Discrete - Time Signals and Systems

FIR Filters-II
Frequency Response

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The Frequency Response

$Consider\ a\ continuous-time\ complex\ exponential$

$signal,\ \ x(t) = Ae^{j\phi}e^{j\omega t},$

$Sample\ x(t)\ to\ get\ the\ discrete-time\ signal,$

$x[n] = x(t)\big|_{=nT_s}$

$x[n] = x(nT_s)\ then\ x[n] = Ae^{j\phi}e^{j\omega(nT_s)},$

$x[n] \ldots discrete-time\ complex\ exponential\ signal$

$discrete-time\ frequency\ \ \hat{\omega} = \omega T_s,\ x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$

Pass the signal x[n] through a simple FIR system described by the difference equation,

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

$$y[n] = Ae^{j\phi}e^{j\hat{\omega}n} + 2Ae^{j\phi}e^{j\hat{\omega}(n-1)} + Ae^{j\phi}e^{j\hat{\omega}(n-2)}$$

$$= Ae^{j\phi}e^{j\hat{\omega}n}\left[1 + 2e^{-j\hat{\omega}} + e^{-j\hat{\omega}2}\right]$$

$$y[n] = x[n]\left[1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right]$$

$$Let \quad H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$H(e^{j\hat{\omega}})$…*is known as the frequency response function*

$$Thus \; y[n] = x[n]\, H(e^{j\hat{\omega}})$$

i.e, output is input multiplied with the frequency response

$$y[n] = x[n]H(e^{j\hat{\omega}})$$

However, This is only true for complex exponential input signals

$H(e^{j\hat{\omega}})$ ... *is a complex function*

$$H(e^{j\hat{\omega}}) = \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

$\left|H(e^{j\hat{\omega}})\right|$ ... *Magnitude or gain*

$\angle H(e^{j\hat{\omega}})$ ... *Phase*

$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}})$$

$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

$$\because -1 \le \cos(\hat{\omega}) \le 1$$

$$0 \le \left(2 + 2\cos(\hat{\omega})\right) \le 4$$

$$\therefore \left(2 + 2\cos(\hat{\omega})\right) \ge 0 \quad \textit{for} \quad -\pi \le \hat{\omega} \le \pi$$

$$\therefore \left|H(e^{j\hat{\omega}})\right| = \left(2 + 2\cos(\hat{\omega})\right)$$

Also phase from $e^{-j\hat{\omega}}$,

$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$$

Plot of Frequency Response $\{b_k\} = \{1,2,1\}$

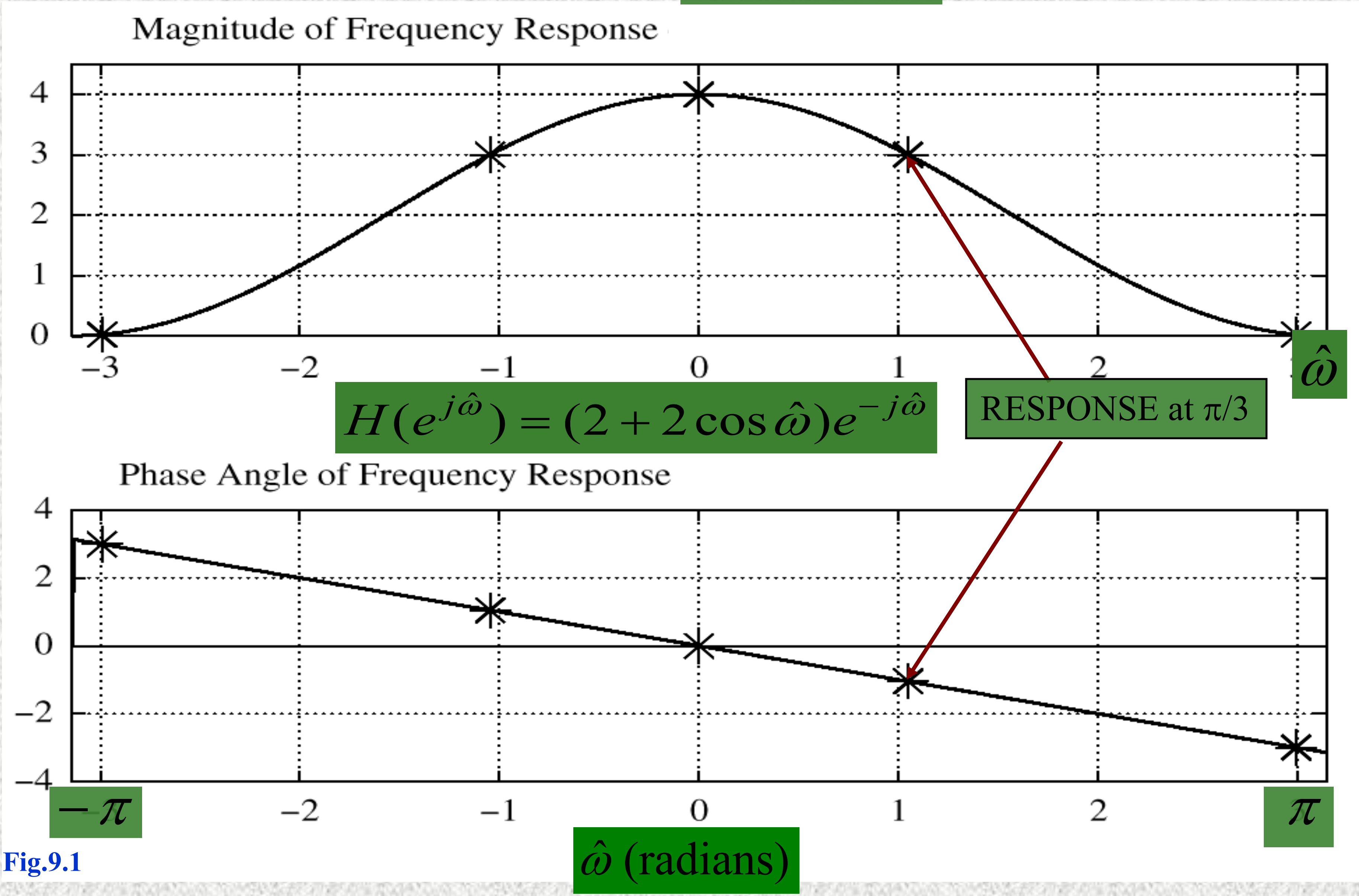


Fig.9.1

Example1: *Frequency Response*

For the system considered so far,

$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

input frequency… $\pi/3$

$$x[n] = 2e^{j(\pi n/3 + \pi/4)} \qquad Find\ \ y[n]?$$

$$y[n] = x[n]H(e^{j\hat{\omega}})$$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$H(e^{j\pi/3}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}\Big|_{\hat{\omega}=\pi/3} = 3e^{-j\pi/3}$$

$$y[n] = \underbrace{\left(2e^{j(\pi n/3 + \pi/4)}\right)}_{x[n]}\underbrace{\left(3e^{-j\pi/3}\right)}_{H(e^{j\hat{\omega}})} = 6e^{j(\pi n/3 - \pi/12)}$$

same output frequency, $\pi/3$

Example 2: *Frequency Response*

$$H(e^{j\hat{\omega}}) = 1 - e^{-j2\hat{\omega}}$$

$$y[n] = e^{j(\pi n/6 + \pi/4)} \qquad Find\ x[n]?$$

$$y[n] = x[n]H(e^{j\hat{\omega}})$$

$$H(e^{j\hat{\omega}}) = (e^{j\hat{\omega}} - e^{-j\hat{\omega}})e^{-j\hat{\omega}}$$

$$= 2j\sin(\hat{\omega})e^{-j\hat{\omega}}$$

$$= 2e^{j\pi/2}\sin(\hat{\omega})e^{-j\hat{\omega}}$$

$$= 2\sin(\hat{\omega})e^{-j(\hat{\omega}-\pi/2)}$$

$$H(e^{j\hat{\omega}}) = 2\sin(\hat{\omega})e^{-j(\hat{\omega}-\pi/2)}$$

$$x[n] = \frac{y[n]}{H(e^{j\hat{\omega}})}$$

$$The\ input\ frequency = The\ output\ frequency = \pi/6$$

$$frequency\ response\ at\ \hat{\omega} = \pi/6$$

$$H(e^{j\hat{\omega}}) = 2\sin(\hat{\omega})e^{-j(\hat{\omega}-\pi/2)}\Big|_{\hat{\omega}=\pi/6}$$

$$= 2.(1/2)e^{-j(\pi/6-\pi/2)} = e^{j\pi/3}$$

$$y[n] = e^{j(\pi n/6+\pi/4)}$$

$$x[n] = \frac{e^{j(\pi n/6+\pi/4)}}{e^{j\pi/3}} = e^{j(\pi n/6-\pi/12)}$$

Generalization of *Frequency Response*

$$y[n] = \sum_{k=0}^{M} b_k x[n-k], \quad FIR \;\; filters$$

$$b_k \ldots Filter \;\; coefficients$$

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n} \ldots Discrete \;\; exponential \;\; signal$$

$$Pass \;\; the \;\; signal \;\; through \;\; FIR \;\; filter$$

$$y[n] = \sum_{k=0}^{M} b_k Ae^{j\phi}e^{j\hat{\omega}(n-k)}$$

$$= \sum_{k=0}^{M} b_k Ae^{j\phi}e^{j\hat{\omega}n}e^{-j\hat{\omega}k}$$

$$= \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \right) A e^{j\phi} e^{j\hat{\omega}n}$$

$$= H(e^{j\hat{\omega}}) x[n]$$

Where,

$H(e^{j\hat{\omega}})$...*frequency response function*

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$= \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

$h[k]$...*impulse response coefficients*

$b_k = h[k]$...*for FIR filters*

Example 1

$$y[n] = \frac{1}{4}\left[x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]\right]$$

$$x[n] = e^{j\hat{\omega}n} \qquad (A = 1, \phi = 0)$$

$$y[n] = \frac{1}{4}\left[e^{j\hat{\omega}n} + e^{j\hat{\omega}(n-1)} + e^{j\hat{\omega}(n-2)} + e^{j\hat{\omega}(n-3)} + e^{j\hat{\omega}(n-4)}\right]$$

$$= e^{j\hat{\omega}n}\frac{1}{4}\left[1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}\right]$$

$$= x[n]H(e^{j\hat{\omega}})$$

$$\therefore H(e^{j\hat{\omega}}) = \frac{1}{4}\left[1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}\right]$$

$$= \frac{1}{4}\left[\frac{1 - e^{-j5\hat{\omega}}}{1 - e^{-j\hat{\omega}}}\right]$$

$$= \frac{1}{4}\frac{e^{-j5\hat{\omega}/2}}{e^{-j\hat{\omega}/2}}\left[\frac{e^{j5\hat{\omega}/2} - e^{-j5\hat{\omega}/2}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}\right]$$

$$= \frac{1}{4}e^{-j4\hat{\omega}/2}\left[\frac{2j\sin(5\hat{\omega}/2)}{2j\sin(\hat{\omega}/2)}\right]$$

$$= \frac{1}{4}e^{-j2\hat{\omega}}\left[\frac{\sin(5\hat{\omega}/2)}{\sin(\hat{\omega}/2)}\right] \ldots \textit{frequency response function}$$

$$H(e^{j\hat{\omega}}) = \frac{1}{4} e^{-j2\hat{\omega}} \left[\frac{\sin(5\hat{\omega}/2)}{\sin(\hat{\omega}/2)} \right]$$

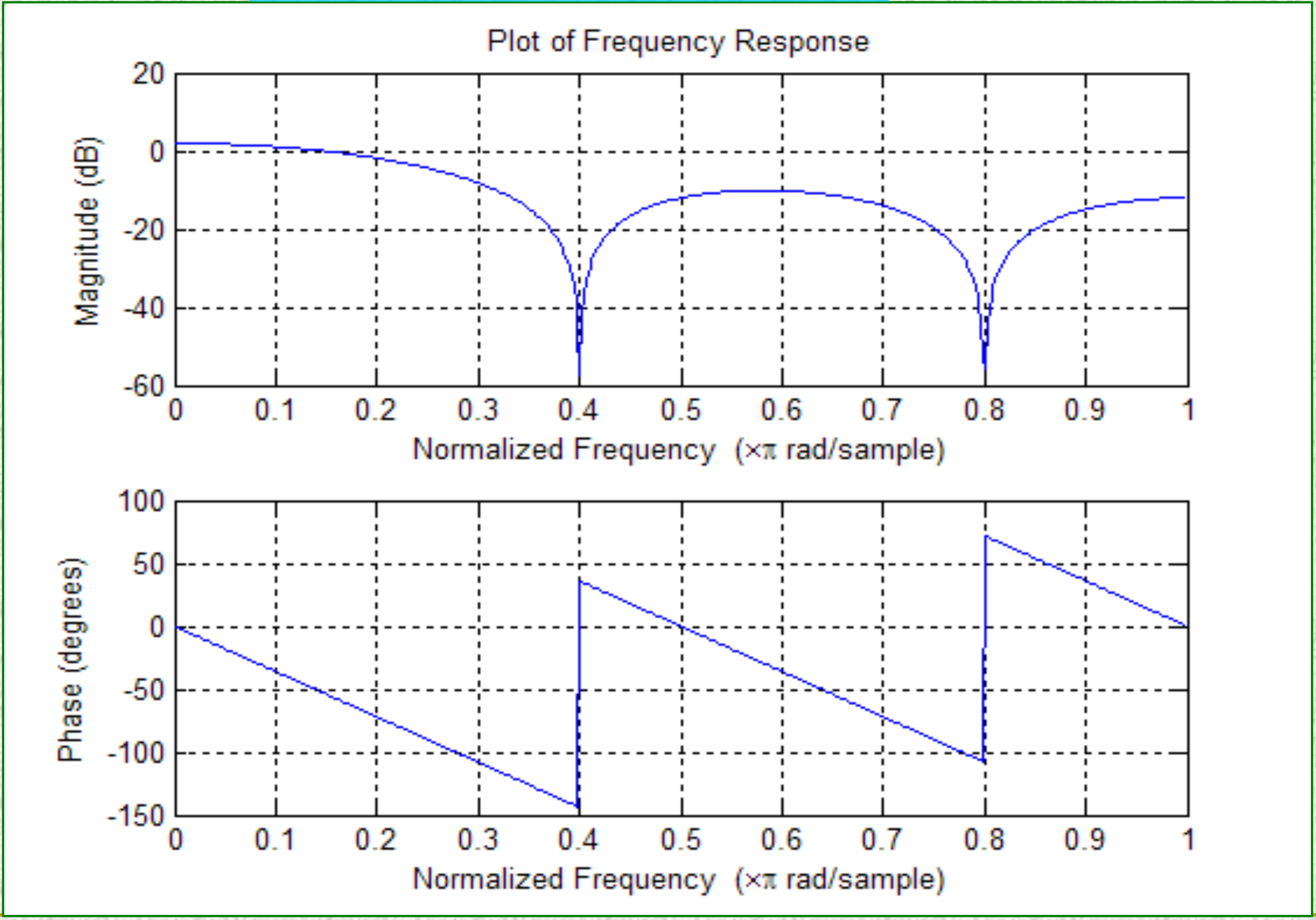


Fig.9.2

Example 2: Generalization---*L-point* averaging

$$y[n]=\frac{1}{L}\sum_{k=0}^{L-1}x[n-k]$$

$$H(e^{j\hat{\omega}})=\frac{1}{L}\sum_{k=0}^{L-1}e^{-j\hat{\omega}k}=\frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right)$$

$$=\frac{1}{L}\frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}}\left(\frac{e^{j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2}}{e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}}\right)=e^{-j\hat{\omega}(L-1)/2}\left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)$$

$$H\left(e^{j\hat{\omega}}\right)=D_L\left(e^{j\hat{\omega}}\right)e^{-j\hat{\omega}(L-1)/2}$$

$$Where,\quad D_L\left(e^{j\hat{\omega}}\right)=\left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)\ldots Dirichlet\ Function$$

Dirichlet function plot with 'L=11'

An even and symmetric function

Periodic with a period of 2π

Maximum value of 1 obtained with L'Hospital's rule

Minimum value at π

The function has 'L-1' zero crossings

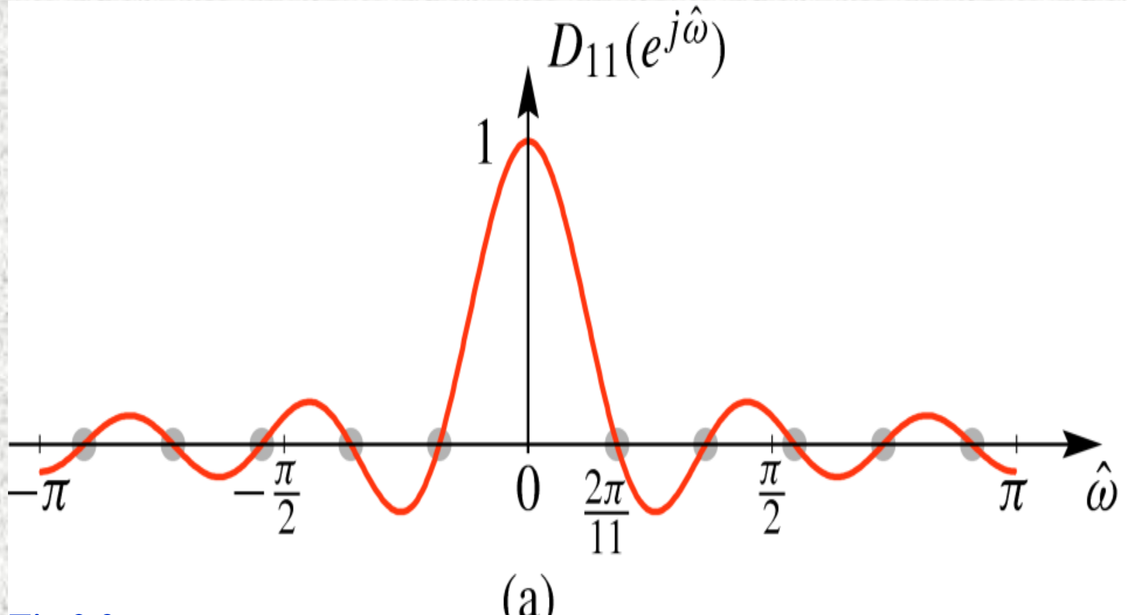


Fig.9.3

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}(L-1)/2}\left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)$$

With $L = 11$

Notice the phase part

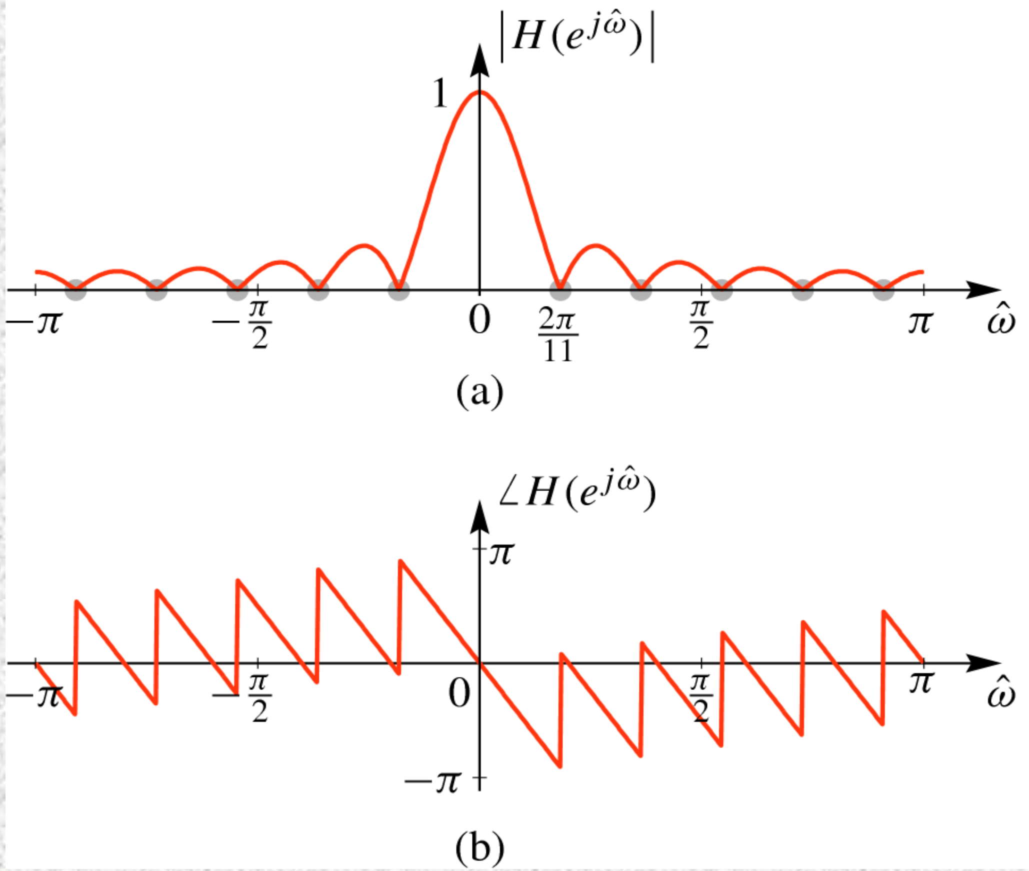


Fig.9.4

Superposition

Output of a FIR system if the input is a sum of complex exponential signals

$$x[n] = A_0 + A_1 \cos(\hat{\omega}_1 n + \phi_1)$$

$$= A_0 e^{j0n} + \frac{A_1}{2} e^{j\phi_1} e^{j\hat{\omega}_1 n} + \frac{A_1}{2} e^{-j\phi_1} e^{-j\hat{\omega}_1 n}$$

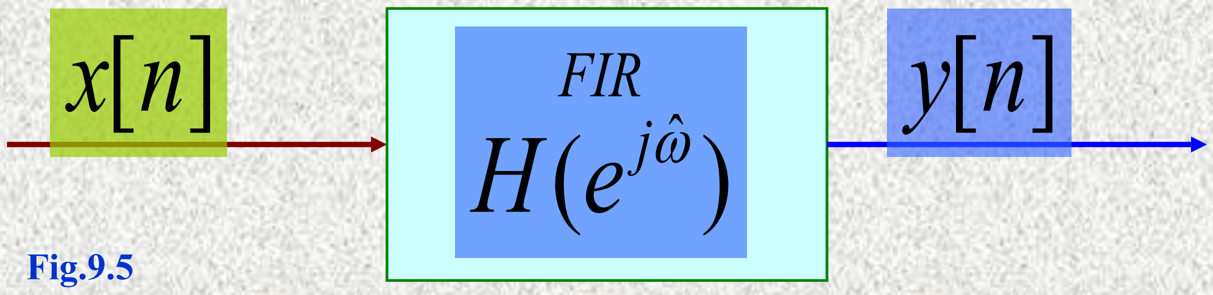


Fig.9.5

Use the principle of superposition to calculate output

$$x[n] = A_0 e^{j0n} + \frac{A_1}{2} e^{j\phi_1} e^{j\hat{\omega}_1 n} + \frac{A_1}{2} e^{-j\phi_1} e^{-j\hat{\omega}_1 n}$$

$$y[n] = H(e^{j0}) A_0 e^{j0n} + H(e^{j\hat{\omega}_1}) \frac{A_1}{2} e^{j\phi_1} e^{j\hat{\omega}_1 n}$$

$$+ H(e^{-j\hat{\omega}_1}) \frac{A_1}{2} e^{-j\phi_1} e^{-j\hat{\omega}_1 n}$$

Notice that the frequencies are preserved in the output, only amplitude is scaled

Example

$$x[n] = 2\cos(\frac{\pi}{3}n - \frac{\pi}{2})$$

$$y[n] = x[n] + 2x[n-1] + x[n-2] \ldots FIR\ filter$$

$$H(e^{j\hat{\omega}}) = \left[1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right]$$

$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

Express the input as complex exponential signals

$$x[n] = e^{j\left(\frac{\pi}{3}n - \frac{\pi}{2}\right)} + e^{-j\left(\frac{\pi}{3}n - \frac{\pi}{2}\right)}$$

Frequency response at $\frac{\pi}{3}$

$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{\pi}{3}}(2+2\cos\frac{\pi}{3}) = 3e^{-j\frac{\pi}{3}}$$

complex conjugates

Frequency response at $-\frac{\pi}{3}$

$$H(e^{-j\frac{\pi}{3}}) = e^{j\frac{\pi}{3}}(2+2\cos-\frac{\pi}{3}) = 3e^{j\frac{\pi}{3}}$$

$$x[n] = e^{j\left(\frac{\pi}{3}n - \frac{\pi}{2}\right)} + e^{-j\left(\frac{\pi}{3}n - \frac{\pi}{2}\right)}$$

$$y[n] = H(e^{j\frac{\pi}{3}})e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{3}n} + H(e^{-j\frac{\pi}{3}})e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{3}n}$$

$$= 3e^{-j\frac{\pi}{3}}e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{3}n} + 3e^{j\frac{\pi}{3}}e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{3}n}$$

$$= 3\left(e^{j\left(\frac{\pi}{3}n - \frac{5\pi}{6}\right)} + e^{-j\left(\frac{\pi}{3}n - \frac{5\pi}{6}\right)}\right)$$

$$= 6\cos\left(\frac{\pi}{3}n - \frac{5\pi}{6}\right)$$

Conclusions

For a sinusoidal input to an FIR system

- **Output frequency is same as input**

- **Amplitude of the output is input's amplitude multiplied by the absolute value of frequency response function**

- **Output sinusoid's phase is shifted by the amount of phase in the frequency response function**

Generalization

If the input signal $x[n]$ *to a linear discrete-time system is a real signal,*

$$x[n] = X_0 + \sum_{k=1}^{N} |X_k| \cos(\hat{\omega}_k n + \angle X_k)$$

$$= X_0 + \sum_{k=1}^{N} \frac{X_k}{2} e^{j\hat{\omega}_k n} + \frac{X_k^*}{2} e^{-j\hat{\omega}_k n}$$

Where,

$$X_k = |X_k| e^{j\angle X_k}$$

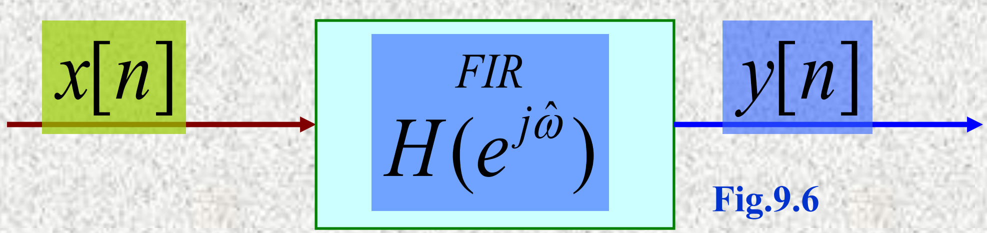


Fig.9.6

The output of the system to this input,

$$x[n] = X_0 + \sum_{k=1}^{N} \frac{X_k}{2} e^{j\hat{\omega}_k n} + \frac{X_k^*}{2} e^{-j\hat{\omega}_k n}$$

$$y[n] = H(e^{j0})X_0 + \sum_{k=1}^{N} \left(\left\{ H(e^{j\hat{\omega}_k}) \frac{X_k}{2} e^{j\hat{\omega}_k n} \right\} + \left\{ H(e^{-j\hat{\omega}_k}) \frac{X_k^*}{2} e^{-j\hat{\omega}_k n} \right\} \right)$$

$$= H(e^{j0})X_0 + \sum_{k=1}^{N} \left| H(e^{j\hat{\omega}_k}) \right| \left| X_k \right| e^{j\hat{\omega}_k n} \cos(\hat{\omega}_k n + \angle X_k + \angle H(e^{j\hat{\omega}_k}))$$

$$\because H(e^{-j\hat{\omega}_k}) = H^*(e^{j\hat{\omega}_k})$$

Example

$$x[n] = 4 + 3\cos(\frac{\pi}{3}n - \frac{\pi}{2}) + 3\cos(\frac{7\pi}{8}n)$$

$$y[n] = x[n] + 2x[n-1] + x[n-2] \ldots FIR\ filter$$

$$H(e^{j\hat{\omega}}) = \left[1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right]$$

$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

$$y[n] = H(e^{j0})X_0 + \sum_{k=1}^{N} \left|H(e^{j\hat{\omega}_k})\right| \left|X_k\right| e^{j\hat{\omega}_k n} \cos(\hat{\omega}_k n + \angle X_k + \angle H(e^{j\hat{\omega}_k}))$$

$$H(e^{j0}) = 4$$

$$H(e^{j0}) = 4$$

$$H(e^{j\pi/3}) = 3e^{-j\pi/3}$$

$$H(e^{j7\pi/8}) = 0.1522e^{-j7\pi/8}$$

$$y[n] = 4.4 + 3.3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2} - \frac{\pi}{3}\right) + 0.1522.3\cos\left(\frac{7\pi}{8}n - \frac{7\pi}{8}\right)$$

$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

This frequency response represents a low pass filter

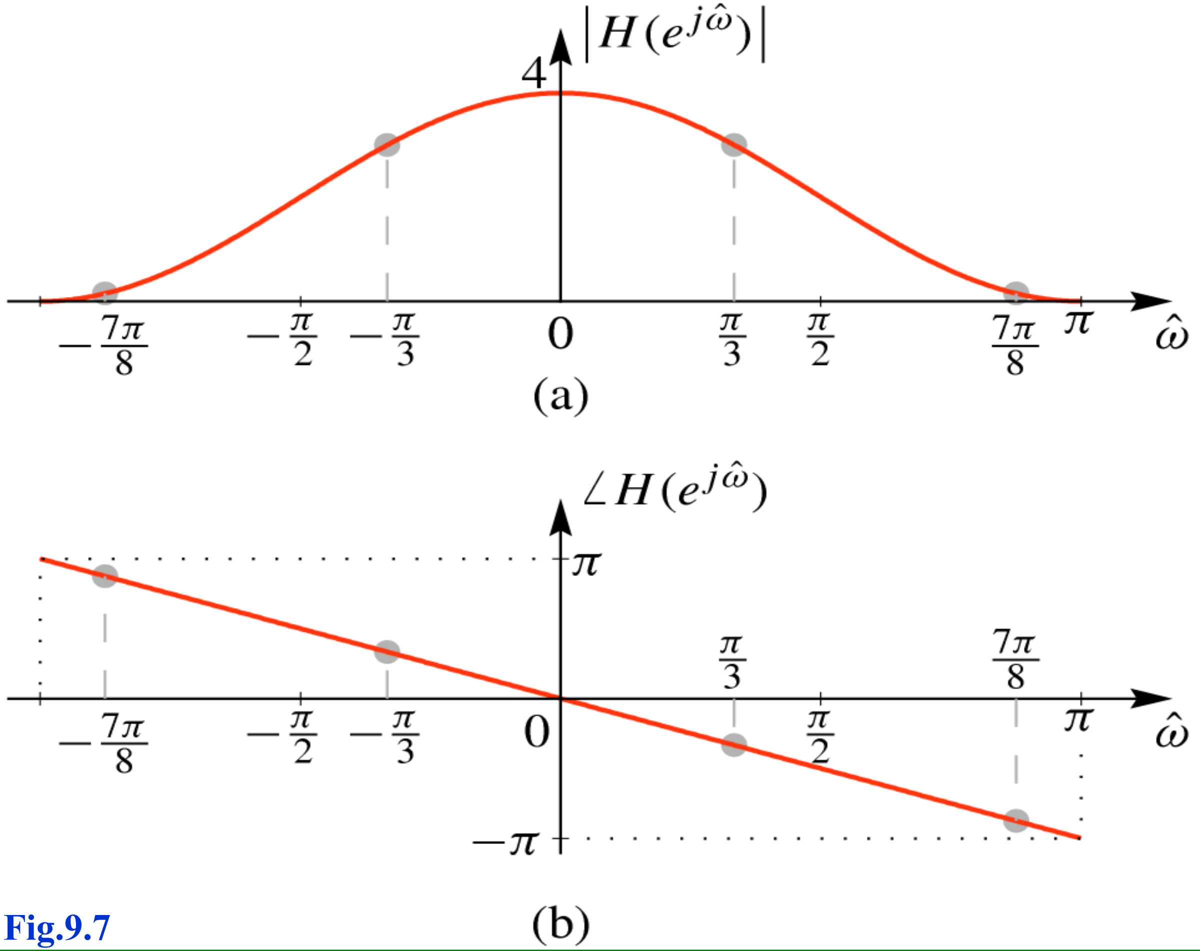


Fig.9.7

All frequencies around pi and -pi are attenuated

A slope of '-1' in the phase plot indicates a delay of 1 sample

$$x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right)$$

Oscillations seem to have an amplitude of 9 and a period of 6, this may not be exactly true as there is some contribution from another frequency

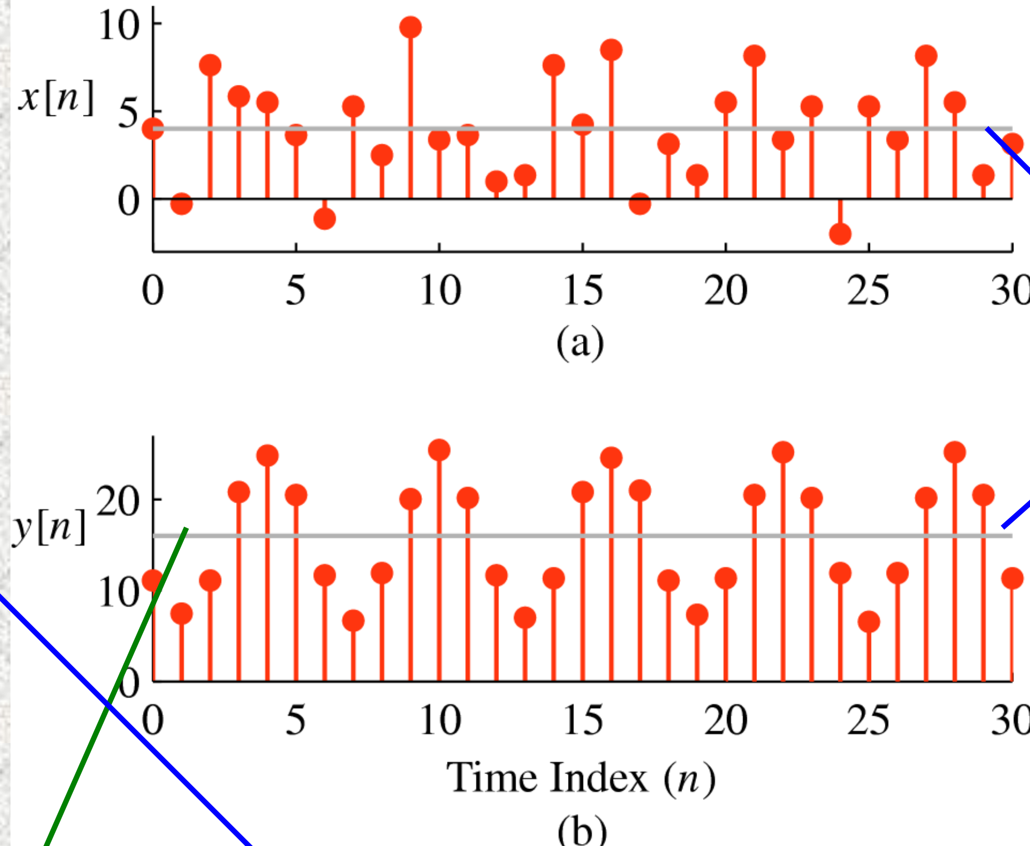


Fig.9.8

$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right)$$

Review

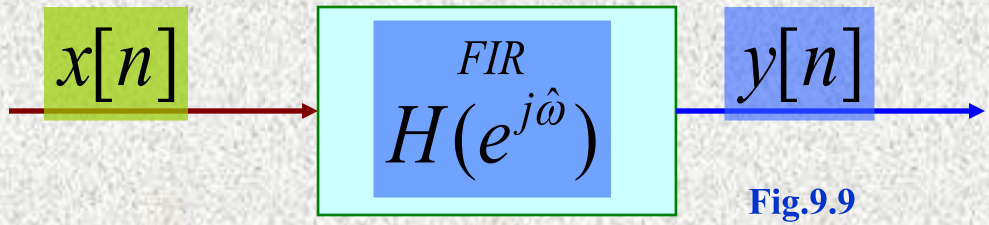


Fig.9.9

Time Domain

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{M} h_k \delta[n-k]$$

Frequency Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h(k) e^{-j\hat{\omega}k}$$

Frequency response function can be directly derived from the impulse response, which in the case of FIR filter are filter coefficients

$$h(k) = b_k$$

Example 1

$$h[n] = \delta[n-1]$$

$$Implies \ \{b_k\} = \{0,1\} = \{h[k]\}$$

$$y[n] = x[n-1]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h(k)e^{-j\hat{\omega}k} = \sum_{k=0}^{1} h(k)e^{-j\hat{\omega}k}$$

$$= e^{-j\hat{\omega}}$$

Example 2

$$y[n] = x[n] - x[n-1]$$

$$\{b_k\} = \{1, -1\},$$

$$Implies \Rightarrow \ \{h[k]\} = \{1, -1\}$$

$$y[n] = \delta[n] - \delta[n-1]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h(k) e^{-j\hat{\omega}k}$$

$$= \sum_{k=0}^{1} h(k) e^{-j\hat{\omega}k} = 1 - e^{-j\hat{\omega}}$$

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$$

$$= e^{-j\hat{\omega}/2}\left[e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}\right]$$

$$= e^{-j\hat{\omega}/2}\, 2j\sin(\hat{\omega}/2)$$

$$= 2e^{-j(\hat{\omega}/2-\pi/2)}\sin(\hat{\omega}/2)$$

$$\left|H(e^{j\hat{\omega}})\right| = \left|2\sin(\hat{\omega}/2)\right|$$

$$\angle H(e^{j\hat{\omega}}) = \begin{cases} \pi/2 - \hat{\omega}/2 & 0 < \hat{\omega} < \pi \\ -\pi/2 - \hat{\omega}/2 & -\pi < \hat{\omega} < 0 \end{cases}$$

Only Magnitude, an example of high pass filter

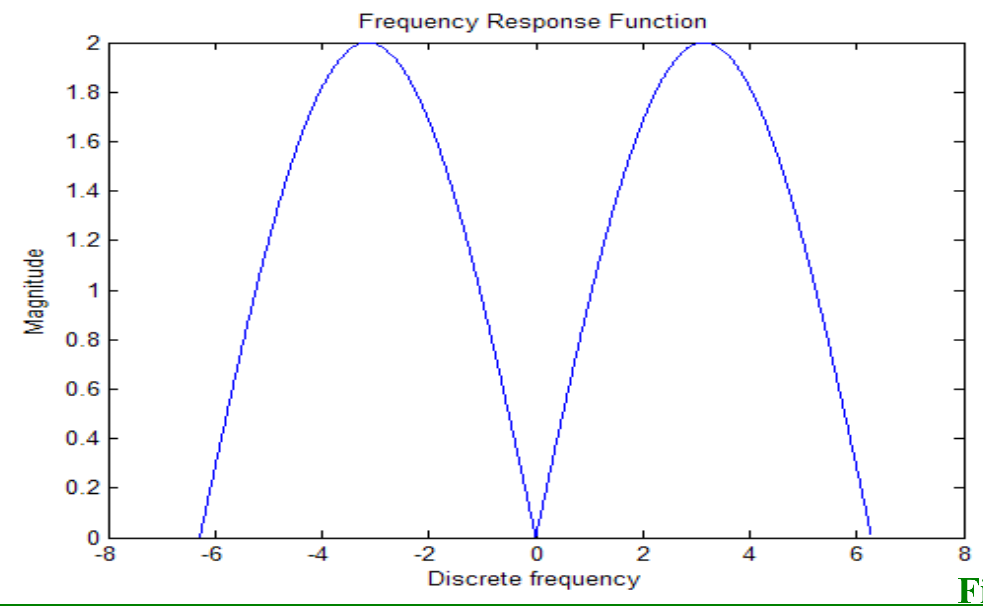
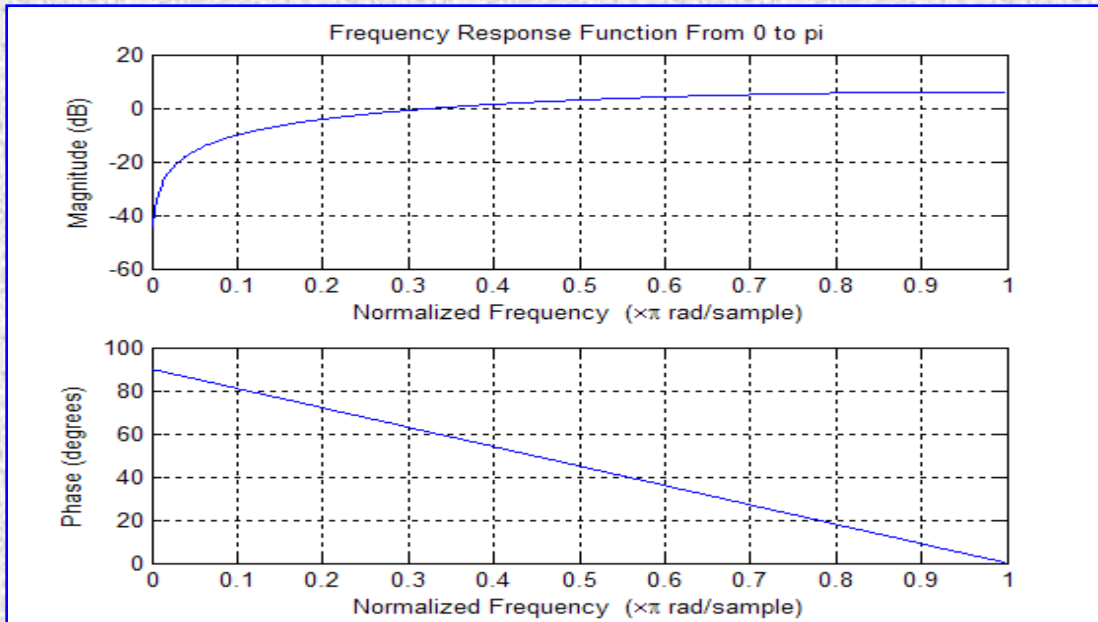


Fig.9.9

Magnitude along with phase for half a cycle ‘0’ to ‘pi’



Example 3

The frequency response of a FIR system is,

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(3 - 2\cos(\hat{\omega}))$$

$$= e^{-j\hat{\omega}}\left\{3 - 2\left(\frac{e^{j\hat{\omega}} + e^{-j\hat{\omega}}}{2}\right)\right\}$$

$$= -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$$

$$\{h[k]\} = \{-1, 3, -1\} = \{b_k\}$$

The difference equation for the filter,

$$y[n] = -x[n] + 3x[n-1] - x[n-2]$$

Summary: Properties of $H(e^{j\hat{\omega}})$

Periodicity

$$H(e^{j(\hat{\omega}+2\pi k)}) = H(e^{j\hat{\omega}})$$

Conjugate Symmetry

$$H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

$$H^*(e^{j\hat{\omega}}) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^* = \sum_{k=0}^{M} b_k^* e^{j\hat{\omega}k}$$

$$= \sum_{k=0}^{M} b_k^* e^{-j(-\hat{\omega})k} = H(e^{-j\hat{\omega}})$$

$$Periodicity \quad H(e^{j(\hat{\omega}+2\pi k)}) = H(e^{j\hat{\omega}})$$

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$$

$$y[n] = x[n] - x[n-1]$$

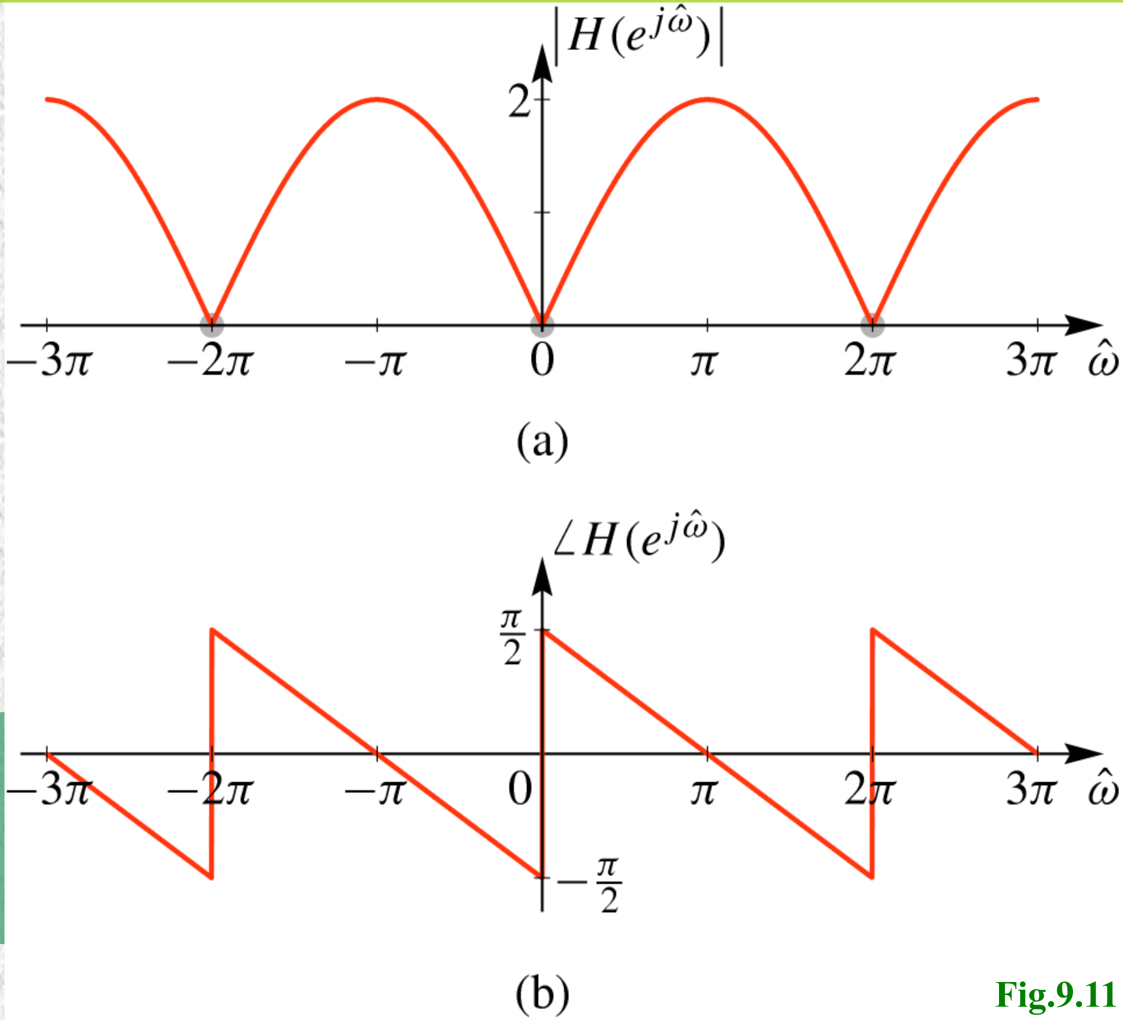


Fig.9.11

Notice the phase

$$\angle H(e^{j\hat{\omega}}) = \begin{cases} \pi/2 - \hat{\omega}/2 & 0 < \hat{\omega} < \pi \\ -\pi/2 - \hat{\omega}/2 & -\pi < \hat{\omega} < 0 \end{cases}$$

$$Conjugate\ Symmetry \quad H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

Summary: Properties of $H(e^{j\hat{\omega}})$

Magnitude is an even function

$$\left|H(e^{-j\hat{\omega}})\right| = \left|H(e^{j\hat{\omega}})\right|$$

Phase is an odd function

$$\angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}})$$

$$\Re e\left\{H(e^{-j\hat{\omega}})\right\} = \Re e\left\{H(e^{j\hat{\omega}})\right\} \ldots cosine\ part$$

$$\Im m\left\{H(e^{-j\hat{\omega}})\right\} = -\Im m\left\{H(e^{j\hat{\omega}})\right\} \ldots sine\ part$$

$$\Re e\left\{H(e^{-j\hat{\omega}})\right\} = \Re e\left\{H(e^{j\hat{\omega}})\right\} \ldots cosine\ part$$

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$$

$$y[n] = x[n] - x[n-1]$$

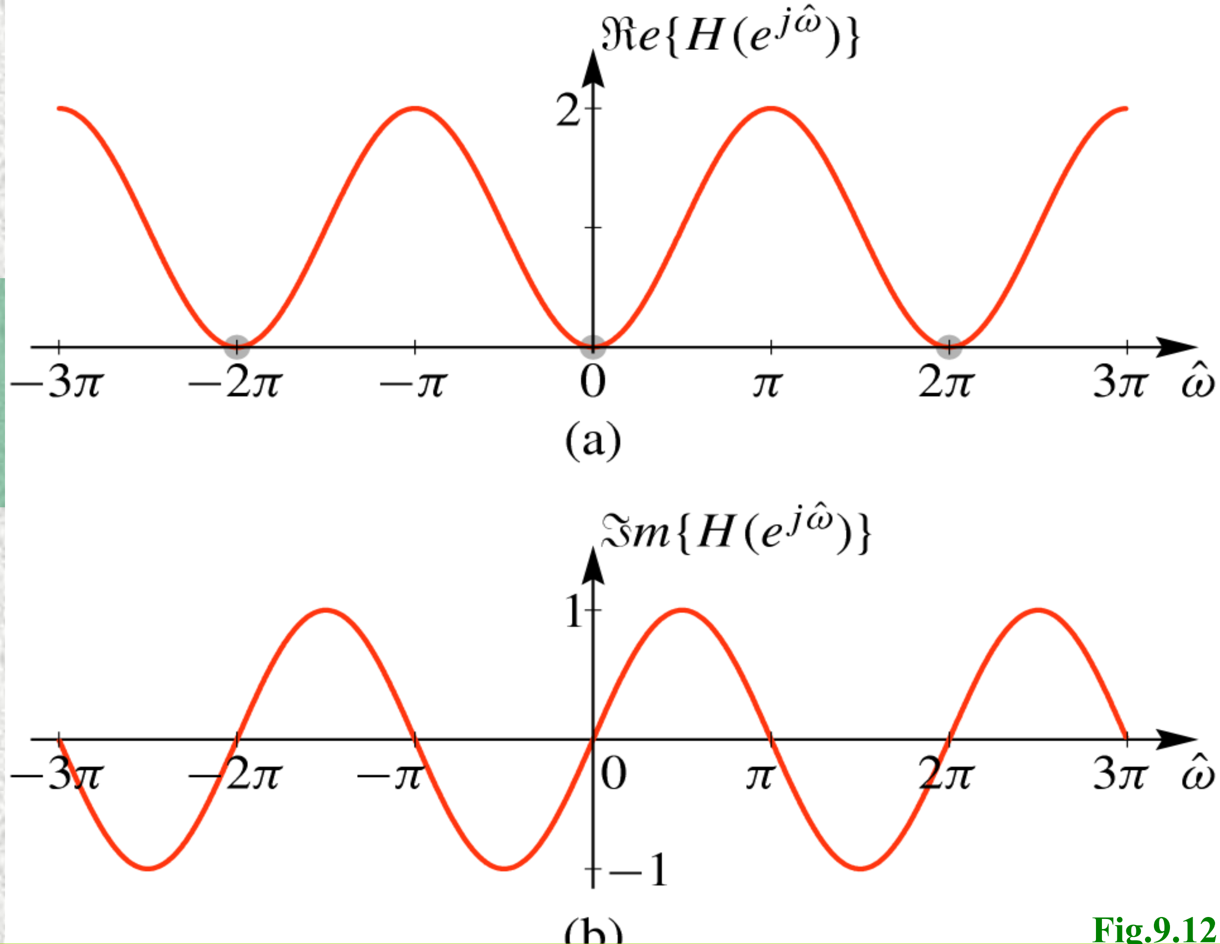


Fig.9.12

$$\Im m\left\{H(e^{-j\hat{\omega}})\right\} = -\Im m\left\{H(e^{j\hat{\omega}})\right\} \ldots sine\ part$$

Reference

James H. McClellan, Ronald W. Schafer and Mark A. Yoder, “ 6.1,6.2, 6.4, 6.5 and 6.7 Signal Processing First”, Prentice Hall, 2003