Textbook problems:

3.2 Level sets of convex, concave, quasiconvex, and quasiconcave functions. Some level sets of a function $f$ are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.

Could $f$ be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.

3.15 A family of concave utility functions. For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\text{dom } u_\alpha = \mathbb{R}_+$. We also define $u_0(x) = \log x$ (with $\text{dom } u_0 = \mathbb{R}_+$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

(b) Show that $u_\alpha$ are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of $u_\alpha$ means that the marginal utility (i.e., the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of satiation.
3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) \( f(x) = e^x - 1 \) on \( \mathbb{R} \).
(b) \( f(x_1, x_2) = x_1x_2 \) on \( \mathbb{R}^2_{++} \).
(c) \( f(x_1, x_2) = 1/(x_1x_2) \) on \( \mathbb{R}^2_{++} \).
(d) \( f(x_1, x_2) = x_1/x_2 \) on \( \mathbb{R}^2_{++} \).
(e) \( f(x_1, x_2) = x_1^2/x_2 \) on \( \mathbb{R} \times \mathbb{R}^+ \).
(f) \( f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha} \), where \( 0 \leq \alpha \leq 1 \), on \( \mathbb{R}^2_{++} \).

4.1 Consider the optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x_1, x_2) \\
\text{subject to} & \quad 2x_1 + x_2 \geq 1 \\
& \quad x_1 + 3x_2 \geq 1 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) \( f_0(x_1, x_2) = x_1 + x_2 \).
(b) \( f_0(x_1, x_2) = -x_1 - x_2 \).
(c) \( f_0(x_1, x_2) = x_1 \).
(d) \( f_0(x_1, x_2) = \max\{x_1, x_2\} \).
(e) \( f_0(x_1, x_2) = x_1^2 + 9x_2^2 \).
Additional exercise problems:

1.7 Dual cones in $\mathbb{R}^2$. Describe the dual cone for each of the following cones.
   (a) $K = \{0\}$.
   (b) $K = \mathbb{R}^2$.
   (c) $K = \{(x_1, x_2) \mid |x_1| < x_2\}$.
   (d) $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

2.2 A general vector composition rule. Suppose

$$f(x) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

where $h : \mathbb{R}^k \to \mathbb{R}$ is convex, and $g_i : \mathbb{R}^n \to \mathbb{R}$. Suppose that for each $i$, one of the following holds:

- $h$ is nondecreasing in the $i$th argument, and $g_i$ is convex
- $h$ is nonincreasing in the $i$th argument, and $g_i$ is concave
- $g_i$ is affine.

Show that $f$ is convex. (This composition rule subsumes all the ones given in the book, and is the one used in software systems such as CVX.) You can assume that $\text{dom } h = \mathbb{R}^k$; the result also holds in the general case when the monotonicity conditions listed above are imposed on $\hat{h}$, the extended-valued extension of $h$.

2.12 Continued fraction function. Show that the function

$$f(x) = \frac{1}{x_1 - \frac{1}{x_2 - \frac{1}{x_3 - \frac{1}{x_4}}}$$

defined where every denominator is positive, is convex and decreasing. (There is nothing special about $n = 4$ here; the same holds for any number of variables.)

3.2 ‘Hello World’ in CVX. Use CVX to verify the optimal values you obtained (analytically) for exercise 4.1 in Convex Optimization.