Discrete - Time Signals and Systems

Sampling – II
Sampling theorem & Reconstruction

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Sampling at diffe- -rent rates

From these figures, it can be concluded that it is very important to sample the signal adequately to avoid problems in reconstruction, which leads us to ***Shannon's sampling theorem***

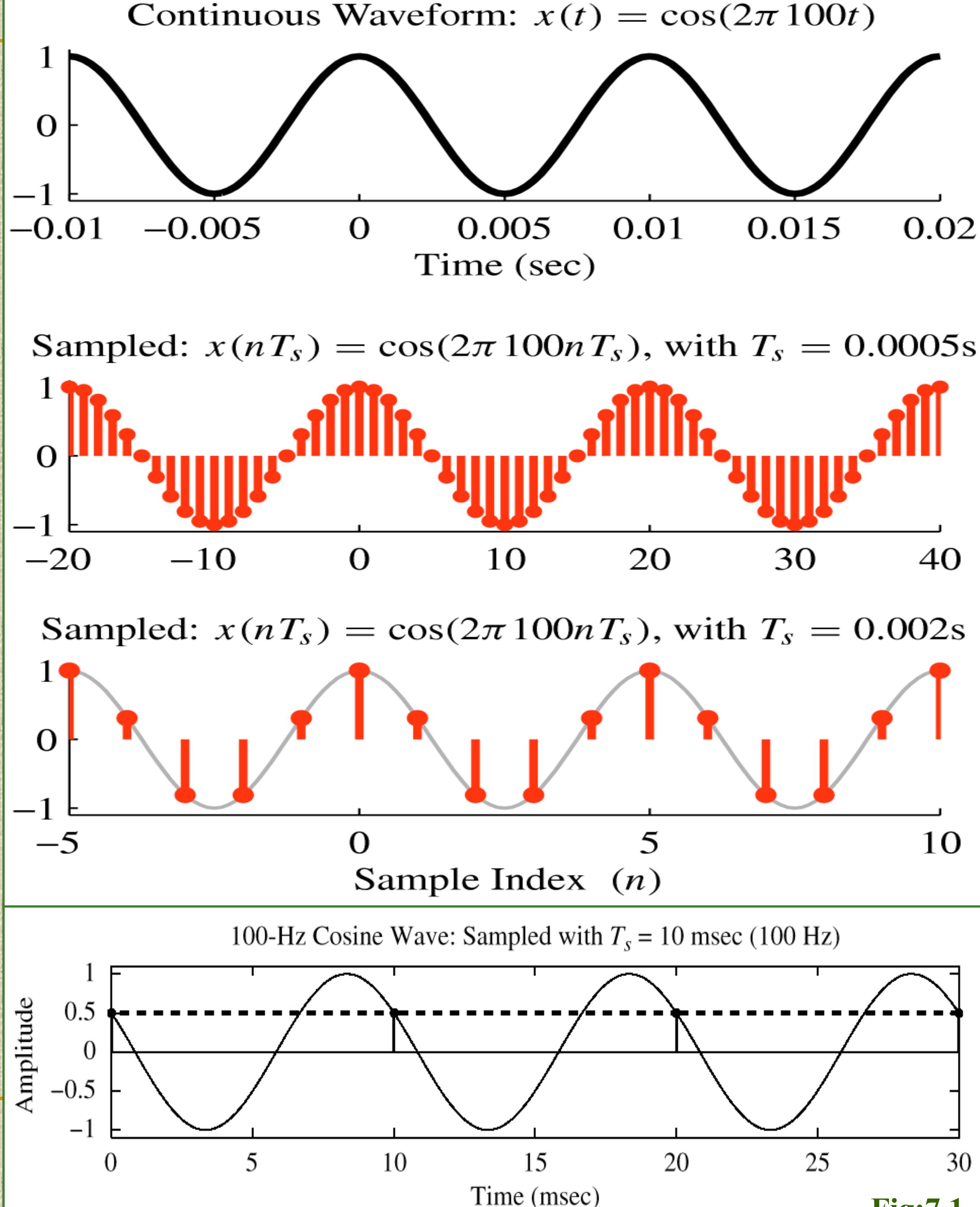
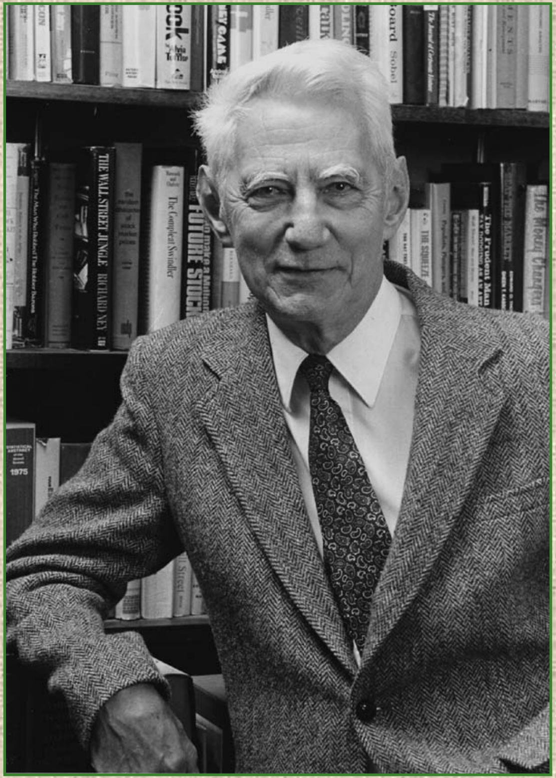


Fig:7.1

Claude Shannon: *The man who started the digital revolution*



1916-2001

Shannon arrived at the revolutionary idea of digital representation by sampling the information source at an appropriate rate, and converting the samples to a bit stream

Before Shannon, it was commonly believed that the only way of achieving arbitrarily small probability of error in a communication channel was to reduce the transmission rate to zero.

All this changed in 1948 with the publication of "A Mathematical Theory of Communication"—Shannon's landmark work

Shannon's Sampling theorem

A continuous signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x[nT_s]$, if the samples are taken at a rate $f_s \geq 2f_{\max}$, where $f_s = 1/T_s$

This simple theorem is one of the theoretical Pillars of digital communications, control and signal processing

Shannon's Sampling theorem,

- **States that reconstruction from the samples is possible, but it doesn't specify any algorithm for reconstruction**
- **It gives a minimum sampling rate that is dependent only on the frequency content of the continuous signal *x(t)***
- **The minimum sampling rate of $2f_{max}$ is called the *"Nyquist rate"***

Example1: Sampling theorem-*Nyquist rate*

$x(t) = 2\cos(20\pi t), \quad find\ the\ Nyquist\ frequency?$

$x(t) = 2\cos(2\pi(10)t)$

$The\ only\ frequency\ in\ the\ continuous-time$

$signal\ is\ 10Hz$

$\therefore\ f_{\max} = 10Hz$

$Nyquist\ sampling\ rate\ Sampling\ rate,$

$f_{snyq} = 2f_{\max} = 20Hz$

Continuous-time sinusoid of frequency 10Hz

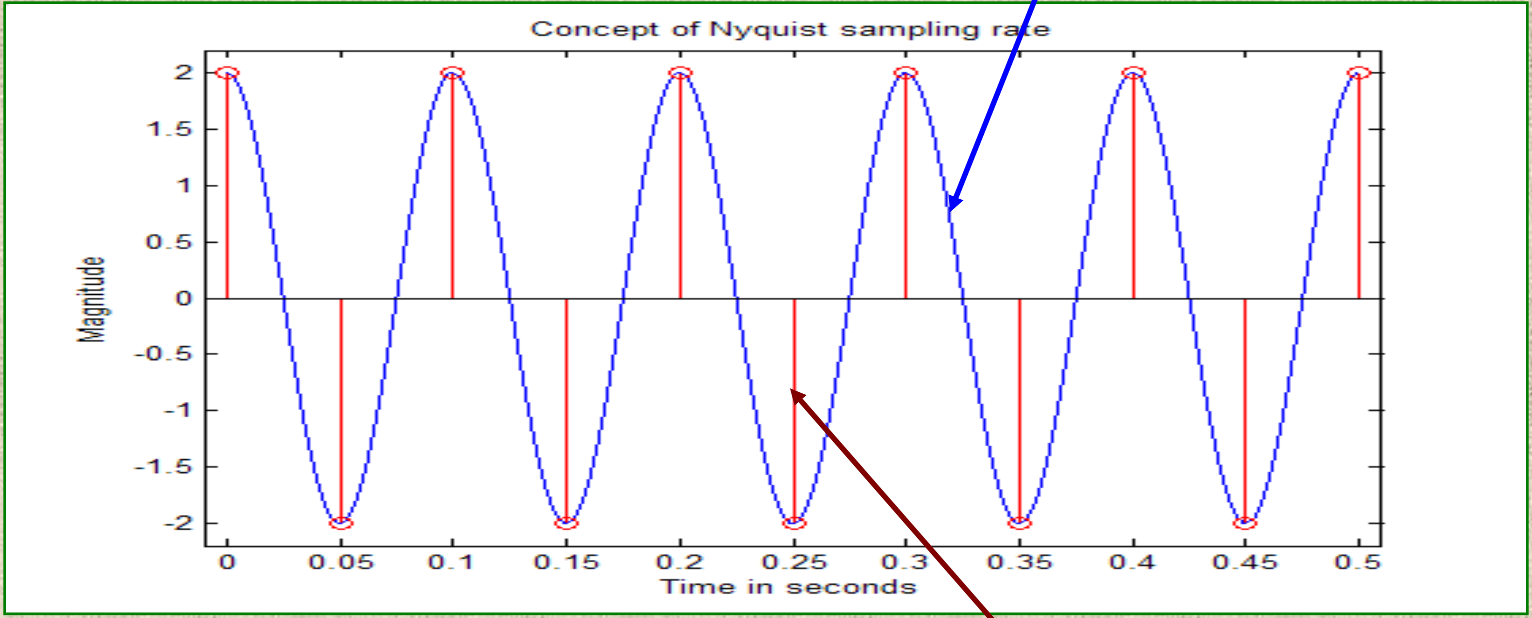


Fig:7.2

Sampled at Nyquist rate, so, the theorem states that 2 samples are enough per period. Intuitively it doesn't seem to be enough, there must be a sophisticated algorithm for reconstruction

Example2: *Nyquist rate*

$$x(t) = \cos(10\pi t) + \sin(22\pi t)$$

$$= \cos(2\pi(5)t) + \sin(2\pi(11)t)$$

The frequencies in the signal are 5 *and* 11,

$$\therefore f_{\max} = 11Hz$$

$$Nyquist\ frequency = 2f_{\max} = 22Hz$$

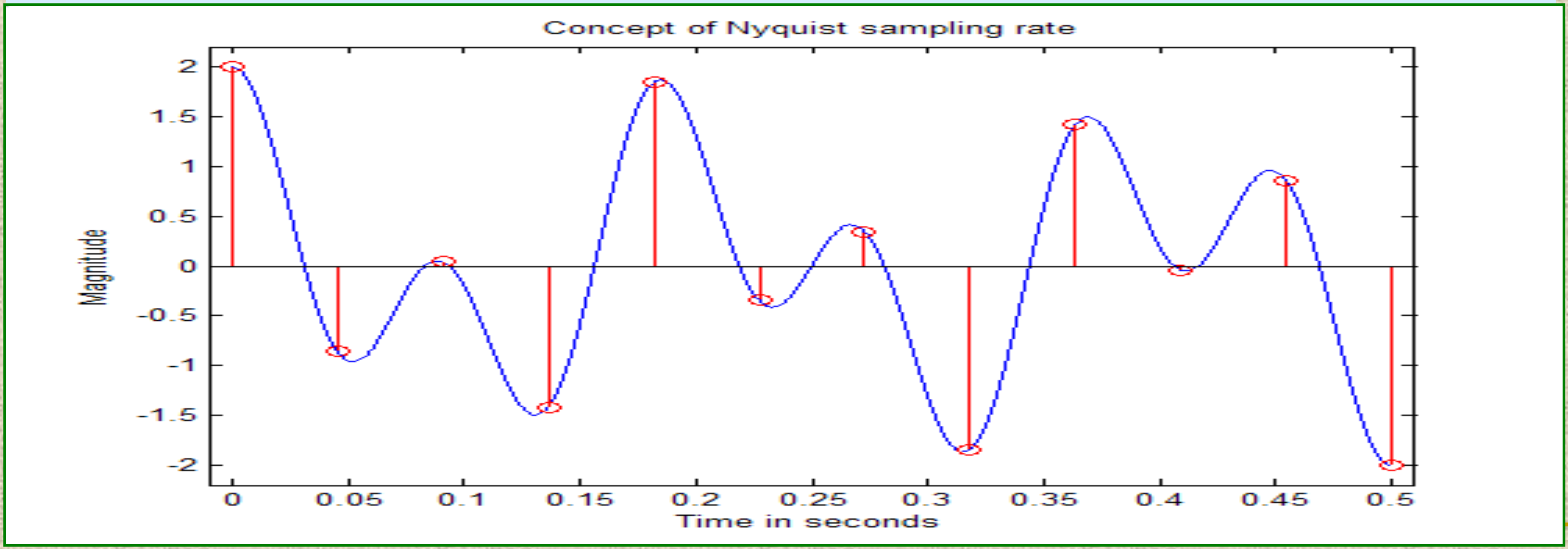


Fig:7.3

Example3: *Nyquist rate*

The positive side of the spectrum is shown below, find the '*Nyquist rate*' of sampling for this signal?

Fig:7.4

The range of the spectrum is from 21 to 48

f_{max}=48Hz

***Nyquist sampling* rate, $f_s = 2 f_{max}$ =96Hz**

Frequency mapping

$$\textit{Continuous signal with frequency } f_0$$

$$x(t) = A\cos(2\pi f_0 t), \textit{ sampled at a rate } f_s$$

$$x[n] = A\cos(2\pi f_0 n / f_s)$$

$$\hat{\omega} = 2\pi f_0 T_s$$

$$\hat{\omega} \quad \textit{discrete} - \textit{time frequency}$$

$$\omega = 2\pi f_0 \quad \textit{continuous} - \textit{time frequency}$$

Frequency mapping *contd….*

$$\therefore \hat{\omega} = \omega T_s$$

$$\omega = \hat{\omega}/T_s = \hat{\omega} f_s$$

Thus there is a corresponding continuous-time frequency for every discrete-time frequency

However, the converse is not true, due to aliasing and folding. Principal aliases are the generally accepted basis for obtaining a continuous-time frequency from a discrete-time one

With the *"Nyquist sampling rate"*:

$$\hat{\omega} = \omega T_s = \omega / f_s$$

Let there be 'i' frequencies in the continuous – time signal, mapping them into discrete – time frequencies

$$\hat{\omega}_i = \omega_i / f_s = \frac{2\pi f_i}{f_s}, \qquad i = 1, 2, 3....$$

$$\because f_s = 2 f_{\max}, \quad Nyquist\ rate$$

$$\hat{\omega}_i = \omega_i / f_s = \frac{2\pi f_i}{2 f_{\max}} = \frac{\pi f_i}{f_{\max}}$$

$$\hat{\omega}_{\max} = \frac{\pi f_{\max}}{f_{\max}} = \pi$$

correspondingly for negative frequencies

$$\hat{\omega}_{\min} = -\pi$$

Thus the discrete - time frequencies are guaranteed to be in the range $-\pi < \hat{\omega} \leq \pi$

The above result is the combined effect of Applying '*Nyquist rate*' to the principal alias domain

$-\pi < \hat{\omega} \leq \pi$ *also implies that the continuous frequency obtained from a discrete one is guaranteed to be in the range of* $-\frac{1}{2} f_s < f_i \leq \frac{1}{2} f_s$ *or* $-f_{\max} < f_i \leq f_{\max}$

Spectrum view of frequency mapping

Consider a continuous-time sinusoid x(t) with frequency f_0

$$x(t) = A\cos(\omega_0 t + \phi) = A\cos(2\pi f_0 t + \phi)$$

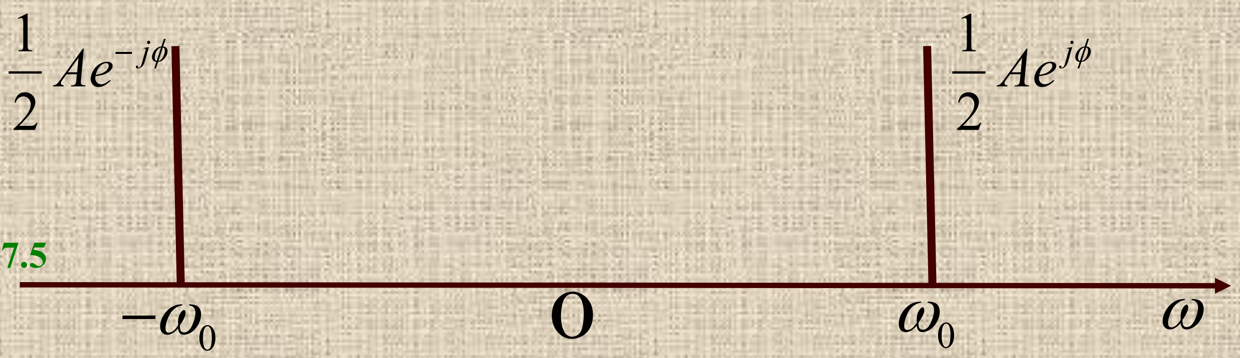


Fig. 7.5

Notice that the spectrum is plotted against ω

The continuous-time signal sampled at a rate f_s

$$x[n] = x(n/f_s) = A\cos((\omega_0/f_s)n + \phi)$$

This discrete-time spectrum has,

$$\hat{\omega} = \omega_0/f_s + 2\pi l, \quad l = 0, \pm 1, \pm 2, \ldots$$

$$\hat{\omega} = -\omega_0/f_s + 2\pi l, l = 0, \pm 1, \pm 2, \ldots$$

Frequencies associated with principal +ve frequency, ω_0/f_s

Principal *aliases*

Frequencies associated with principal –ve frequency, $-\omega_0/f_s$ notice conjugate for magnitude. *folded aliases* are also associated with –ve frequencies

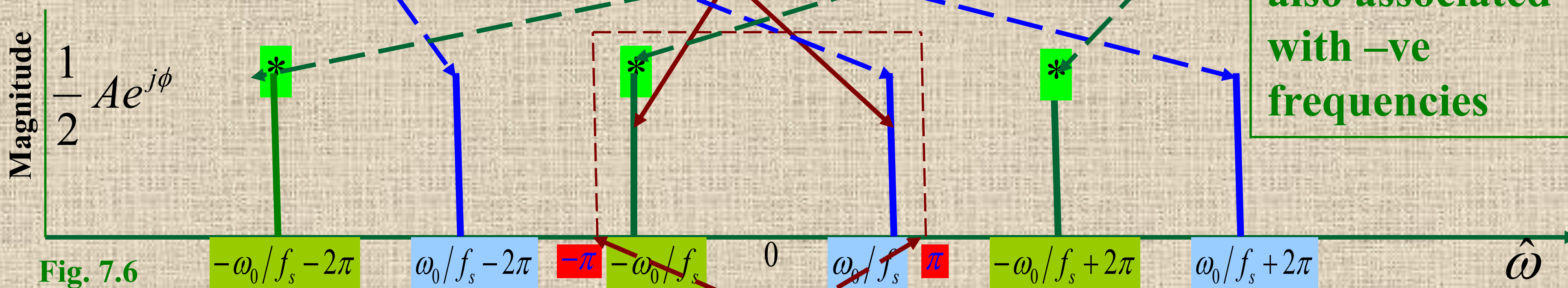


Fig. 7.6

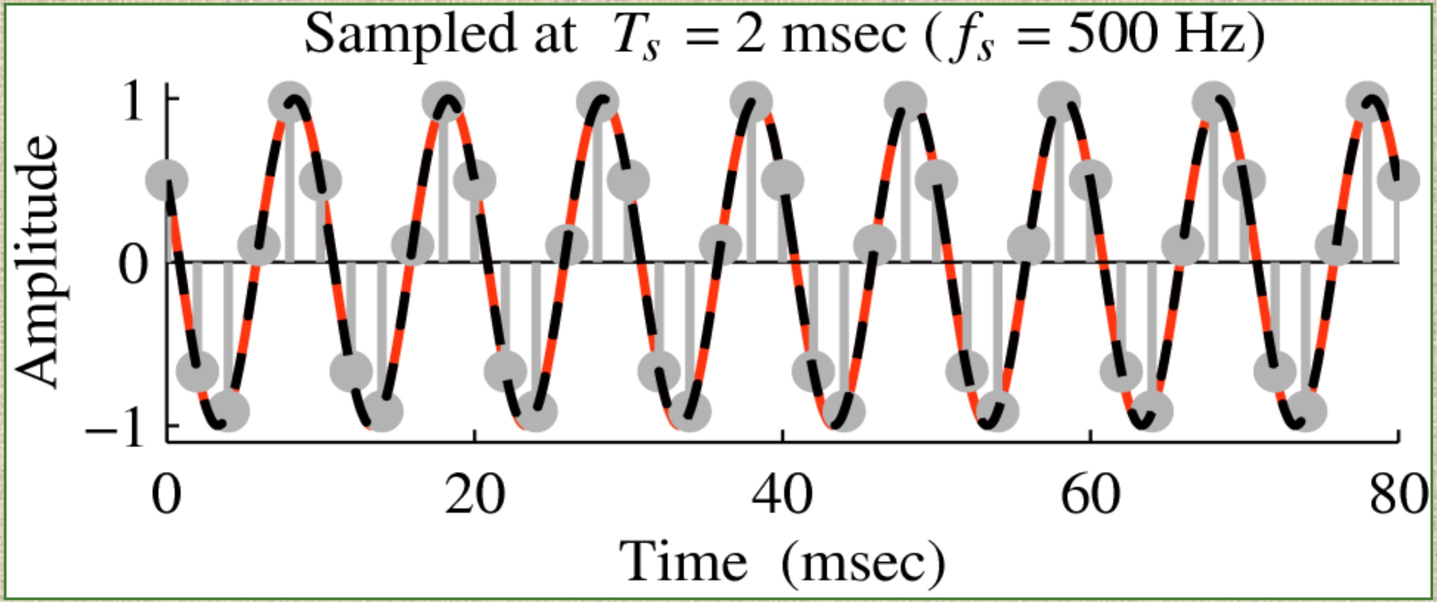
fundamental domain obtained with *Nyquist rate*

Example1: *Over Sampling*

- In most applications sampling rate is chosen to be higher than Nyquist rate to avoid problems in reconstruction

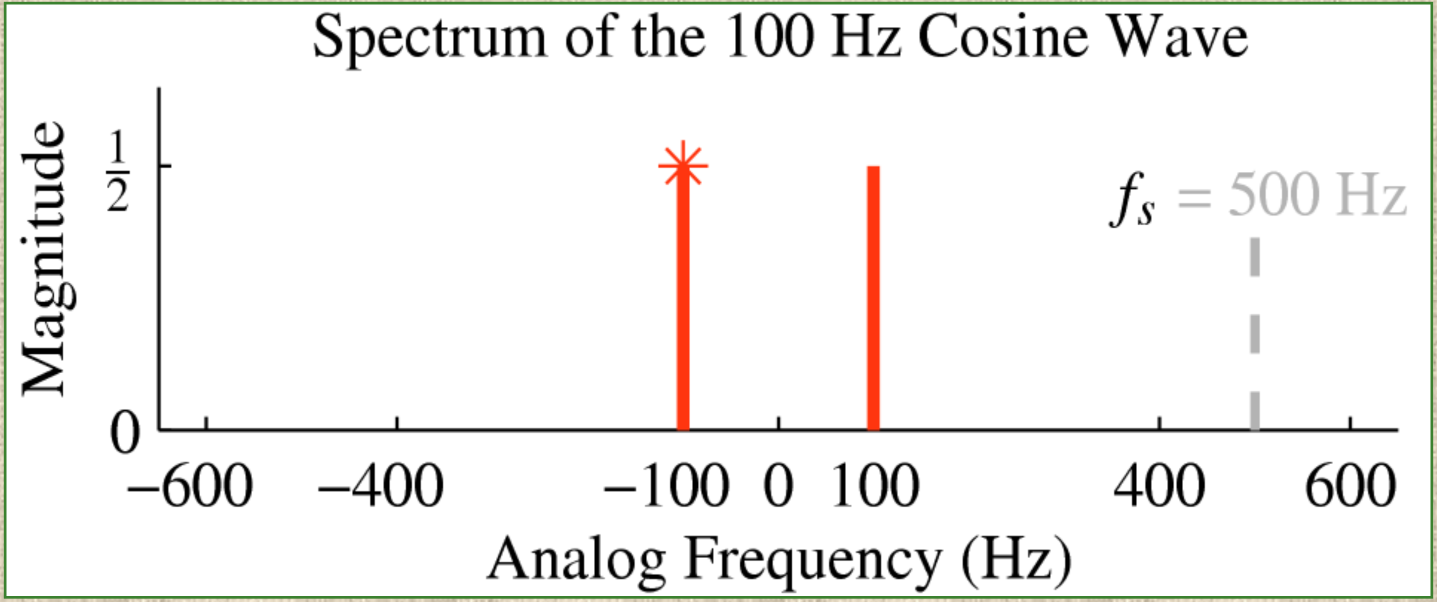
- The sampling rate in CD's is 44.1kHz. The highest frequency we can hear is 20kHz, so sampling rate is slightly higher than 40kHz

- Consider sampling a 100Hz sinusoid at 500*samples/sec*



Sampled at $T_s = 2$ msec ($f_s = 500$ Hz)
Amplitude
1
0
−1
0
20
40
60
80
Time (msec)

Fig:7.7



Spectrum of the 100 Hz Cosine Wave
Magnitude
$\frac{1}{2}$
0
$f_s = 500$ Hz
−600
−400
−100
0
100
400
600
Analog Frequency (Hz)

Fig:7.8

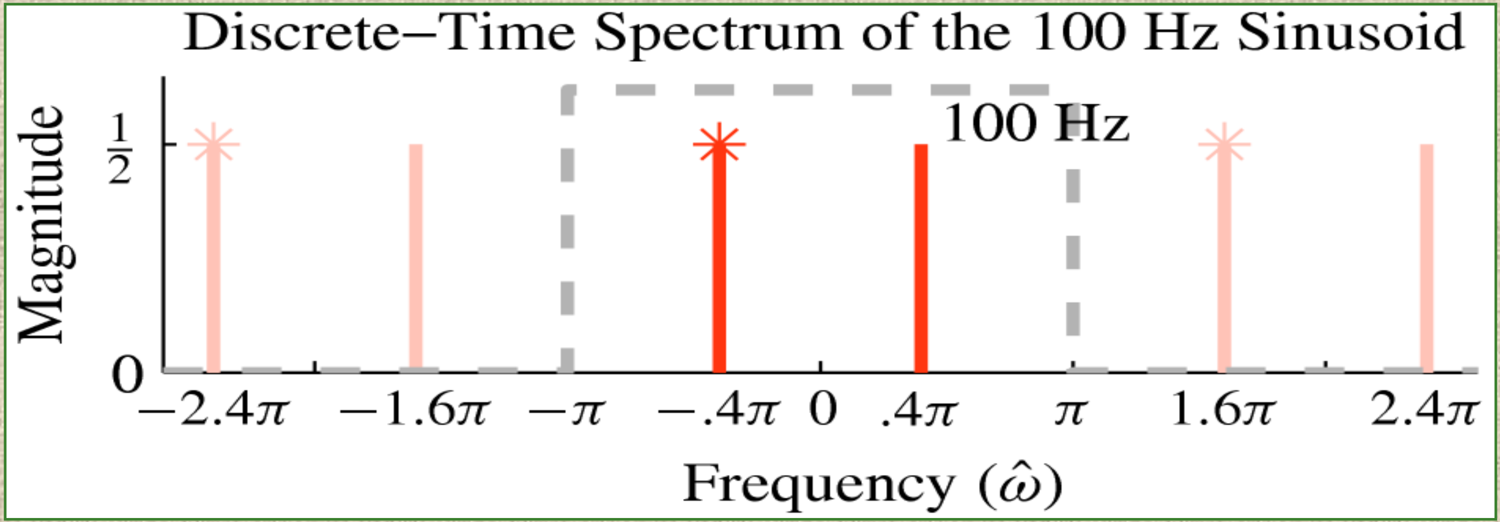


Fig:7.9

$$\hat{\omega} = \omega T_s = \omega / f_s = \frac{2\pi 100}{500} = 0.4\pi$$

$$Nyquist\ rate = 2(100) = 200Hz,\ but\ f_s = 500Hz$$

$$f_s >> Nyquist\ rate$$

Notice that both principal aliases are well with in the limit of $-\pi < \hat{\omega} \leq \pi$ **In the reconstruction only frequencies in this range are used to get continuous frequencies**

Example2: Under Sampling, *aliasing*

The effect of under sampling can be seen in the time-domain plot itself

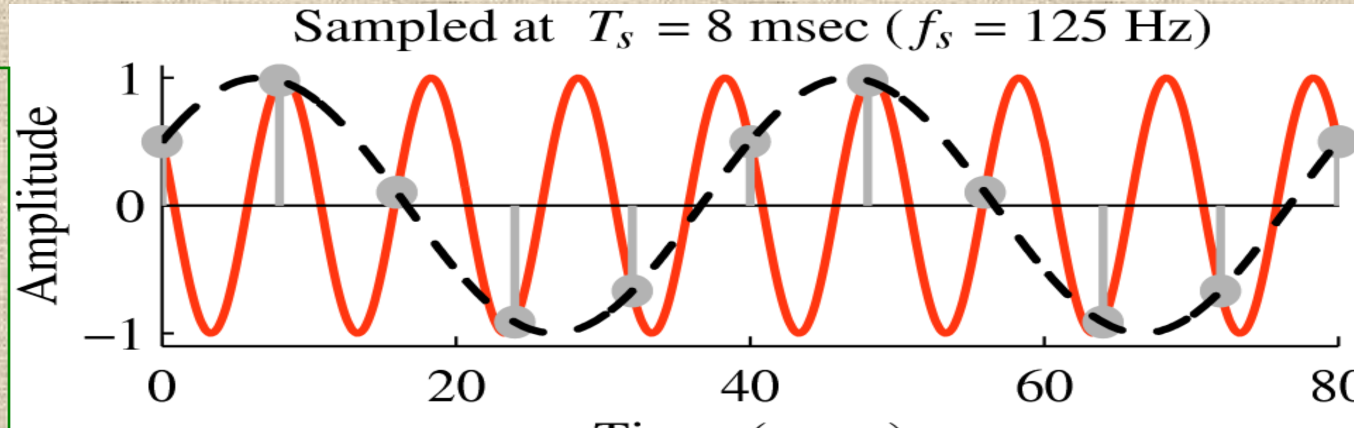


Fig:7.10

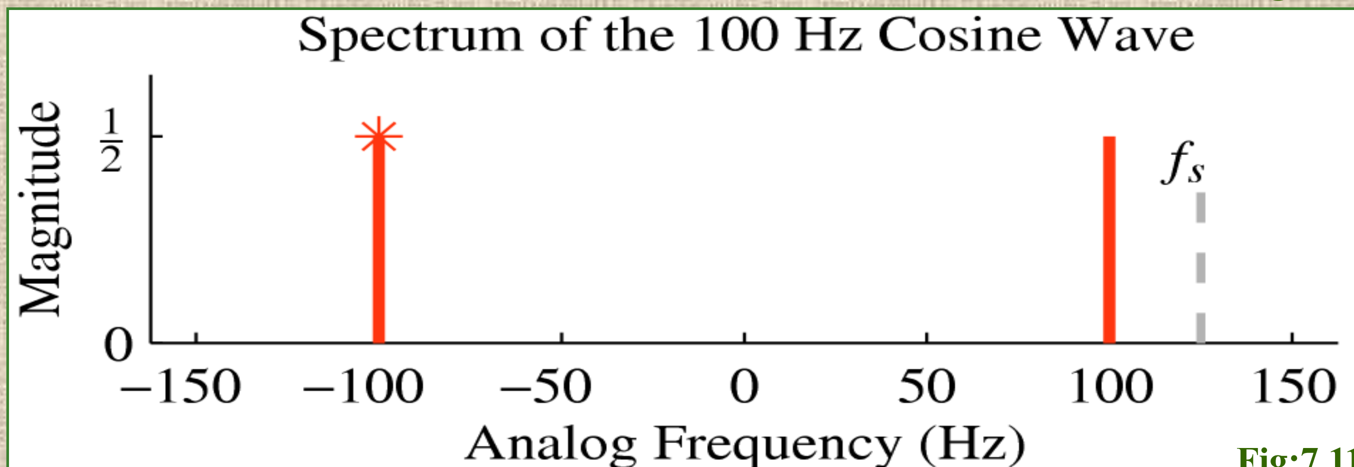


Fig:7.11

$$Nyquist\ rate = 2(100) = 200Hz,\ but\ f_s = 125Hz$$

$$f_s << Nyquist\ rate$$

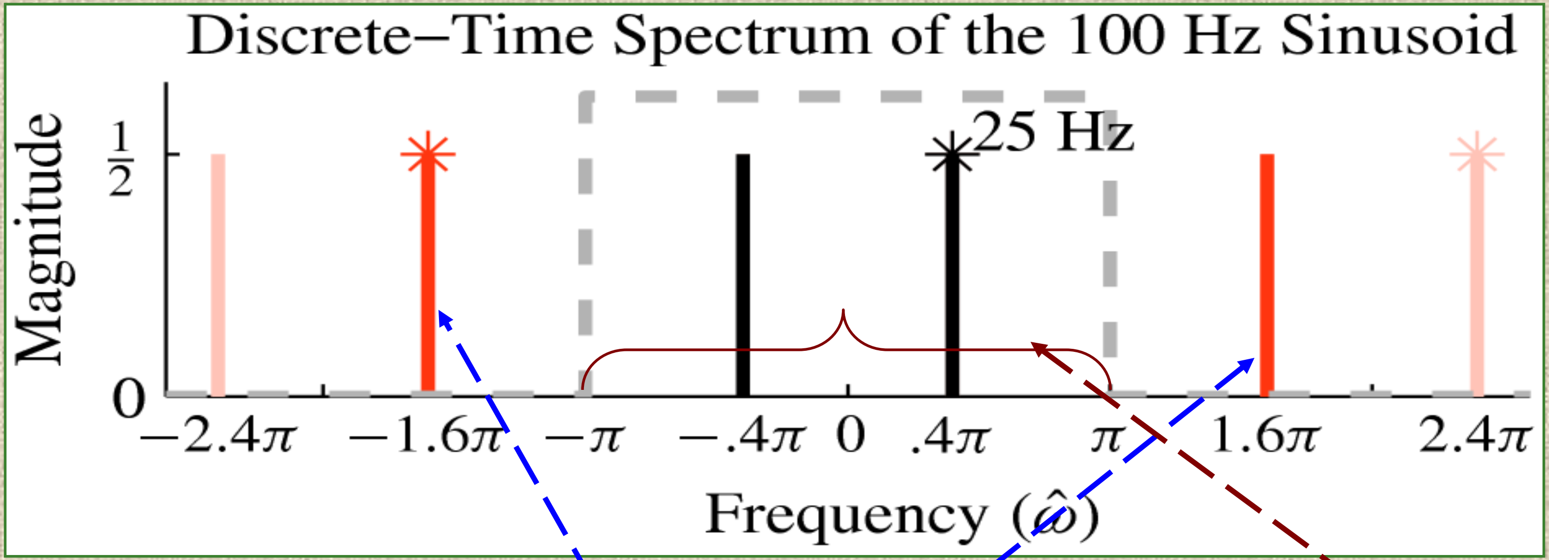


Fig:7.12

$$\hat{\omega} = \omega T_s = \omega / f_s = \frac{2\pi 100}{125} = 1.6\pi$$

Notice that prinipal aliases are beyond the range $(-\pi, \pi]$

And their aliases are in the primary range $-\pi < \hat{\omega} \le \pi$

Mapping 0.4π *results* in *an analog frequency of* $25Hz$

Example3: Sampling at the rate of signal frequency

sampling at the signal frequency means picking up the same value from each cycle

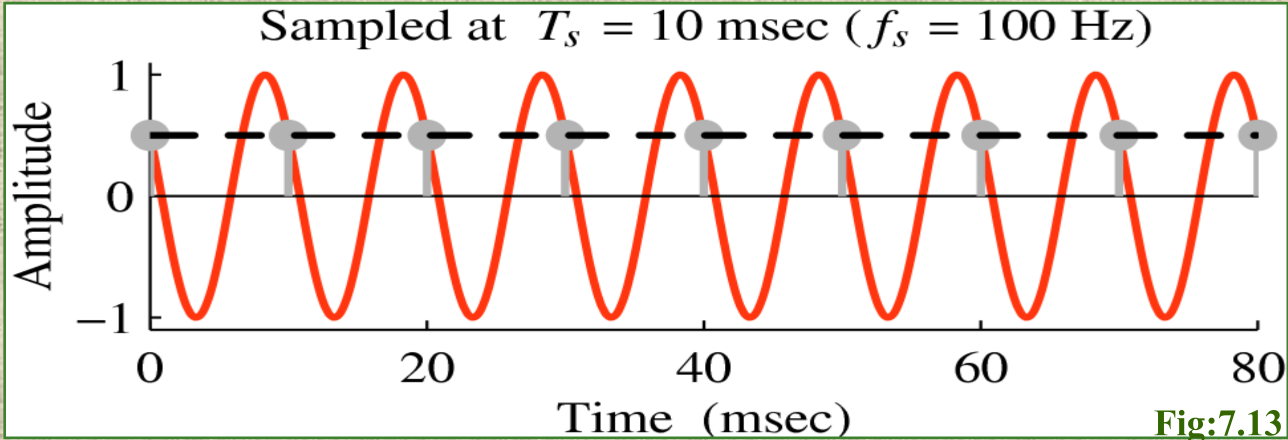


Fig:7.13

Analog spectrum, notice that the signal and the sampling frequencies are same, 100Hz

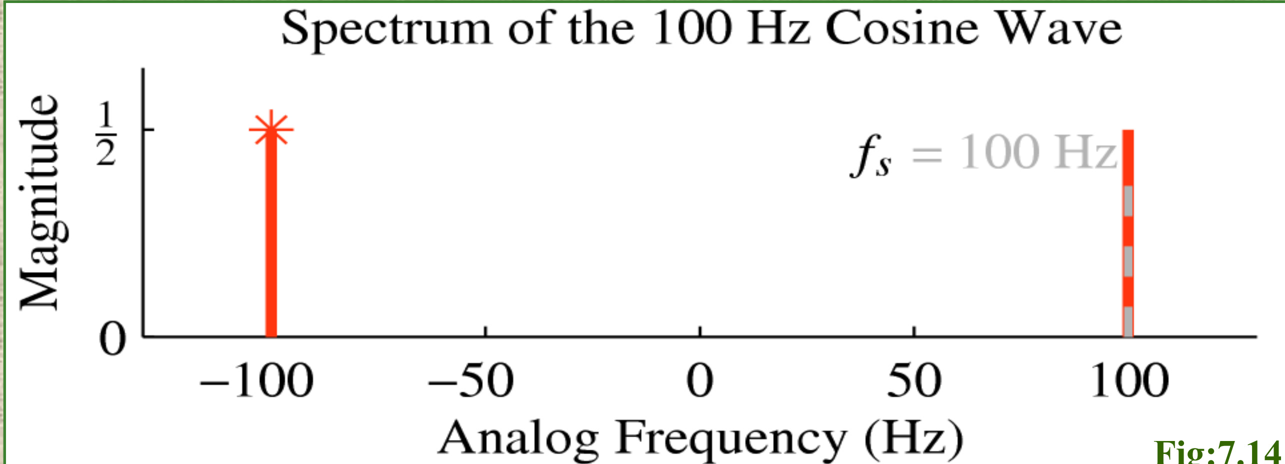


Fig:7.14

$$Nyquist\ rate = 2(100) = 200Hz,\ but\ f_s = 100Hz$$

$$f_s <<< Nyquist\ rate$$

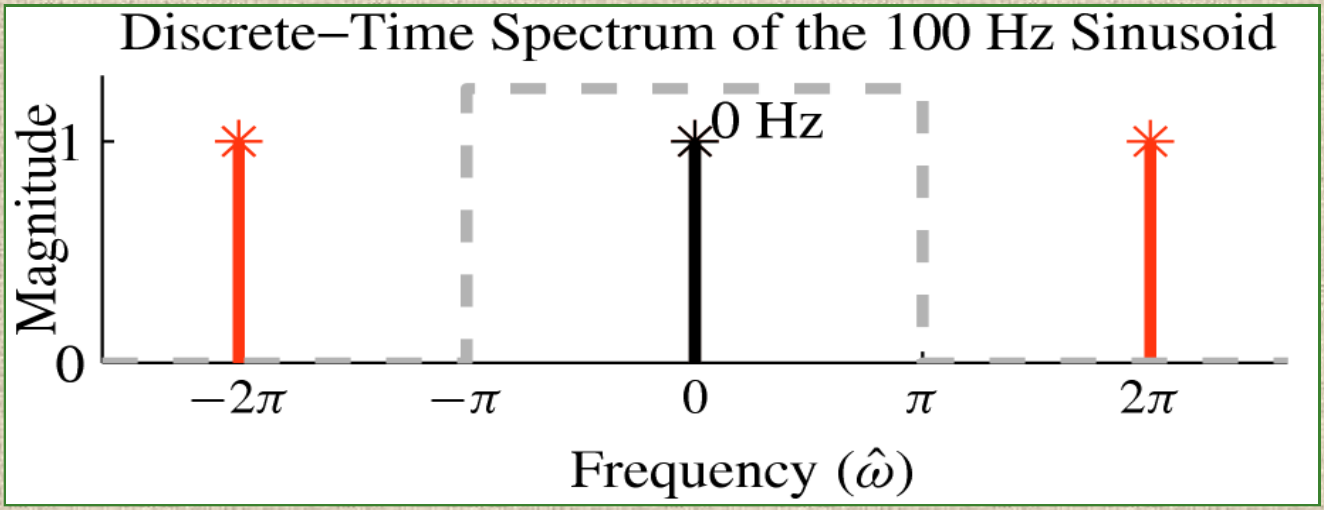


Fig:7.15

$$\hat{\omega} = \omega T_s = \omega / f_s = \frac{2\pi 100}{100} = 2\pi$$

Notice that prinipal aliases are at 2π *and*, -2π

And '0' is their common alias, which is in the range $-\pi < \hat{\omega} \leq \pi$

The result is obvious as we have the same value for each sample

This movie illustrates the phenomenon of aliasing. A 600 Hz sinusoid is sampled at 500 samples per second.

This movie illustrates the phenomenon of folding. A 600 Hz sinusoid is sampled at 750 samples per second.

A 600 Hz sinusoid is sampled at 2000 samples per second. Since the samples are taken at more than two times the frequency of the cosine wave, there is no aliasing.

Apparent frequency

If the Nyquist rate is not followed, the apparent frequency will not be the actual frequency

$$f_s \geq 2f_{\max}, \quad f_{apparant} = f_{actual}$$

$$0 \leq f_s < 2f_{\max}, \quad folding\ occurs\ at\ f_s/2$$

If the sampling rate is not enough, the chirp signal could sound like this **, remember how an actual chirp is supposed to sound** 

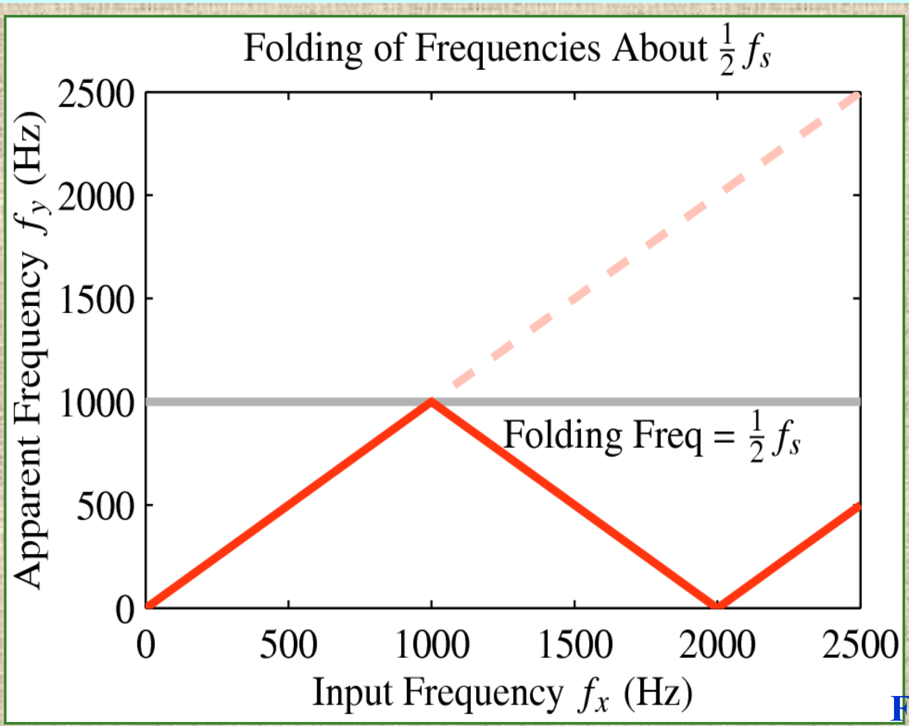
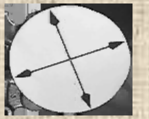


Fig:7.16

A Mechanical viewpoint of sampling

In this movie the video camera is sampling at a fixed rate of 30 frames/second. Observe how the rotating phasor aliases to different speeds as it spins faster.

In this movie the video camera is sampling at a fixed rate of 30 frames/second. Observe how the rotating phasors alias to a different speed as the disk spins faster. The fact that the four phasors are identical further contributes to the aliasing effect.



Samples of continuous time signal are processed as required in this stage, output is also discrete samples

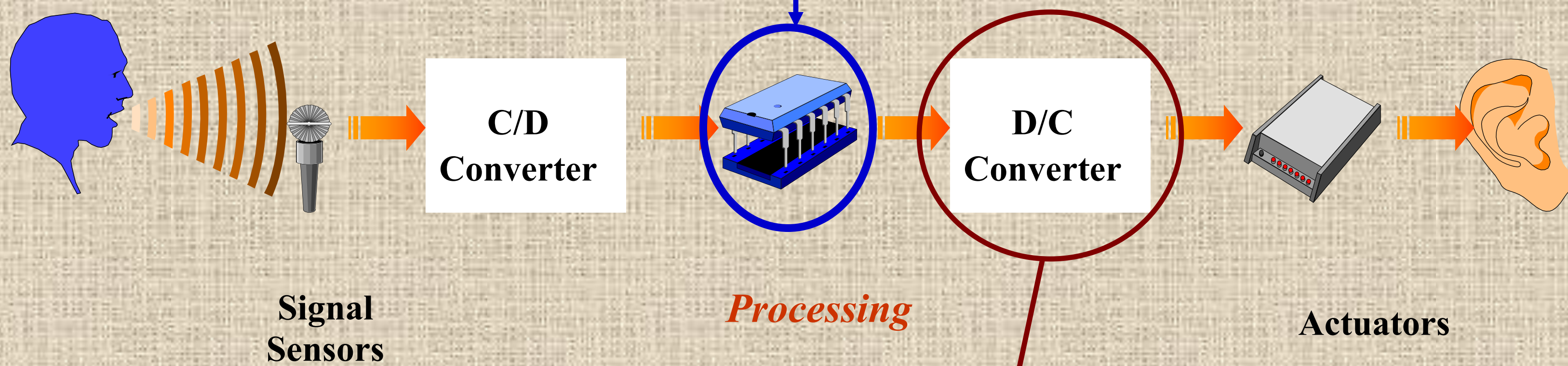


Fig:7.17, General System

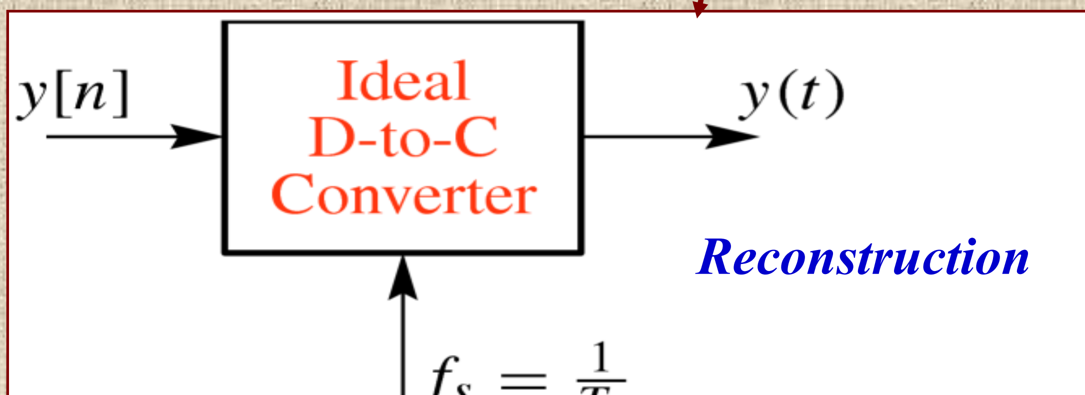


Fig:7.18

Ideal Reconstruction

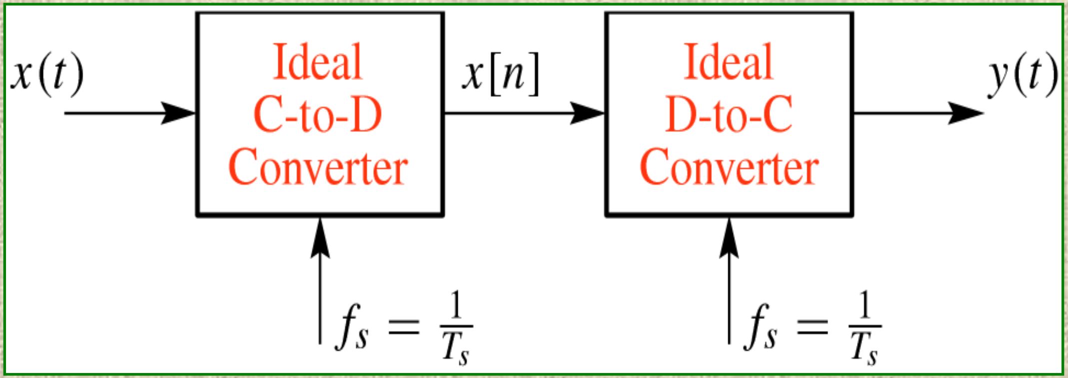


Fig:7.19

Ideal C-D converter was defined as:

$$x(t) = x[nT_s] = x[n/f_s]$$

Ideal D-C converter is governed by an inverse relation to that of C-D converter

$$y(t) = y[n]\big|_{n=f_s t} \qquad -\infty < n < \infty$$

Ideal Reconstruction *contd*….

The simple substitution of '$n = f_s t$', is only valid if the signal consists of one or more sinusoids or if the signal can be expressed as a mathematical formula

$$y[n] = A\cos(2\pi f_0 n T_s + \phi)$$

$$y(t) = A\cos(2\pi f_0 t + \phi)$$

Most of the real world signals can't be reduced into a simple mathematical equation from the discrete samples

- **D-C conversion involves filling in the signal values between sampling instances $t_n = nT_s$**
- ***"Interpolation"* can be used to approximate the behavior of ideal D-C converter**

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]\,p(t - nT_s)$$

The above equation describes a broad class of D-C converters. Where *p(t)* is the characteristic Pulse shape of the converter

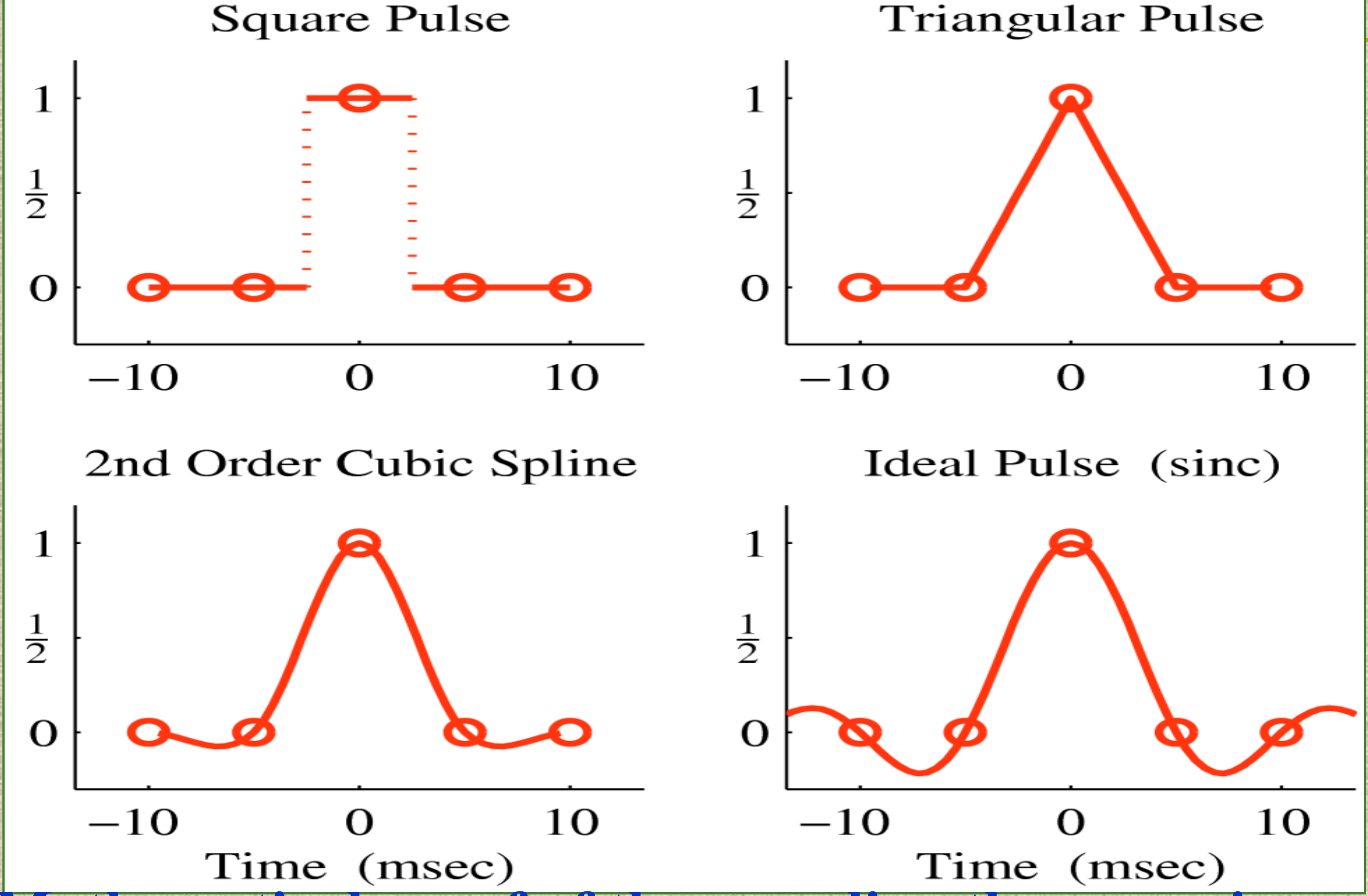


Fig:7.20

Mathematical proof of the sampling theorem gives the ideal pulse shape, which is a sinc function

$$p(t) = \frac{\sin(\pi t/T_s)}{(\pi t/T_s)}$$

Square Pulse

The simplest of all is the 'square pulse' defined as,

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \le \frac{1}{2}T_s \\ 0 & otherwise \end{cases}$$

Each term $y[n]p(t - nT_s)$ will create flat region of Amplitude y[n] centered at $t = nT_s$

Since the effect of the flat pulse is to hold or replicate each sample for T_s seconds, it is also known as '*zero-order hold reconstruction*'

In all the examples sampling rate is greater than Nyquist rate

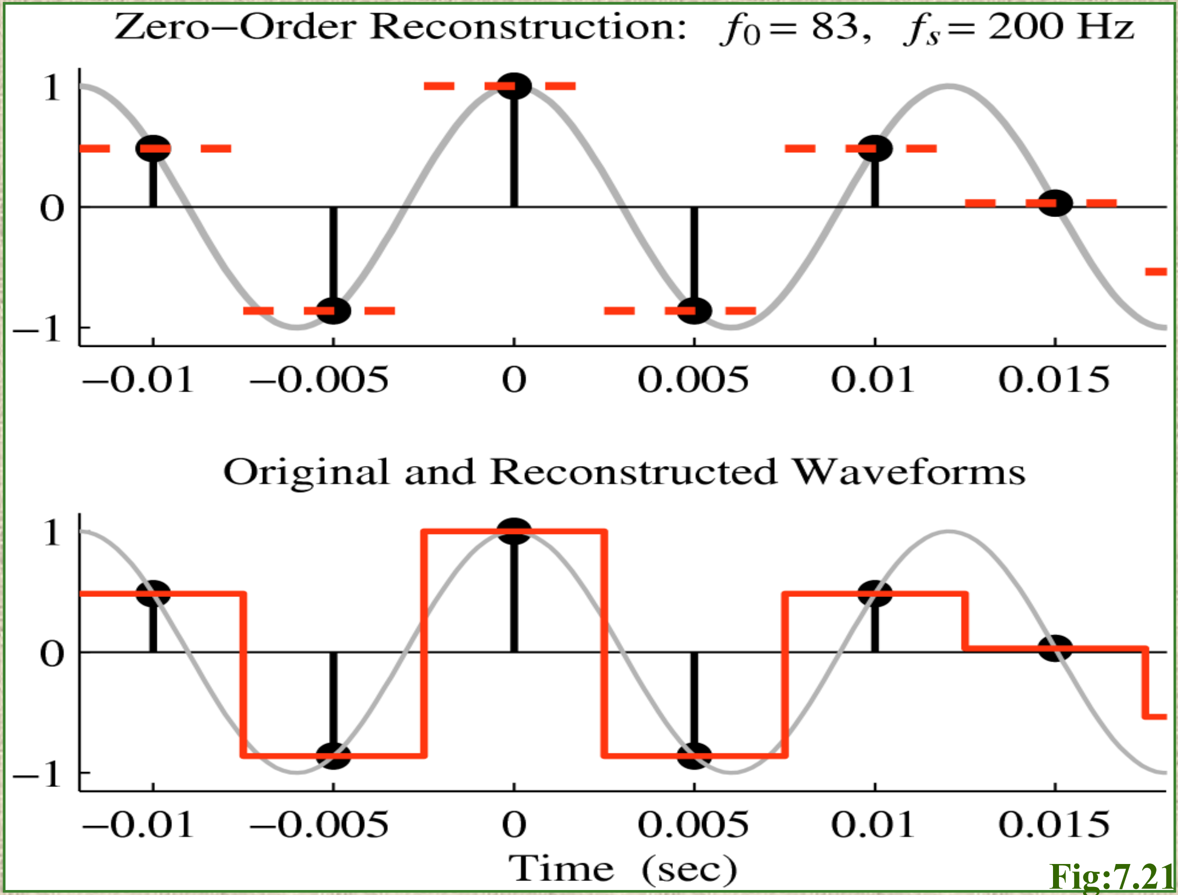
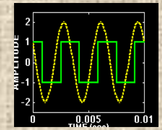


Fig:7.21



Movie Illustrates a similar process

Obviously the more no. of samples we have the better we should be able to reconstruct the signal

Notice that the sampling is much higher than previous case

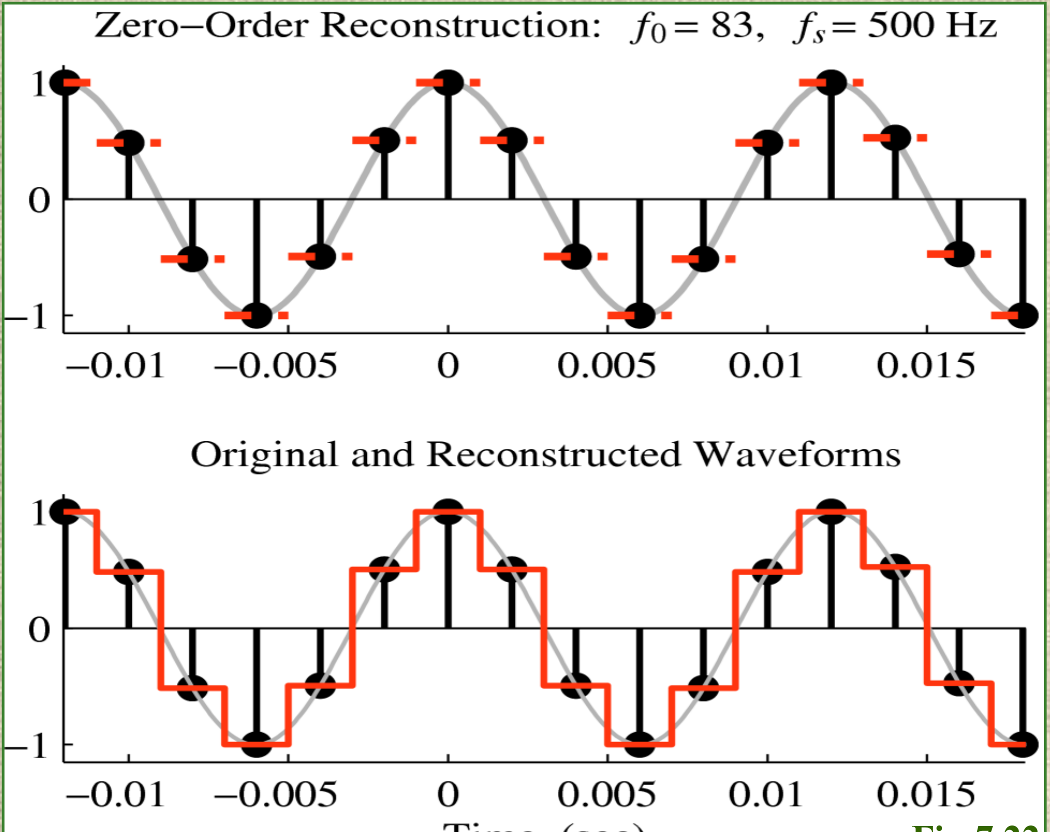
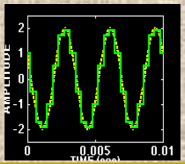


Fig:7.22



Movie Illustrates a similar process

Triangular Pulse

It is a pulse of 1st order polynomial, defined as,

$$p(t) = \begin{cases} 1 - |t|/T_s & -T_s < t \le T_s \\ 0 & otherwise \end{cases}$$

In this case the output *y(t)* of the D-C converter at any time '*t*' is the sum of the scaled pulses that overlap at that time instant. The performance is better than a square pulse

More than Nyquist

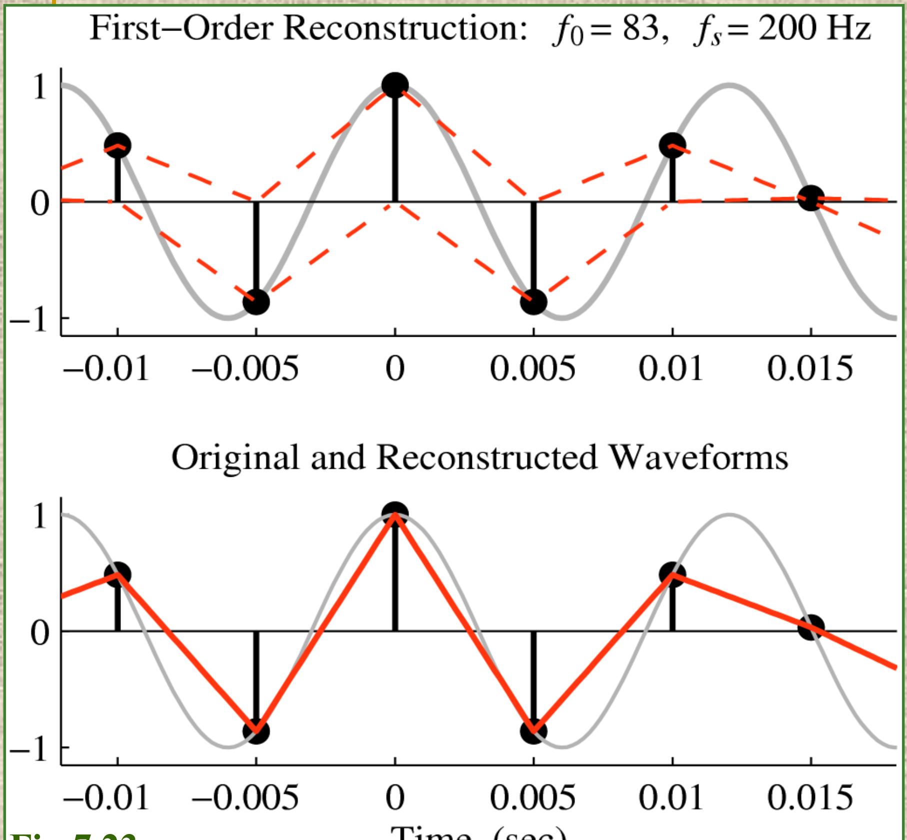
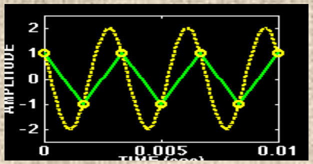


Fig:7.23



Movie

4 times Nyquist

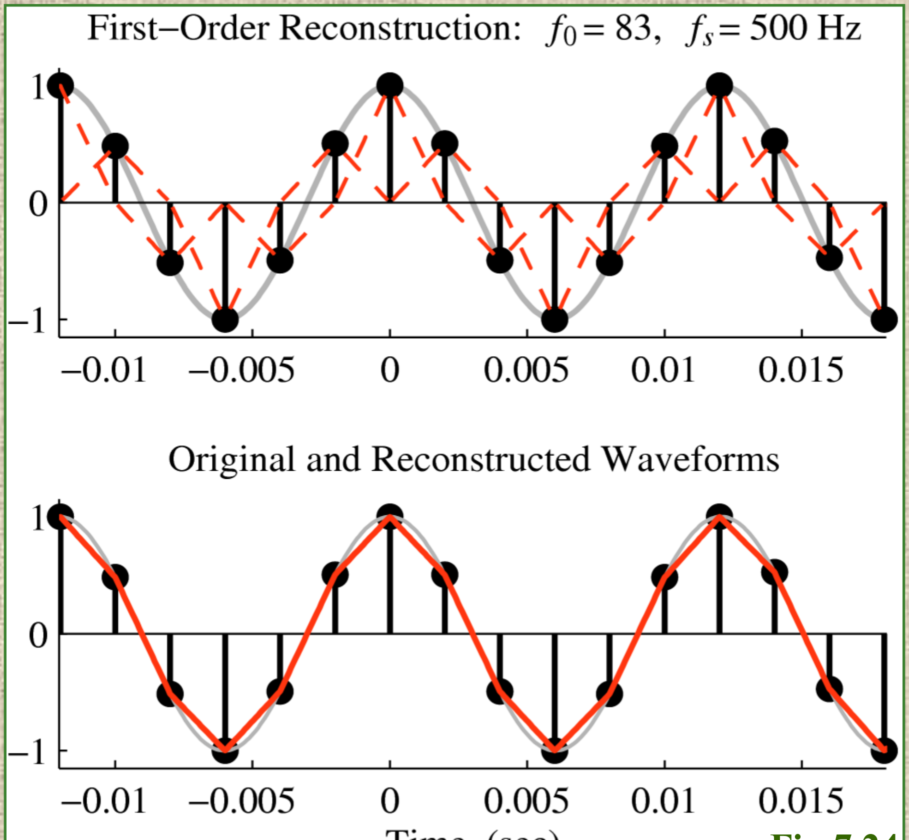
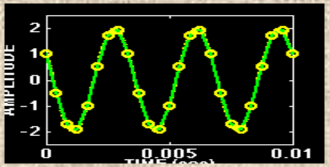


Fig:7.24



Movie

Cubic spline interpolation: A third order polynomial with exactly a similar process as a triangular pulse

More than Nyquist

4 times Nyquist

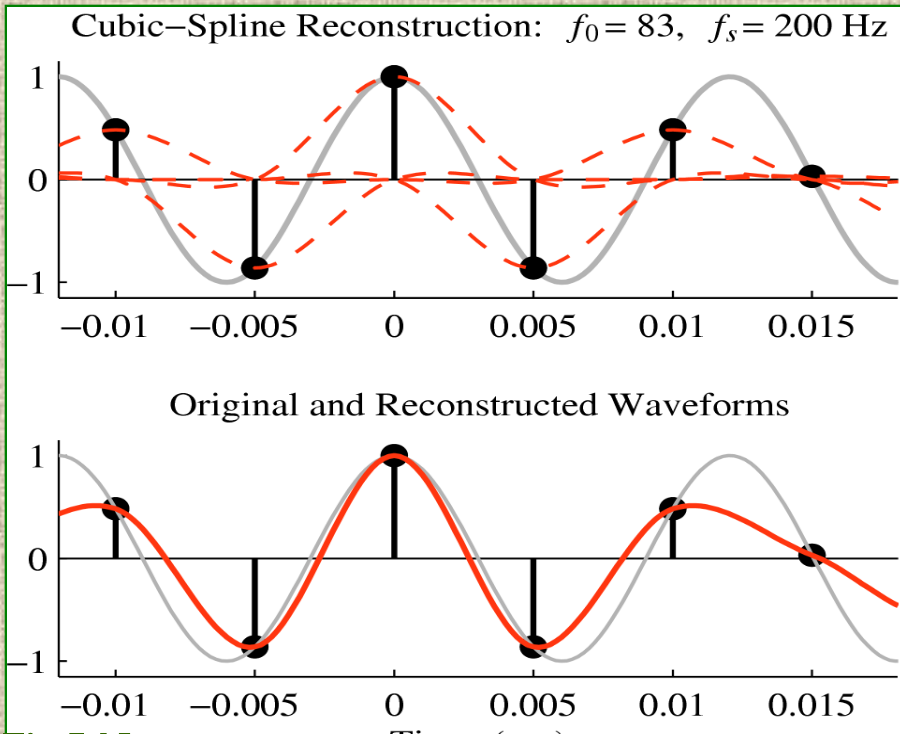


Fig:7.25

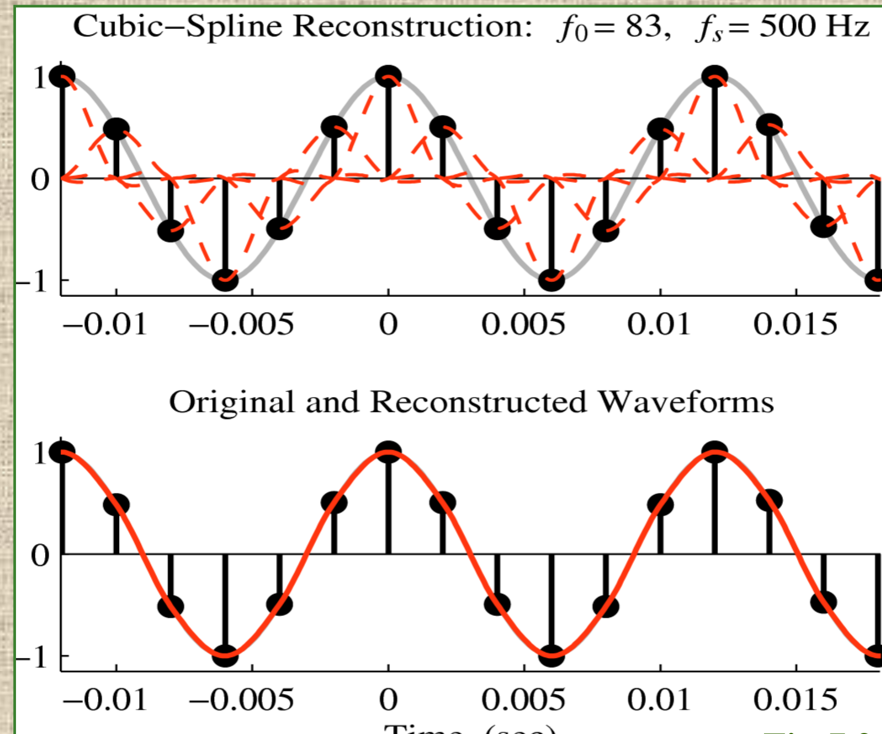
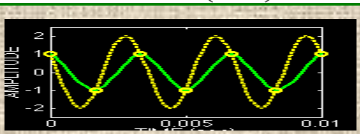
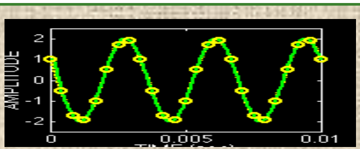


Fig:7.26

Movie





Movie

The results with this pulse are extremely close to original !!

Reference

1.James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “Signal Processing First”, Prentice Hall, 2003

2. http://www.research.att.com/~njas/doc/ces5.html, Shannon’s Biography