Problem 1 Provide the proof of Lemma 4.5 in the book under the weakened assumption that $x \mapsto f(x, 0)$ is globally Lipschitz and there exists $L > 0$ such that $|f(x, u) - f(x, 0)| \leq L|u|$ for all $(x, u)$. Show, by an example, that the combination of these two conditions is weaker than asking that $f$ is globally Lipschitz, which is the assumption in Lemma 4.5.

Problem 2 Consider $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2 \times 2}$ given by

$$A(t) := \begin{bmatrix} -1 + 1.5 \cos^2(t) & 1 - 1.5 \sin(t) \cos(t) \\ -1 - 1.5 \sin(t) \cos(t) & -1 + 1.5 \sin^2(t) \end{bmatrix} \quad \forall t \geq 0.$$  

1. Prove that the eigenvalues of $A(t)$ are constant with negative real part.

2. Show that the solution of $\dot{x} = A(t)x$ starting from $t_0 = 0$ and $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is equal to $e^{0.5t} \begin{bmatrix} \cos(t) & -\sin(t) \end{bmatrix}^T$. What does this imply about uniform global exponential stability of the origin for $\dot{x} = A(t)x$?

3. Use a Lyapunov function argument to show that the origin of $\dot{x} = \mu A(t)x$ is uniformly globally exponentially stable for $\mu \gg 1$. You may use the fact that the pointwise symmetric, positive definite solution $P : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2 \times 2}$ of the equation $A^T(t)P(t) + P(t)A(t) = -I$ is continuously differentiable for the given $A(\cdot)$.

Problem 3 Consider Problem 4.6 in the text but prove uniform global asymptotic stability via the Matrosov conditions under the alternative assumption that $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is bounded and persistently exciting, in the sense that there exists $p \geq 1$, $T > 0$, and $\varepsilon > 0$ such that

$$\int_t^{t+T} g^p(\tau) d\tau \geq \varepsilon \quad \forall t \geq 0.$$

Problem 4 Text problem 4.14(8).

Problem 5 Text problem 5.4.

Problem 6 Text problem 5.8.

Problem 7 Text problem 6.10.