ECE 235 Final Exam Solutions

**Time:** 3 hours. **Instructions:** Do all five problems. You may use your text, notes and a calculator.

(20 points) 1. Let a two-dimensional probability density function be defined by

\[ f_{X,Y}(x,y) = \begin{cases} \frac{1}{c} \left(x^2 + 2y\right), & -1 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

(a) Find the value of the normalization constant c.

(b) Find the marginal densities \( f_X(y) \) and \( f_Y(x) \).

(c) Find an expression for \( f_{X\mid Y}(x \mid y) \). Are x and y independent? Explain.

**Ans:**

(a) 
\[
\int_{-1}^{1} dx \int_{0}^{1} dy \frac{1}{c} \left(x^2 + 2y\right) = \frac{1}{c} \int_{-1}^{1} dx \int_{0}^{1} dy \left(x^2 + 2y\right)
\]
\[
= \frac{1}{c} \int_{-1}^{1} dx \left[ \frac{1}{2} x^2 + 1 \right] = \frac{1}{c} \left[ \frac{1}{2} \left( \frac{2}{3} \right) + 2 \right] = 1
\]
\[
\Rightarrow \frac{17}{3} = 1 \Rightarrow c = \frac{7}{3}
\]

(b) 
\[
f_X(y) = \int_{-1}^{1} \frac{3}{7} \left(x^2 + 2y\right) dx = \frac{3}{7} \left( y \frac{2^3}{3^2} + 4y \right) = 2y, \ y \in [0,1]
\]
\[
f_Y(x) = \int_{0}^{1} \frac{3}{7} \left(x^2 + 2y\right) dy = \frac{3}{7} \left( x^2 \frac{1}{2} + 2 \frac{1}{2} \right)
\]
\[
= \frac{3}{14} x^2 + \frac{3}{7}
\]

(c)
\[ f_{x|y}(x \mid y) = \frac{f_{x,y}(x, y)}{f_y(y)} \]
\[ = \frac{3}{7} \left( x^2 y + 2y \right) / (2y) \]
\[ = \frac{3}{14} x^2 + \frac{3}{7} \]

Since \( f_{x|y}(x \mid y) = f_x(x) \), we conclude that \( x \) and \( y \) are independent.

(20 points) 2. Let \( x \) and \( n \) be independent Gaussian random variables, such that

\[ E\{\bar{x}\} = 0, \quad E\{\bar{n}\} = 0, \quad Var\{\bar{x}\} = \sigma_x^2, \quad Var\{\bar{n}\} = \sigma_n^2 \]

A third r.v. is generated according to

\[ \bar{y} = \frac{\bar{x}}{\bar{n}} \]

(Find the density function \( f_{\bar{y}}(y) \). \( \text{(Hint: Consider a transformation of the pair of r.v.s} (x,n) \text{ to a pair of r.v.s} (y,n) \text{ and find the joint density of} (y,n).) \)

Ans:

\[ f_{\bar{y},n}(y,n) \frac{dy}{dx} \frac{dy}{dn} = f_{x,n}(y,n) = f_x(yn)f_n(n) \]

\[ \Rightarrow f_{\bar{y},n}(y,n) = \left| \frac{n}{2\pi \sigma_x \sigma_y} \right| \exp \left( -\frac{1}{2\sigma_x^2} (yn)^2 \right) \frac{1}{\sqrt{2\pi \sigma_n^2}} \exp \left( -\frac{1}{2\sigma_n^2} n^2 \right) \]

\[ = \left| \frac{n}{(2\pi \sigma_x \sigma_y)} \right| \exp \left( -\frac{1}{2} n^2 \left( y^2 / \sigma_x^2 + 1/ \sigma_n^2 \right) \right) \]

To find \( f_{\bar{y}}(y) \), integrate over \( n \).
(20 points) 3. A random sequence is given by

\[ \tilde{x}(n) = \tilde{x}(n-1) + \tilde{v}(n) \]

with \( x(0) = 0 \) and \( v(0) = 0 \). The first nonzero \( v(n) \) is \( v(1) \). \( v(n) \) is an independent, but not identically distributed sequence, such that

\[ E[v(n)] = 0, \quad E[\tilde{v}(n)^2] = n(1/2)^n \]

(a) Find \( E\{x(n)\} \) and the correlation function \( E\{x(n)x(m)\} \) in as compact a form as possible (i.e. sums in closed form.) Is \( x(n) \) wide-sense stationary? Explain.

(b) Does \( x(n) \) converge in mean-square? And if so describe the mean-square limit in terms of either its deterministic value or mean and variance.

Ans: (a)
\[ \tilde{x}(n) = \sum_{k=1}^{n} \tilde{v}(k) \]

\[ E\{\tilde{x}(n)\} = \sum_{k=1}^{n} E\{\tilde{v}(k)\} = 0 \]

\[ E\{\tilde{x}(n)\tilde{x}(m)\} = E\left( \sum_{k=1}^{n} \tilde{v}(k) \sum_{j=1}^{m} \tilde{v}(j) \right) = \sum_{k=1}^{n} \sum_{j=1}^{m} E\{\tilde{v}(k)\tilde{v}(j)\} \]

\[ = \sum_{k=1}^{n} \sum_{j=1}^{m} k(1/2)^k \delta_{k,j} = \sum_{k=1}^{\min(n,m)} k \left( \frac{1}{2} \right)^k \]

Note

\[ \sum_{k=0}^{n} k\alpha^k = \alpha \frac{d}{d\alpha} \sum_{k=0}^{n} \alpha^k = \alpha \frac{d}{d\alpha} \frac{1 - \alpha^{n+1}}{1 - \alpha} \]

\[ = \alpha \frac{1 - \alpha^{n+1}}{(1 - \alpha)^2} - \alpha \frac{(n+1)\alpha^n}{1 - \alpha} \]

\[ \Rightarrow E\{\tilde{x}(n)\tilde{x}(m)\} = (1/2)^2 \frac{1 - (1/2)^{\min(n,m)+1}}{(1 - (1/2))^2} - (1/2)^2 \frac{(\min(n,m)+1)(1/2)^{\min(n,m)}}{1 - (1/2)} \]

\[ x(n) \text{ is not wide-sense stationary, since the correlation depends on } \min(n,m), \text{ not } n-m. \]

(b) Using the Loève criterion,

\[ \lim_{n \to \infty, m \to \infty} E\{\tilde{x}(n)\tilde{x}(m)\} = \left( \frac{1}{2} \right)^2 \frac{1}{(1 - (1/2))^2} = 2 \]

This is a finite constant, hence \( x(n) \) converges in mean-square. Convergence is to a random variable with mean zero and variance \( 2/3 \).

(20 points) 4. A zero-mean, discrete-time Gaussian process \( x(n) \) has correlation and covariance function

\[ E\{x(n_1)x(n_2)\}R_{xx}(n_1,n_2) = \alpha^{\max(n_1-n_2,0)\min(n_1,n_2)} \]

where \( 0 < \alpha < 1 \).

(a) Find the predictable increment \( \Delta y(n) \) and Martingale increment \( \Delta u(n) \) in as compact a form as possible for this process \( x(n) \). (Hint: Is \( x(n) \) Markov?)

(b) Define the innovations as \( \tilde{v}(n) = \tilde{x}(n) - E\{\tilde{x}(n)|\tilde{x}(n-1),...,\tilde{x}(0)\} \). Find the mean and variance of this innovations in closed form for the process \( x(n) \) defined above.
(c) Does the process $x(n)$ converge in mean-square and if so to what r.v. or point? Explain.

Ans: (a)

Note first that $x(n)$ is Markov.

$$R_{xx}(n_3, n_1) = \alpha^{|n_3-n_1|} \alpha^{n_1} = \frac{R_{xx}(n_3, n_2) R_{xx}(n_2, n_1)}{R_{xx}(n_2, n_2)} = \frac{\alpha^{n_3-n_2} \alpha^{n_2} \alpha^{n_2-n_1}}{\alpha^{n_2}}$$

$$= \alpha^{n_3-n_1} \alpha^{n_1}$$

Then

$$\Delta y(n) = E\{\tilde{x}(n+1) | \tilde{x}(n), ..., \tilde{x}(0)\} - \tilde{x}(n)$$

$$= E\{\tilde{x}(n+1) | \tilde{x}(n)\} - \tilde{x}(n)$$

$$= \frac{R_{xx}(n+1,n)}{R_{xx}(n)} \tilde{x}(n) - \tilde{x}(n) = \frac{\alpha^2}{\alpha} \tilde{x}(n) - \tilde{x}(n) = (\alpha - 1) \tilde{x}(n)$$

$$\Delta u(n) = \Delta \tilde{x}(n) - \Delta \tilde{y}(n) = (x(n+1) - x(n)) - (\alpha - 1)x(n)$$

$$= x(n+1) - \alpha x(n)$$

Ans: (b)

$$E\{v(n)\} = 0$$

$$E\{v(n+1)^2\} = E\{\Delta u(n)^2\} = E\{(x(n+1) - \alpha x(n))^2\}$$

$$= R_{xx}(n+1,n+1) - 2\alpha R_{xx}(n+1,n) + \alpha^2 R_{xx}(n,n)$$

$$= \alpha^{n+1} - 2\alpha^2 \alpha^n + \alpha^2 \alpha^n$$

$$= \alpha^{n+1} - \alpha^{n+2}$$

(c) From the Loeve criterion, we have

$$\lim_{n,m \to \infty} R_{xx}(n,m) = \lim_{n,m \to \infty} \alpha^{n-m} \alpha_{\min(n,m)} \to 0, \quad 0 < \alpha < 1$$

$$= 0$$

Thus $x(n)$ converges in mean-square. Furthermore

$$E\{(x(n) - 0)^2\} = \alpha^n \to 0 \quad n \to \infty$$

So $x(n)$ converges in m.s. to zero.

(20 points) 5. A discrete-time filter has a complex-valued impulse response
The filter is in steady, with input a \textit{real-valued} white Gaussian noise process $x(n)$ such that

$$E\{x(n)\} = 0, \ E\{x(n)x(m)\} = \delta_{n,m}$$

Where $\delta_{n,m}$ is the Kronecker delta.

(a) Find the correlation function of the filter output $y(n)$, defined by $R_{yy}(n) = E\{y(n)y^*(n-k)\}$, where $y^*(n)$ is the conjugate of $y(n)$. Express in as compact a form as possible.

(b) Find the power spectral density of the filter output, $\Phi_{yy}(e^{j\omega})$.

(c) Find the limit of $\Phi_{yy}(e^{j\omega})$ as $N \to \infty$ again in as compact a form as possible.

Ans:

(a)

$$E\{y(n)y^*(n-k)\} = E\left\{ \sum_{l=0}^{N-1} h(l)x(n-l) \sum_{l'=0}^{N-1} h^*(l')x(n-k-l') \right\} = \sum_{l=0}^{N-1} h(l) \sum_{l'=0}^{N-1} h^*(l') \delta_{n-l,n-k-l'}$$

$$= \sum_{l=0}^{N-1} h(l)h^*(l-k) = \sum_{l=k}^{N-1} h(l)h^*(l-k), \ k > 0$$

$$= \frac{1}{N} \sum_{l=k}^{N-1} \exp(i\pi/4)\exp(-i\pi/4)(l-k))$$

$$= \frac{1}{N} \sum_{j=k}^{N-1} \exp(i\pi/4)k) = \begin{cases} \exp(i\pi/4)k)(1-|k|/N), & |k| \leq N \\ 0 & |k| > N \end{cases}$$

(b) Note $H(z)$ is given by

$$H(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp(i\pi/4)k)z^{-k} = \frac{1}{\sqrt{N}} \frac{1-e^{i\pi/4}N}{1-e^{i\pi/4}z^{-1}}$$

Hence
\[
\Phi_y(e^{j\omega}) = H(e^{j\omega})H^*(e^{j\omega})
\]
\[
= \frac{1}{N} \left[ \frac{1-e^{j(\pi/4)N-i\omega N}}{1-e^{j(\pi/4)-i\omega}} \right] \left[ \frac{1-e^{-j(\pi/4)N+i\omega N}}{1-e^{-j(\pi/4)+i\omega}} \right]
\]
\[
= \frac{1}{N} \left[ \frac{\sin((N/2)(\omega - \pi/4))}{\sin((1/2)(\omega - \pi/4))} \right]^2
\]

(c) From part (b),

\[ R_{yy}(k) = \exp(i\pi/4)(1-|k|/N) \]
\[ \xrightarrow{N \to \infty} \exp(i\pi/4) \]
\[ \Rightarrow \Phi_y(e^{j\omega}) \xrightarrow{N \to \infty} 2\pi \delta(\omega - \pi/4) \]