier coefficients

$$a_k = \begin{cases} \dfrac{1}{j\pi k} & \text{for } k = \pm 1,\ \pm 3,\ \pm 5,.... \\ 0 & \text{for } k = \pm 2,\ \pm 4,\ \pm 6,.... \\ \dfrac{1}{2} & \text{for } k = 0 \end{cases}$$

Trigonometric form

$$A_0 = a_0 = 1/2$$

$$A_k = 2\Re e(a_k) = 0, \quad k = odd$$

$$B_k = -2\,\mathrm{Im}(a_k) = 2/\pi k, \quad k = odd$$

$$x(t) = 1/2 + \sum_{\substack{k=1 \\ k,odd}}^{\infty} \left(2/\pi k\right) \sin(k\omega_0 t)$$

Example : Periodic train of rectangular pulse of width(τ) and period(T_0)

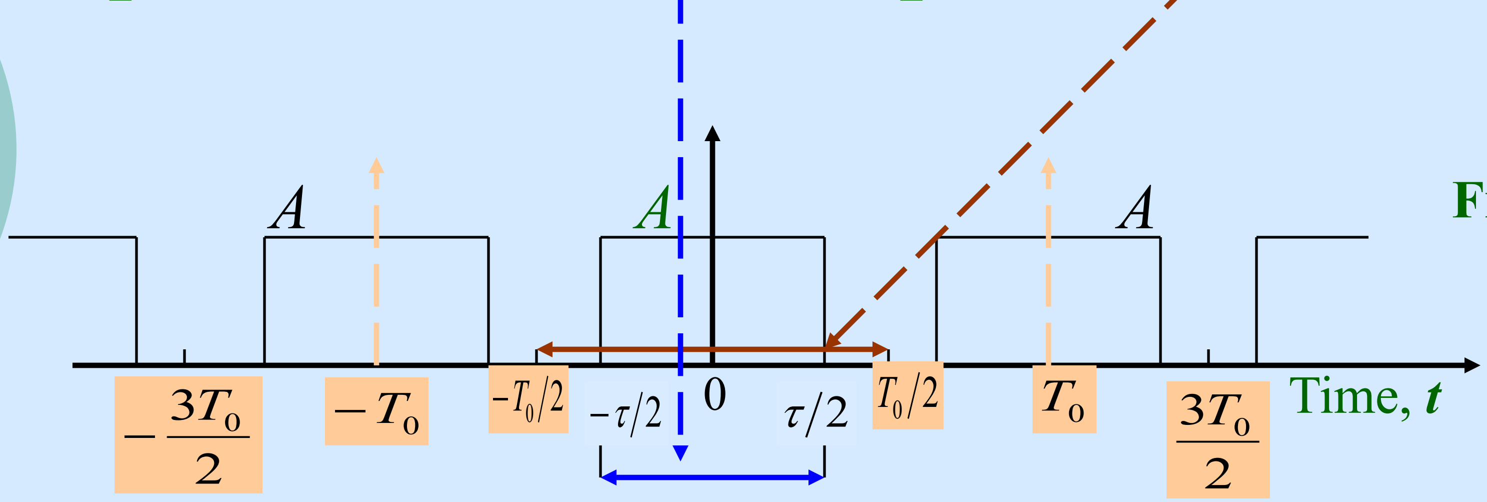


Fig. 4.16

Fourier coefficients

$$a_0 = \frac{A\tau}{T_0}$$

$$a_k = \frac{A}{k\pi}\sin\left(\frac{k\pi n}{T_0}\right), \quad k \neq 0$$

$$Let\ \ T_0 = 1/4\,,\ \tau = 1/20,\ \ A = 1$$

$$a_0 = 4/20 = \frac{1}{5}$$

$$a_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{5}\right),\quad k \neq 0$$

a) Spectrum

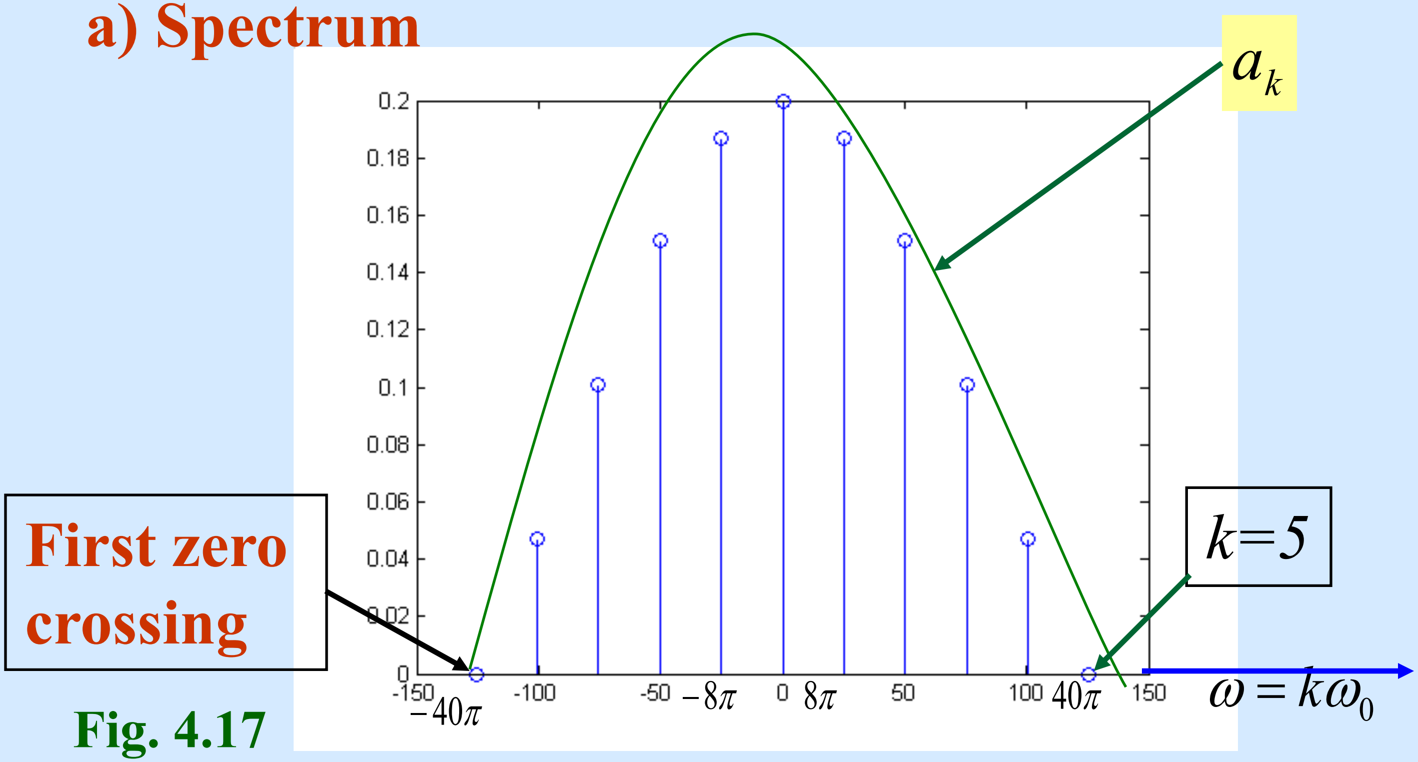


Fig. 4.17

b) Average Power

$$P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t)dt$$

$$= 4 \int_{-1/40}^{1/40} 1dt = 4(1/40 + 1/40) = 0.2$$

c) Average Power contained between 1st zero crossing

$$P_N = \sum_{K=-5}^{5} |a_k|^2 = a_0^2 + 2\sum_{K=1}^{4} |a_k|^2$$

$$= (0.2)^2 + \frac{2}{\pi^2}\left[|\sin(\pi/5)|^2/1^2 + |\sin(2\pi/5)|^2/2^2\right]$$

$$+ \frac{2}{\pi^2}\left[|\sin(3\pi/5)|^2/3^2 + |\sin(4\pi/5)|^2/4^2\right] = 0.181$$

$$P_N/P = 0.181/0.2 = 90.5\%$$

This is one way of defining the bandwidth of a signal

Fourier series practice example

a)

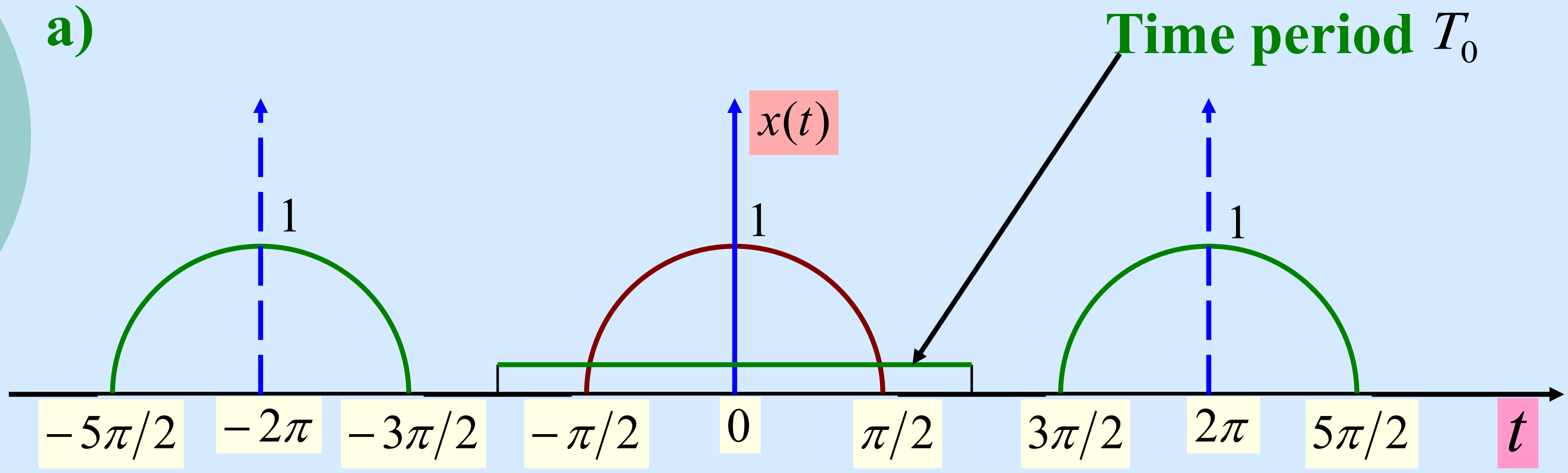


Fig. 4.18

$$x(t)=\begin{cases} 0 & -\pi<t<-\pi/2 \\ \cos t & -\pi/2<t<\pi/2 \\ 0 & \pi/2<t<\pi \end{cases}$$

Calculation of Fourier coefficients, a_k

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

$$\because\ T_0 = 2\pi, \omega_0 = \frac{2\pi}{T_0} = 1$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(t)e^{-jkt}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{\left(e^{jt} + e^{-jt}\right)}{2} e^{-jkt}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{\left(e^{jt(1-k)} + e^{-jt(1+k)}\right)}{2}\,dt$$

$$= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} e^{jt(1-k)}dt + \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} e^{-jt(1+k)}dt$$

$$= \frac{1}{4\pi}\frac{1}{j(1-k)}\left(e^{jt(1-k)}\right)\Big|_{-\pi/2}^{\pi/2} + \frac{1}{4\pi}\frac{1}{j(1+k)}\left(e^{jt(1+k)}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{j4\pi(1-k)}\left(e^{j\pi/2(1-k)} - e^{-j\pi/2(1-k)}\right) + \frac{1}{j4\pi(1+k)}\left(e^{j\pi/2(1+k)} - e^{-j\pi/2(1+k)}\right)$$

$$= \frac{1}{j4\pi(1-k)}\left(2j\sin(\pi/2(1-k))\right) + \frac{1}{j4\pi(1+k)}\left(2j\sin(\pi/2(1+k))\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{(1-k)}\left(\sin(\pi/2(1-k))\right) + \frac{1}{(1+k)}\left(\sin(\pi/2(1+k))\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(1-k)}\left(\cos(\pi k/2)\right) + \frac{1}{(1+k)}\left(\cos(\pi k/2)\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(1-k)}\left(\cos\left(\pi k/2\right)\right) + \frac{1}{(1+k)}\left(\cos\left(\pi k/2\right)\right)\right]$$

$$= \frac{\cos\left(\pi k/2\right)}{2\pi}\left[\frac{1}{(1-k)} + \frac{1}{(1+k)}\right]$$

$$= \frac{\cos\left(\pi k/2\right)}{\pi(1-k^2)}, \quad for\ |k| \neq 1$$

$$for\ \ k = 1\, and\ -1,$$

$$a_1 = a_{-1} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos(t)e^{-jt}dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \frac{\left(e^{jt} + e^{-jt}\right)}{2} e^{-jt}dt = \frac{1}{4} + \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} e^{-2jt}dt = \frac{1}{4}$$

$$a_1 = a_{-1} = \frac{1}{4}$$

Fourier series practice examples *continued*…..

b)

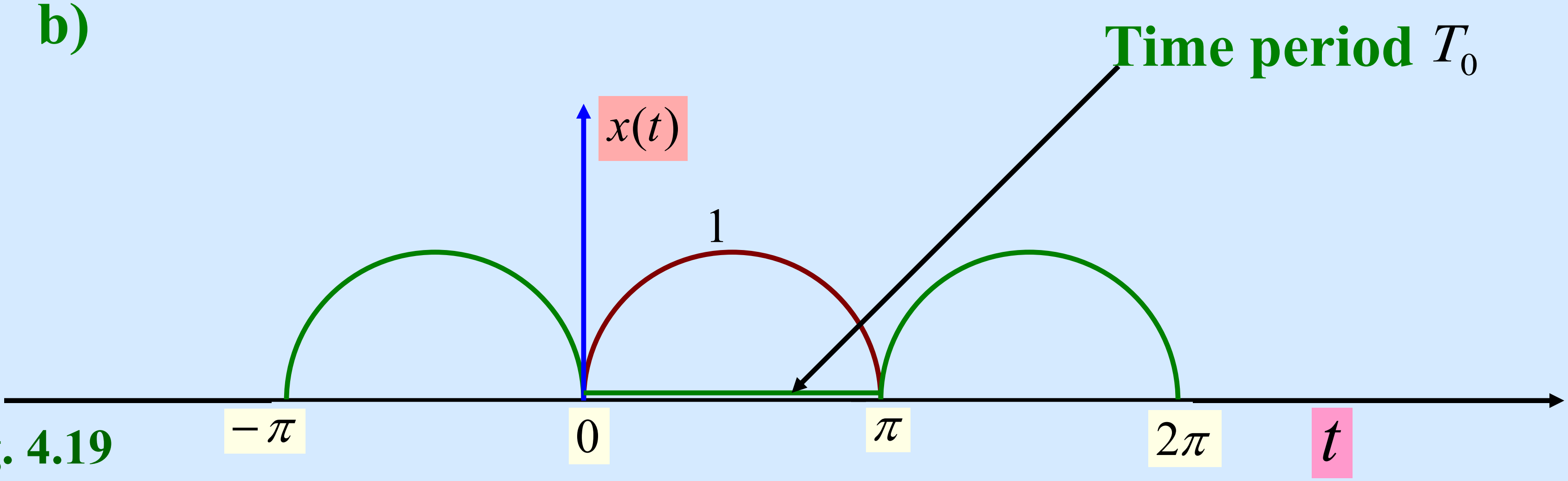


Fig. 4.19

$$x(t) = \sin t \quad 0 < t < \pi$$

$$The\ period\ \ T_0 = \pi$$

Calculation of Fourier coefficients, a_k

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

$$\because\ T_0 = \pi, \omega_0 = \frac{2\pi}{T_0} = 2$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(t)e^{-j2kt}\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\left(e^{jt} - e^{-jt}\right)}{2j} e^{-j2kt}\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\left(e^{jt(1-2k)} + e^{-jt(1+2k)}\right)}{2j}\,dt$$

$$= \frac{1}{2\pi j}\int_0^{\pi} e^{jt(1-2k)}dt + \frac{1}{2\pi j}\int_0^{\pi} e^{-jt(1+2k)}dt$$

$$= \frac{1}{2\pi j}\frac{1}{j(1-2k)}\left(e^{jt(1-2k)}\right)\Big|_0^{\pi} + \frac{1}{2\pi j}\frac{1}{j(1+2k)}\left(e^{jt(1+2k)}\right)\Big|_0^{\pi}$$

$$= \frac{-1}{2\pi}\left[\frac{1}{(1-2k)}\left(e^{j\pi(1-2k)}-1\right) + \frac{1}{(1+2k)}\left(e^{j\pi(1+2k)}-1\right)\right]$$

$$= \frac{-\left(e^{j\pi}-1\right)}{2\pi}\left[\frac{1}{(1-2k)} + \frac{1}{(1+2k)}\right]$$

$$= \frac{\left(1-e^{j\pi}\right)}{2\pi}\left[\frac{2}{(1-4k^2)}\right]$$

$$a_k = \frac{2}{\pi(1-4k^2)}, \; for \; all \;\; 'k'$$

Fourier series practice examples *continued*…..

c): Saw tooth waveform

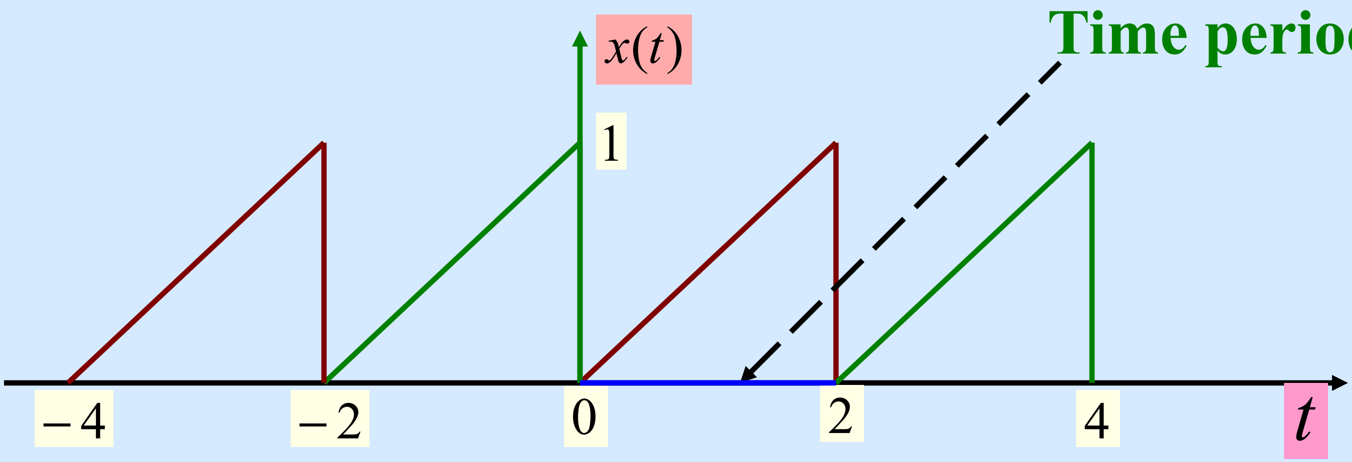


Fig. 4.20

$$x(t) = \frac{t}{2} \quad 0 < t < 2$$

$$x(t+2) = x(t)$$

$$The\ period\ \ T_0 = 2$$

Calculation of Fourier coefficients, a_k

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

$$\because\ T_0 = 2, \omega_0 = \frac{2\pi}{T_0} = \pi$$

$$= \frac{1}{2}\int_0^{2}\left(\frac{t}{2}\right)e^{-j\pi kt}\,dt$$

$$= \frac{1}{4}\int_0^{2} te^{-j\pi kt}\,dt$$

As demonstrated earlier,

$$\int te^{mt} = \frac{e^{mt}}{m}\left(t + \frac{1}{m}\right)$$

$$a_k = \frac{1}{4}\int_0^2 t e^{-j\pi k t} dt$$

$$= \frac{1}{4}\left(\frac{e^{-j\pi k t}}{-j\pi k}\left(t - \frac{1}{j\pi k} \right) \right)\Bigg|_0^2$$

$$= \frac{1}{4}\left(\frac{e^{-j2\pi k}}{-j\pi k}\left(2 - \frac{1}{j\pi k} \right) + \frac{1}{\pi^2 k^2} \right)$$

$$= \frac{1}{4}\left(\frac{2}{-j\pi k} - \frac{1}{\pi^2 k^2} + \frac{1}{\pi^2 k^2} \right) = \frac{j}{2\pi k}$$

$$a_k = \frac{j}{2\pi k}$$

$$a_0 = \frac{1}{4}\int_0^2 t dt = \frac{1}{4}\left[\frac{t^2}{2}\Bigg|_0^2 \right] = \frac{1}{4}[2] = \frac{1}{2}$$

Spectrograms: Introduction

From the perspective of the independent variable, *'Frequency'*, signals can be classified into two:

1. **Signals with constant spectrum**

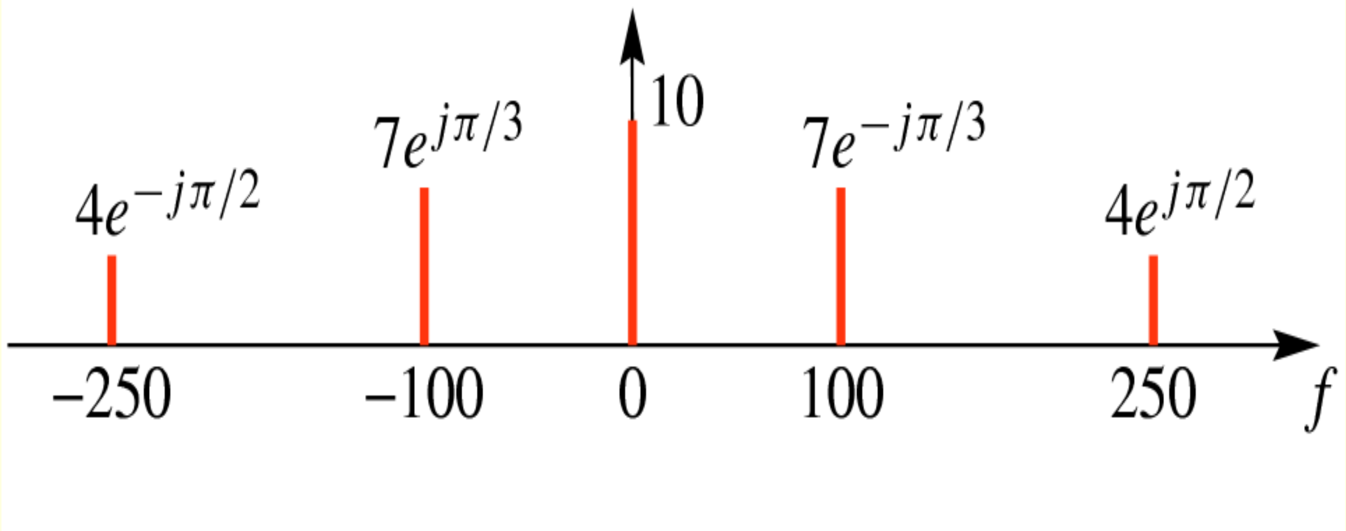
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \phi_k)$$

- As shown earlier, with the help of above equation, using different values for amplitude, fundamental frequency and phase, periodic signals can be generated
- If the frequencies chosen are not harmonically related, a non-periodic signal will result

Example: *Signal with constant spectrum*

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Spectrum



As it can be seen, the spectrum doesn't change with time

2. Signals with varying spectrum

Most of the real world signals like music and speech doesn’t have a constant spectrum

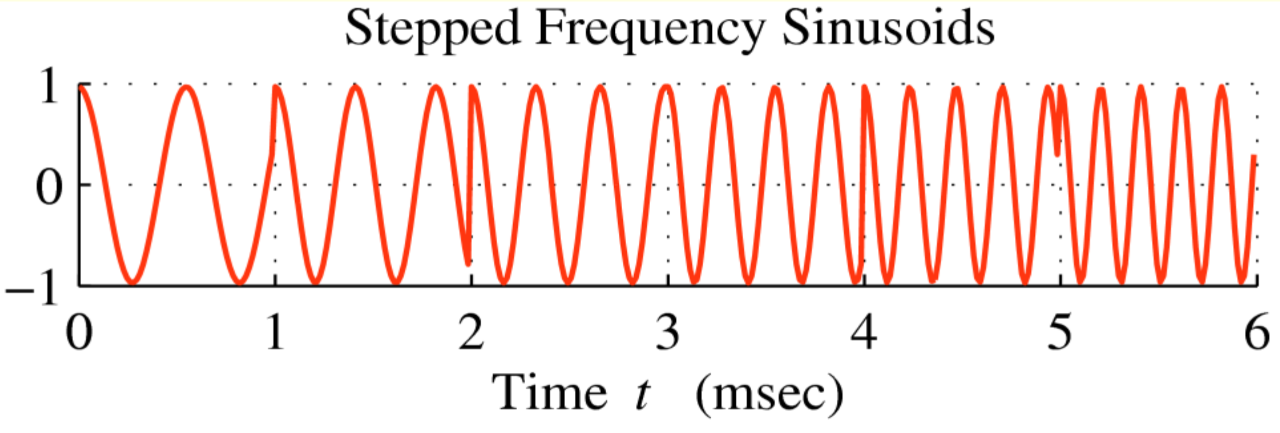
For a very short time music or speech spectrum can be constant. However, when the tone changes or when we speak a different word, the frequency content changes. In fact without a change in frequency no information can be transmitted

Most real world signals can be synthesized if we let the frequencies, amplitudes and phases vary with time, this leads to the concept of *time-frequency spectrum* or *spectrogram*

Example: Signals with varying spectrum

Frequency changes linearly from a low to high in Chirp signals

Concatenating a large number of short constant-frequency sinusoids produces such a signal



Six frequencies from 30 to 80 Hz, notice the discontinuities

As it can be seen, care should be taken to see that the waveform doesn't have discontinuities

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$x(t) = A\cos(\psi(t))$$

$$Where,\ \psi(t) = \omega_0 t + \phi$$

In the standard sinusoid, the angle function changes linearly with time, making it a quadratic function produces the effect of variable frequency, this class of Signals are also called '*FM signals*'

$$\psi(t) = 2\pi\mu t^2 + 2\pi f_0 t + \phi$$

Instantaneous frequency is an important measurement For the varying frequency signals

$$\psi(t) = 2\pi\mu t^2 + 2\pi f_0 t + \phi$$

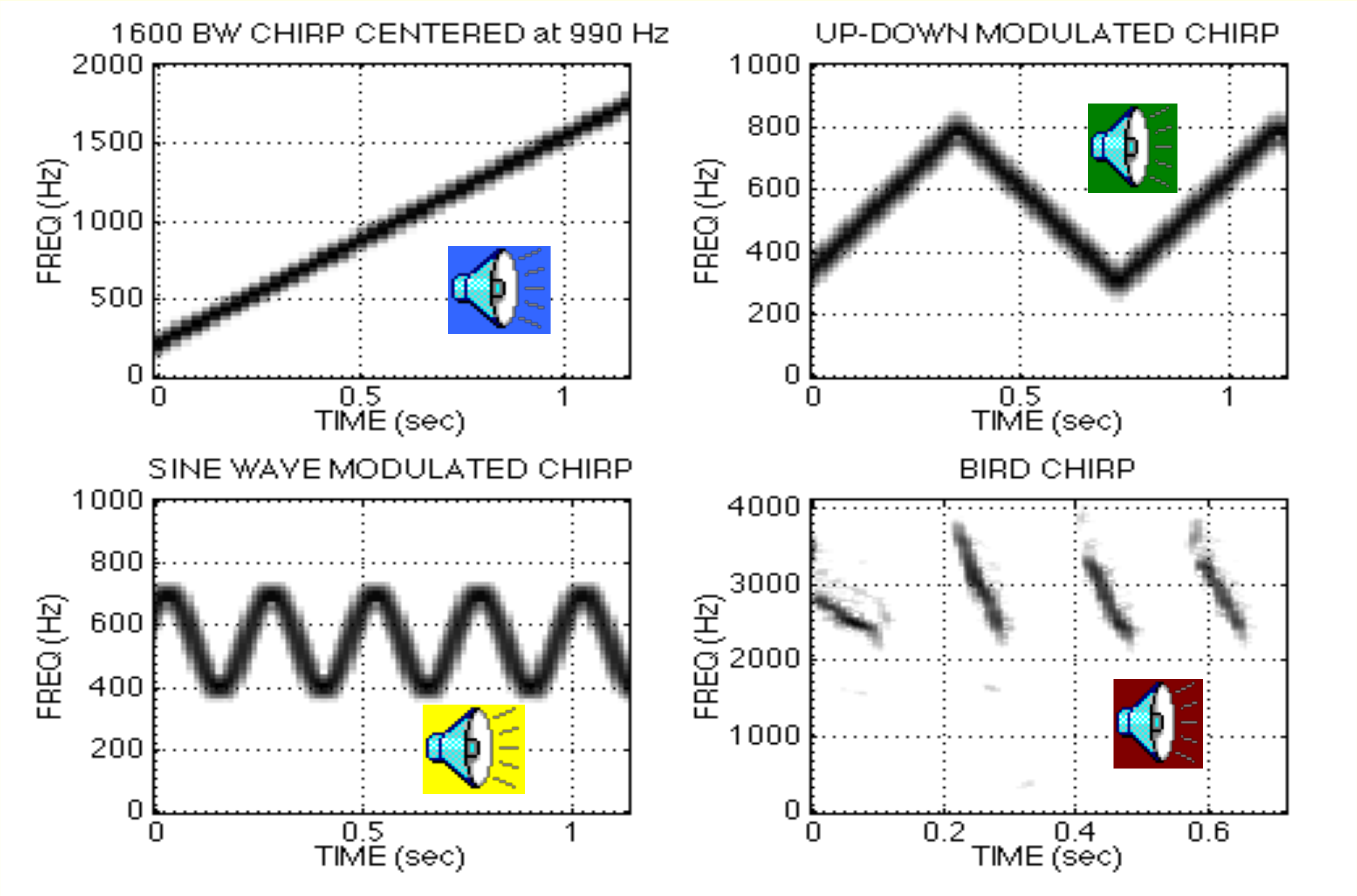
$$\omega_i(t) = \frac{d\psi(t)}{dt}\left(rad / \sec\right)$$

$$in\ cycles,\ f_i(t) = \frac{1}{2\pi}\left(\frac{d\psi(t)}{dt}\right)$$

$$f_i(t) = \left(2\mu t + f_0\ \right)Hz$$

$$Where,\ f_0\ is\ the\ starting\ \ frequency$$

Some examples produced with the help of equation



Example: Synthesizing a chirp Signal

Synthesizing a frequency sweep from f_1=220 Hz to f_2 =2320Hz, over a 3 second time interval

$$f_i(t) = \frac{(f_2 - f_1)}{(t_2 - t_1)} t + f_1 = \frac{(2320 - 220)}{3} t + 220$$

$$\omega_i(t) = \frac{d\psi(t)}{dt}$$

$$\psi(t) = \int_0^t \omega_i(t)dt = 2\pi \int_0^t f_i(t)dt$$

$$\psi(t) = 2\pi \int_0^t \left(\frac{(2320 - 220)}{3} t + 220 \right) dt = 700\pi t^2 + 440\pi t + \phi,$$

ϕ, *is an arbitrary constant*

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “Signal Processing First”, Prentice Hall, 2003