The Zero-Error Capacity of Compound Channels

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Outline

• Preliminaries
• Problem definition
• Capacity
  – Neither side has side-information
  – Encoder has side-information
  – Decoder has side-information
• Comparison with vanishing-error case
Zero-Error vs. Vanishing-Error

• Vanishing-error Case
  – $P_e$ tends to zero as block length tends to infinity.

• Zero-error case
  – $P_e$ zero at all block lengths
  – Tools: graph theory and combinatorics
Discrete Memoryless Channel

Channel transition probability $p(y|x), x \in \mathcal{X}, y \in \mathcal{Y}$

$F(x) = \{ y \in \mathcal{Y} : p(y|x) > 0 \}$

$G_X = (\mathcal{X}, E_X)$ Characteristic graph of the channel

$(x_1, x_2) \in E_X \iff F(x_1) \cap F(x_2) = \emptyset$  

Undirected

Example

\[ \mathcal{X} \quad \mathcal{Y} \]

\[ 1 \quad 2 \quad 3 \]

\[ a \quad b \quad c \quad d \quad e \]

\[ G_X \]
Channel Code

• Code ⇒ symbols pair-wise connected = clique
• Example (cont.)
  – Scalar code: \{1, 3\}
• 1-use capacity = \(\log_2 \omega(G_X)\) bits
  – \(\omega(G_X)\) is the clique number
Channel Code

- \( n \) uses of the channel
  - Fan-out set is Cartesian product of fan-out sets at each coordinate

\[
\begin{array}{cccc}
  x_1^1 & x_2^1 & \cdots & x_i^1 & \cdots & x_n^1 \\
  x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots & x_n^2 \\
\end{array}
\]

- \( N(G_x, n) \) = size of largest pairwise connected set
Channel Code

• Zero error capacity (Shannon ’56)

\[ C^0 = \lim_{n \to \infty} \frac{1}{n} \log_2 \left[ N(G_X, n) \right] \text{ bits/channel use} \]

• Depends only on characteristic graph
  – Shannon capacity of a graph \( G_X \)
Other Graph Capacities

• Shannon capacity of a set of graphs: $C(\mathcal{G})$
  – Asymptotic size of set of sequences connected with respect to every graph from $\mathcal{G}$
  – Defined by Cohen et al. ‘88

• Directed graphs

\[ \begin{align*}
x_1^1 & | x_2^1 \\
& \downarrow \\
x_1^2 & | x_2^2 \\
\end{align*} \quad \begin{align*}
\cdots & \quad \begin{align*}
x_i^1 & | x_j^1 \\
& \downarrow \\
x_i^2 & | x_j^2 \\
\cdots & \quad \begin{align*}
x_n^1 & | x_n^2 \\
\end{align*} \quad \begin{align*}
\quad \text{Connected}
\end{align*}
\]
Other Graph Capacities

• Sperner capacity of a graph $\Sigma(G)$
  – largest pairwise connected set
• $\Sigma(G)$: connected with respect to every graph
• Sperner capacities defined by Gargano et al. ’94
  – Motivated by extremal set theory
The Compound Channel

\[ \mathcal{C} = \{ p(y|x, s) : x \in \mathcal{X}, y \in \mathcal{Y}, s \in \mathcal{S} \} \]

- Once chosen, DMC remains constant throughout the transmission.
Capacity

• Conventional Case:
  – Dobrushin ‘59, Wolfowitz ‘60, Breiman ‘59

• Zero-Error Case:
  – Cohen et al. ‘88, Gargano et al. ‘94
  – Capacity expression accurate only when decoder is informed.
Characteristic Set of Graphs

- Fan-out set $\bigcup_{s \in S} F_s(x)$

  $\left( \bigcup_{s \in S} F_s(x) \right) \cap \left( \bigcup_{s' \in S} F_{s'}(x') \right) = \bigcup_{s \in S} \bigcup_{s' \in S} F_s(x) \cap F_{s'}(x')$

- $G(\mathcal{C}) = \{ G_{ss'}, \forall s, s' \in S \}$
  - Vertex set $\mathcal{X}$
  - $x \rightarrow x'$ in $G_{ss'} \iff F_s(x) \cap F_{s'}(x') = \emptyset$
  - Directed graph
Capacity of Compound Channel

- Every code is a pairwise connected set with respect to $\mathcal{G}(\mathcal{C})$ and vice versa

$$C^0(\mathcal{C}) = \sum(\mathcal{G}(\mathcal{C}))$$
Encoder Informed of $S$

$$C_{enc}^0(\mathcal{C}) = \begin{cases} 
\min_{s \in S} C(G_s) & \text{if } C^0(\mathcal{C}) > 0 \\
0 & \text{if } C^0(\mathcal{C}) = 0
\end{cases}$$

- If $C^0(\mathcal{C}) > 0$, encoder sends side-information to decoder.
- Needs constant number of channel uses.
- $C^0(\mathcal{C}) = 0$ case:
  - $\mathcal{G}(\mathcal{C})$ contains an edge-free graph
Decoder Informed of $S$

- Decoder can change with channel
  \[ F_s(x) \cap F_s(x') = \emptyset, \forall s \in S \]
- Therefore
  \[ C_{dec}^0 (\mathcal{C}) = C(\mathcal{G}_S) \]
  \[ \mathcal{G}_S = \{ G_s, s \in S \} \]
Conventional vs. Zero-Error

• Conventional Capacities
  \[
  \min_s C_s > 0 \Rightarrow C(\mathcal{C}) > 0
  \]
  \[
  C(\mathcal{C}) = C_{dec}(\mathcal{C}) \leq C_{enc}(\mathcal{C})
  \]

• Zero-Error Capacities
  \[
  \min_s C(G_s) > 0 \text{ but } C^0(\mathcal{C}) = 0
  \]
  \[
  C^0(\mathcal{C}) \leq C^0_{dec}(\mathcal{C}); C^0(\mathcal{C}) \leq C^0_{enc}(\mathcal{C})
  \]

• Reason: Decoder can identify channel with high probability
Thank you