Lecture 2 Supplemental Notes:

1. **Jacobian**: A matrix relating instantaneous joint velocities to end effector velocities.

Example: 2-link, planar arm.

- **Kinematics**: Geometric relation between joints and end effector.
  
  \[ \begin{align*}
  x_e &= \text{expression} \\
  y_e &= \text{expression} \\
  \theta_e &= \text{expression}
  \end{align*} \]

- **Velocity relationship**: Just differentiate the above expressions, with respect to (wrt) time:
  
  \[ \begin{align*}
  \dot{x}_e &= \text{expression} \\
  \dot{y}_e &= \text{expression} \\
  \dot{\theta}_e &= \text{expression}
  \end{align*} \]
Let Greek letter “xi”, \( \xi \), represent the task space coordinates of the end effector, and let \( q \) represent the joint space coordinates, which are the two joint angles:

\[
\xi = \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix}, \quad q = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}
\]

The JACOBAN is the matrix that relates the velocities of the coordinates in \( \xi \) to the velocities of the coordinates in \( q \):

\[
\dot{\xi} = J \dot{q}
\]

QUESTION: Since \( \xi \) is a (3x1) vector, and \( q \) is a (2x1) vector in this example, what size must matrix \( J \) be?

QUESTION: From the expressions we derived on page 1, what is \( J \)?

\[
J =
\]

QUESTION: If we can always control the velocities \( \theta_1 \) and \( \theta_2 \) independently, does this imply we can always control the velocities of coordinates \( x_e \) and \( y_e \) independently, too?

QUESTION: What happens to \( J \) if \( \theta_1 \) and \( \theta_2 \) are both zero?

QUESTION: How does this affect our ability to set the horizontal velocity, \( \frac{dx_e}{dt} \), of the end effector?
2. **Singularity:** (all statements below are true)

- Loss of an independent DOF (direction of instantaneous motion) of the end effector always occurs at a singularity.

- Rank of $J$ is reduced (i.e., less than its maximal possible value) at a configuration that is a singularity.

- When at the edge of the workspace (of the end effector), this is always a singularity; the converse is not necessarily true…

- i.e., singularities can also occur when the end effector is not at the boundary of the workspace.

- Near a singularity, matrix inversion techniques (e.g., to control the dynamics) can “blow up”! (If $J$ is usually invertible, it becomes dangerous to do so as the system approaches a singularity…)

  e.g., if $x_i$ contains only $x_e$ and $y_e$ (not $\theta_e$), then $J$ would be 2x2 and would be invertible when the system in NOT in a singularity:

  \[
  \dot{\xi}_{ee} = J \dot{\theta}_q, \quad \text{so we can invert the relationship:} \quad \dot{\theta}_q = J^{-1} \dot{\xi}_{ee}
  \]

- Mechanical advantage: Near a singularity, the system can sometimes have improved load bearing. For example, when you “lock your knees” to stand, your two-link leg systems are each at a singularity, which reduces the torque you need to supply to remain standing.

- The Jacobian must also describes the static force-torque relationship between actuator effort and output at the end effector:

  \[
  \tau_q = J^T F_{ee}
  \]

- Finally, remember that $J$ is a function of the joint space configuration, $q$, so we should really always be writing $J(q)$ above to indicate this as a reminder, instead of just “$J$”, alone!
3. Work: force, displace, and energy

Terms

Force: a measure of “effort”.

Work: a measure of “accomplishment” (i.e., force time distance).

Energy: ability to do work. (Joules and Newton-meters are the same units!)

Power: Rate of doing work (work per unit time), and/or rate of using energy.

Inertia: Defined with respect to some point (such as center of mass), an inertia is the sum of “mass times radius squared” for all particles in a rigid object.

4. Impedance.

Impedance: The ratio of “Effort”/”Flow”. In general, this can be either a static relationship, such as a resistor value, R, or it can by a dynamic relationship, described by a differential equation.

Mechanical Impedance: Force/Velocity (Effort/Flow). \( Z_m(s) = \frac{F(s)}{sX(s)} \).
Examples: mass, damper, spring.

Electrical Impedance: Voltage/Current (Effort/Flow). \( Z_e(s) = \frac{V(s)}{I(s)} \).
Examples: inductor, resistor, capacitor.