**Sampling and Quantization**

**Spatial Resolution (Sampling)**
- Determines the smallest perceivable image detail.
- What is the best sampling rate?

**Gray-level resolution (Quantization)**
- Smallest discernible change in the gray level value.
- Is there an optimal quantizer?

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**Image sampling and quantization**

**In 1-D**

- **Sampler**: \( f(m,n) \)
- **Quantizer**: \( u(m,n) \)
- **To Computer**: \( x(t) \)

**1-D**

\[
x(t) = \sum x(kt) \delta(t-kT)
\]

**Sampled Image**

\[
f_x(x,y) = f(x,y) \text{comb}(x,y;\Delta x,\Delta y)
\]

\[
= \sum_{m,n} f(m\Delta x, n\Delta y) \delta(x-m\Delta x, y-n\Delta y)
\]

\[
\text{comb}(x,y;\Delta x,\Delta y) \rightarrow \text{COMB}(u,v) = \frac{1}{\Delta x \Delta y} \text{comb}(u,v;\frac{1}{\Delta x},\frac{1}{\Delta y})
\]
Sampled Spectrum

\[ F_s(u, v) = F(u, v) \ast \text{COMB}(u, v) \]

\[ = \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F(u, v) \ast \delta \left( u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y} \right) \]

\[ = \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F \left( u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y} \right) \]

Bandlimited Images

A function \( f(x, y) \) is said to be band limited if the Fourier transform \( F(u, v) = 0 \) for \( |u| > u_0, |v| > v_0 \).

\( u_0, v_0 \) — Band width of the image in the \( x \)- and \( y \)-directions.

Foldover Frequencies

Sampling frequencies:

Let \( u_s \) and \( v_s \) be the sampling frequencies

Then \( u_s > 2u_0 \); \( v_s > 2v_0 \)

or \( \Delta x < \frac{1}{2u_0} \); \( \Delta y < \frac{1}{2v_0} \)

Frequencies above half the sampling frequencies are called fold over frequencies.

Sampling Theorem

A band limited image \( f(x, y) \) with \( F(u, v) \) as its Fourier transform; and \( F(u, v) = 0 \) for \( |u| > u_0, |v| > v_0 \); and sampled uniformly on a rectangular grid with spacing \( \Delta x \) and \( \Delta y \), can be recovered without error from the sample values \( f(m \Delta x, n \Delta y) \) provided the sampling rate is greater than the nyquist rate.

\[ \frac{1}{\Delta x} = u_s > 2u_0, \quad \frac{1}{\Delta y} = v_s > 2v_0 \]

The reconstructed image is given by the interpolation formula:

\[ f(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m \Delta x, n \Delta y) \frac{\sin(xu_0 - m \Delta x)}{(xu_0 - m \Delta x) \pi} \frac{\sin(yv_0 - n \Delta y)}{(yv_0 - n \Delta y) \pi} \]

Reconstruction

\( R_1 \)

\( R_2 \)
Reconstruction via LPF

\[ F(u,v) \text{ can be recovered by a LPF with} \]
\[ H(u,v) = \begin{cases} \Delta x \Delta y & (u,v) \in R \\ 0 & \text{Otherwise} \end{cases} \]
\[ R \text{ is any region whose boundary } \partial R \text{ is contained within the annular ring between the rectangles } R_1 \text{ and } R_2 \text{ in the figure. Reconstructed signal is} \]
\[ \tilde{F}(u,v) = H(u,v) F(u,v) = F(u,v) \]
\[ f(x,y) = \mathcal{F}^{-1}[F(u,v)] \]

Aliasing

Note: If \( u_s \) and \( v_s \) are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below \( u_s/2, v_s/2 \) in the sampled image. This is called aliasing.

Example

\[ f(x,y) = 2 \cos (2\pi (3x + 4y)) \]
\[ F(u,v) = \delta(u - 3, v - 4) + \delta(u + 3, v + 4) \]

\[ \Rightarrow u_0 = 3, \quad v_0 = 4 \]

Let \( \Delta x = \Delta y = 0.2 \), \( u_s = v_s = \frac{1}{0.2} = 5 < 2u_0, < 2v_0 \)

there will be aliasing.

Example: (contd.)

\[ F(u,v) = 25 \sum_{k,l=-\infty}^{\infty} F(u - ku_s, v - lv_s) \]
\[ = 25 \sum_{k,l=-\infty}^{\infty} [\delta(u - 3 - 5k, v - 4 - 5l) + \delta(u + 3 - 5k, v + 4 - 5l)] \]

Let \[ H(u,v) = \begin{cases} \frac{1}{5} & -2.5 \leq u \leq 2.5, \quad -2.5 \leq v \leq 2.5 \\ 0 & \text{Otherwise} \end{cases} \]

\[ . \quad F(u,v) = H(u,v) F(u,v) \]
\[ = \delta(u + 2, v + 1) + \delta(u - 2, v - 1) \]
\[ \Rightarrow f(x,y) = 2 \cos (2\pi (2x + y)) \]

Examples

Original and the reconstructed image from samples.
Another example

Sampling filter

sampled image

Aliasing Problems (real images!)