Sampling and Quantization

Lecture #5
January 21, 2002
Sampling and Quantization

• Spatial Resolution (Sampling)
  – Determines the smallest perceivable image detail.
  – What is the best sampling rate?

• Gray-level resolution (Quantization)
  – Smallest discernible change in the gray level value.
  – Is there an optimal quantizer?
In 1-D

\[ f(x, y) \rightarrow f_s(m, n) \rightarrow u(m, n) \rightarrow \text{To Computer} \]

(Continuous image)
1-D

Time domain

\[ s(t) = \sum x(kt) \delta(t-kT) \]

Frequency

\[ X_s(f) \]

Jan 21, 2003

Sampling
2-D: Comb function

\[ \text{Comb}(x, y; \Delta x, \Delta y) \equiv \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \]
\[ f_s(x, y) = f(x, y) \text{comb}(x, y; \Delta x, \Delta y) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y) \delta(x - m\Delta x, y - n\Delta y) \]

\[ \text{comb}(x, y; \Delta x, \Delta y) \longleftrightarrow \mathcal{S} \rightarrow \text{COMB}(u, v) = \frac{1}{\Delta x \Delta y} \text{comb}(u, v; \frac{1}{\Delta x}, \frac{1}{\Delta y}) \]
Sampled Spectrum

\[ F_s(u, v) = F(u, v) \ast \text{COMB}(u, v) \]

\[ = \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F(u, v) \ast \delta \left( u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y} \right) \]

\[ = \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F \left( u - \frac{k}{\Delta x}, v - \frac{l}{\Delta y} \right) \]
Sampled Spectrum: Example
Bandlimited Images

A function $f(x,y)$ is said to be band limited if the Fourier transform

$$F(u,v) = 0 \text{ for } |u| > u_0, |v| > v_0$$

$u_0, v_0$ → Bandwidth of the image in the x- and y-directions

**region of support**
Foldover Frequencies

Sampling frequencies:
Let $u_s$ and $v_s$ be the sampling frequencies

Then $u_s > 2u_0$ ; $v_s > 2v_0$

or \[ \Delta x < \frac{1}{2u_0} ; \Delta y < \frac{1}{2v_0} \]

Frequencies above half the sampling frequencies are called fold over frequencies.
Sampling Theorem

A band limited image $f(x,y)$ with $F(u,v)$ as its Fourier transform; and $F(u,v) = 0$ if $|u| > u_0$, $|v| > v_0$; and sampled uniformly on a rectangular grid with spacing $\Delta x$ and $\Delta y$, can be recovered without error from the sample values $f(m \Delta x, n \Delta y)$ provided the sampling rate is greater than the nyquist rate.

i.e. $1/ \Delta x = u_s > 2 u_0$, $1/ \Delta y = v_s > 2 v_0$

The reconstructed image is given by the interpolation formula:

$$f(x,y) = \sum_{m,n=-\infty}^{\infty} f(m \Delta x, n \Delta y) \frac{\sin(x u_s - m)\pi}{(x u_s - m)\pi} \frac{\sin(y v_s - n)\pi}{(y v_s - n)\pi}$$
Reconstruction
Reconstruction via LPF

F(u,v) can be recovered by a LPF with

\[ H(u, v) = \begin{cases} \Delta x \Delta y & (u, v) \in R \\ 0 & \text{Other wise} \end{cases} \]

R is any region whose boundary \( \partial R \) is contained within the annular ring between the rectangles \( R_1 \) and \( R_2 \) in the figure. Reconstructed signal is

\[ \tilde{F}(u, v) = H(u, v) F_s(u, v) = F(u, v) \]

\[ f(x, y) = \mathcal{F}^{-1}[F(u, v)] \]
Note: If $u_s$ and $v_s$ are below the Nyquist rate, the periodic replications will overlap, resulting in a distorted spectrum.

This overlapping of successive periods of the spectrum causes the foldover frequencies in the original image to appear as frequencies below $u_s/2$, $v_s/2$ in the sampled image. This is called aliasing.
Fig. 9  (a) The sampling lattice in Experiment 1 is used to sample a picture whose Fourier transform is nonzero over a larger region than would lead to error-free reconstruction. (b) Three of the terms in Eq. (31) are pictorially illustrated here for \( F(\omega') \) shown in Fig. 9a. These three terms correspond to \((m, n)\) equal to \((0, 0), (1, -1), \) and \((1, 1)\).
Example

\[ f(x, y) = 2 \cos (2\pi (3x + 4y)) \]
\[ F(u, v) = \delta(u - 3, v - 4) + \delta(u + 3, v + 4) \]
\[ \Rightarrow u_0 = 3, \quad v_0 = 4 \]

Let \( \Delta x = \Delta y = 0.2, \Rightarrow u_s = v_s = \frac{1}{0.2} = 5 \quad < 2u_0, < 2v_0 \)

there will be aliasing.
Example:(contd.)

\[ F_s(u, v) = 25 \sum_{k,l=-\infty}^{\infty} F(u-ku_v, v-lv_s) \]

\[ = 25 \sum_{k,l=-\infty}^{\infty} [\delta(u-3-5k, v-4-5l) + \delta(u+3-5k, v+4-5l)] \]

Let \[ H(u, v) = \begin{cases} \frac{1}{25} & -2.5 \leq u \leq 2.5, \quad -2.5 \leq u \leq 2.5 \\ 0 & \text{Otherwise} \end{cases} \]

\[ \therefore F(u, v) = H(u, v) F_s(u, v) \]

\[ = \delta(u + 2, v + 1) + \delta(u - 2, v - 1) \]

\[ \therefore \tilde{f}(x, y) = 2 \cos(2\pi(2x + y)) \]
Examples

Original and the reconstructed image from samples.
Another example

Sampling filter

Sampled image
Aliasing Problems (real images!)