Quantization (Jan 21, 2003)

- Optimal Quantizer
- Uniform Quantizer

Image Quantization

\[ u \in \{r_1, r_2, r_3, \ldots, r_L\} \]

\[ u' \epsilon \{r_1, r_2, r_3, \ldots, r_L\} \]

Decision/Reconstruction Levels

\[ u \in [t_k, t_{k+1}] \rightarrow r_k \]

\( \{t_k : k = 1, 2, \ldots, L+1\} \quad \text{Transition or decision levels} \)

\( r_k \quad \text{kth reconstruction level} \)

**Example:** Uniform quantizer \( u \in [0, 10.0] \)

We want \( u' \in \{0, 1, \ldots, 255\} \)

\( t_1 = 0; \quad t_{257} = 10.0; \quad \text{uniformly spaced, } t_k = (k-1) \cdot 10/256 \quad (k = 1, 2, \ldots, 257) \)

Example: quantization

\[ r_k = t_k + \frac{1}{2} \left( \frac{10}{256} \right) = t_k + \frac{5}{256} \]

Quantization interval

\[ q_k = t_k - t_{k-1} = t_k - t_{k-1} \]

= Constant \( \Rightarrow \) Uniform quantizer

MMSE Quantizer

Minimise the mean squared error, \( \text{MSE} = \)

Expected value of \( (u-u')^2 \) given the number of quantization levels \( L \).

Assume that the density function \( p_u(u) \) is known (or can be approximated by a normalised histogram).

Note that for images, \( u \equiv \text{image intensity} \). \( p_u(u) \) is the image intensity distribution.

Optimum MSE quantizer

\[ E(u) = \text{Expected value of } u = \int_{-\infty}^{\infty} p_u(u) \, du \]

\[ \text{MSE}, \varepsilon = E((u-u')^2) = \int_{-\infty}^{\infty} (u-u')^2 \, p_u(u) \, du = \int_{-\infty}^{\infty} (u-u')^2 \, p_u(u) \, du \]

Since \( u' = r_k \) if \( u \in [t_k, t_{k+1}] \), we can rewrite this as

\[ \varepsilon = \sum_{k=1}^{L} \int_{t_k}^{t_{k+1}} (u-r_k)^2 \, p_u(u) \, du \]

Conditions for minimisation of \( \varepsilon \) are:

\[ \frac{\partial \varepsilon}{\partial r_k} = 0; \quad \frac{\partial \varepsilon}{\partial t_k} = 0 \]
MMSE (contd.)

\[
\frac{\partial E}{\partial t_i} = (t_i - t_{i+1})^2 p_i(t_i) - (t_i - t_{i+2})^2 p_i(t_{i+2}) = 0
\]

Now \( t_i \leq t_j < t_{i+1} \) \( \Rightarrow t_i - t_{i+1} = t_i - t_j \Rightarrow t_i = \left( \frac{t_i + t_{i+1}}{2} \right) \) \( \cdots (A) \)

\[
\frac{\partial E}{\partial t_i} = \int_{-\infty}^{\infty} 2(u-t_j) (-1) p_i(u) \, du = 0
\]

\[\Rightarrow t_i = \frac{1}{2} \int_{-\infty}^{\infty} u \, p_i(u) \, du \]

\[\Rightarrow E(u | u \in [t_i, t_{i+1}]) \quad \cdots \cdots (B)
\]

Optimum transition/reconst.

(1) Optimal transition levels lie halfway between the optimum reconstruction levels.

(2) Optimum reconstruction levels lie at the center of mass of the probability density in between the transition levels.

(3) A and B are simultaneous non-linear equations (in general)

Closed form solutions normally don't exist

use numerical techniques

Uniform optimal quantizer

Consider \( p_i(u) = \begin{cases} \frac{1}{t_{i+1} - t_i} & t_i \leq u \leq t_{i+1} \\ 0 & \text{Otherwise} \end{cases} \)

Then \( t_i = \frac{1}{2} \int_{-\infty}^{\infty} \frac{u}{t_{i+1} - t_i} \, p_i(u) \, du \)

\[t_i = \frac{1}{2} \left( t_{i+1} + t_j \right); \quad t_j = \frac{r_j + t_{i+1}}{2} = \frac{1}{2} \left( t_{i+1} + t_i + t_{i+1} \right) = \frac{t_{i+1} + t_{i+1}}{2}
\]

Uniform Quantizer

\( t_i - t_{i-1} = t_{i+1} - t_i = \text{Constant} \quad \Rightarrow q = \frac{t_{i+1} - t_i}{L} \)

\[t_j = t_{i+1} + q \quad ; \quad r_j = t_j + \frac{q}{2}
\]

Quantization error \( e = (u - u') \) is uniformly distributed over the interval \( \left( \frac{-q}{2}, \frac{q}{2} \right) \)

Mean squared error \( E((u - u')^2) = \frac{1}{q} \int_{-\frac{q}{2}}^{\frac{q}{2}} u^2 \, du = \frac{q^2}{12} \)