The Design of Wide-Band Recursive and Nonrecursive Digital Differentiators

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Abstract

Designs for recursive and nonrecursive wide-band differentiators are presented. The coefficients for the recursive differentiators were optimally chosen to minimize a square-error criterion based on the magnitude of the frequency response. The coefficients for the nonrecursive differentiators were chosen using a frequency sampling technique. One or more of the coefficients were optimally selected to minimize the peak absolute error between the obtained frequency response and the response of an ideal differentiator. The frequency response characteristics of the recursive differentiators had small magnitude errors but significant phase errors. The nonrecursive differentiators required on the order of 16 to 32 terms for the magnitude error of the frequency response to be as small as the magnitude errors for the recursive differentiators; however, there were *no* phase errors for the nonrecursive case. The delay of the recursive differentiators was small compared to the delay of the nonrecursive differentiators.

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Introduction

A differentiator forms an integral part of many physical systems. Therefore, the design of adequate wide-band differentiators has always been of considerable interest. With the increased trend towards digital simulation of systems, optimal techniques for designing wide-band digital differentiators are being more widely investigated. Kaiser [1] has presented a review of several techniques for designing both nonrecursive and recursive differentiators.

Recent work by Steiglitz [2] has concentrated on the optimal design of recursive digital filters (i.e., filters synthesized with both poles and zeros) with the aid of a large digital computer. The computer optimally chose z-plane positions of poles and zeros to minimize a square-error criterion based on the magnitude of the frequency response. The design problem for nonrecursive digital filters (i.e., filters synthesized with only zeros) has recently been considered by Gold and Jordan [3], and Rabiner, Gold, and McGonegal [4]. They used a digital computer to determine optimal values of a few samples of the discrete Fourier transform of the finite impulse response in order to minimize peak magnitude deviation from the prescribed frequency response.

The work done by Steiglitz indicated that by designing ideal differentiators and allowing a noninteger number of samples of delay, differentiators could be designed with usable bandwidths up to 100 percent full band. In this paper we present specific designs for several recursive and nonrecursive differentiators using the optimal design techniques of the earlier work. These designs are evaluated and compared with respect to approximation errors and realizations, whenever possible.

Theory

The ideal frequency response characteristics of a digital differentiator are shown in Fig. 1. The first two curves in the top line show the magnitude and phase of the frequency response, and the third curve shows the resulting imaginary part of the frequency response (the real part is identically zero in this case). The magnitude response increases linearly up to a normalized frequency of 1.0 (the Nyquist frequency[1]) and then decreases linearly back to 0.0 at the sampling frequency. The magnitude response is periodic in frequency, as shown, because of the discreteness property. The phase is $\pi/2$ radians for frequencies up to the Nyquist frequency and $-\pi/2$ radians from the Nyquist frequency to the sampling frequency, and is also periodic. The resulting imaginary part of the frequency response increases linearly to 1.0 at the Nyquist frequency, jumps discontinuously to -1.0, and then increases linearly to 0.0 at the sampling frequency.

[1] The Nyquist frequency is the highest frequency allowable at the input to the digital system. Thus the input is sampled at twice the Nyquist frequency.

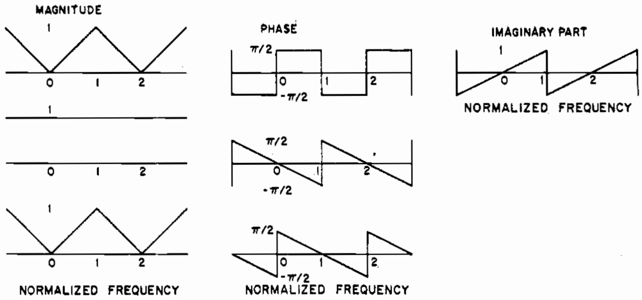


Fig. 1. The frequency response curves for the ideal differentiator. In the first line are shown the magnitude curve, phase curve, and imaginary part of the frequency response. The second line shows the magnitude and phase curves for an ideal one-half sample delay network. The third line shows the magnitude and phase curves for an ideal differentiator with half a sample of delay.

It is the discontinuity in the imaginary part of the response at the Nyquist frequency which makes it difficult to design very wide-band differentiators. It is impossible to obtain a discontinuity in the imaginary part, like the one shown at the top of Fig. 1, in a zero bandwidth region. Therefore a certain amount of the frequency band is designated to be a transition region. Typical designs which result from this method are shown in [1] and [4]. The widest bandwidth differentiators obtained (with reasonable approximation error) are about 95 percent.

One way around the problem of a discontinuity at the Nyquist frequency is to add a delay of a half sample to the differentiator, i.e., to consider the design of an ideal differentiator with half a sample delay. The magnitude response and phase response curves of an ideal half-sample delay network are shown in the middle of Fig. 1. The magnitude is 1 for all frequencies; the phase response is linear with a discontinuity of π radians at the Nyquist frequency. The overall differentiator frequency response with half a sample delay is shown at the bottom of Fig. 1. The magnitude response curve is identical to the original differentiator magnitude response curve, but the phase response curve is now linear with a discontinuity of π radians at 0 frequency. At first thought it would seem that all that this procedure has accomplished is a shifting of a discontinuity from the Nyquist frequency to 0 frequency. However, if one takes into consideration that the magnitude at 0 frequency is exactly 0, it is seen that there must be a zero of the differentiator on the unit circle at zero frequency. A zero on the unit circle will automatically give a phase discontinuity of π radians. Thus the shifting of the discontinuity to 0 frequency, along with the zero of the magnitude response at this frequency, has alleviated the approximation difficulties.

Using this result, both recursive and nonrecursive approximations to the half-sample delay differentiator can be designed with bandwidths up to 100 percent full band. Some specific designs are presented in the following sections.

Recursive Designs

The canonic form used to describe the transfer function of a recursive differentiator is seen in (1):

$$H(z) = A \prod_{i=1}^{K} \frac{(1 - z^{-1}a_{i1})(1 - z^{-1}a_{i2})}{(1 - z^{-1}b_{i1})(z - z^{-1}b_{i2})} \cdot \qquad (1)$$

The transfer function of (1) describes a cascade of K sections each containing two zeros (at $z = a_{i1}, a_{i2}$) and two poles (at $z = b_{i1}, b_{i2}$). The poles and zeros are chosen optimally by computer to minimize a square-error criterion based on the magnitude of the frequency response. The sum of the squares of the magnitude error at 21 equally spaced frequencies from 0 to the Nyquist frequency was minimized with respect to positions of the fixed number of poles and zeros used in the approximation. In Figs. 2 through 4 are shown the error curves (both magnitude and phase error) for one-section ($K=1$), two-section ($K=2$), and three-section ($K=3$) differentiators. It should be noted that these designs are for full-band differentiators. The actual values of the poles and zeros, as well as the constant multiplier A, are given in Table I.

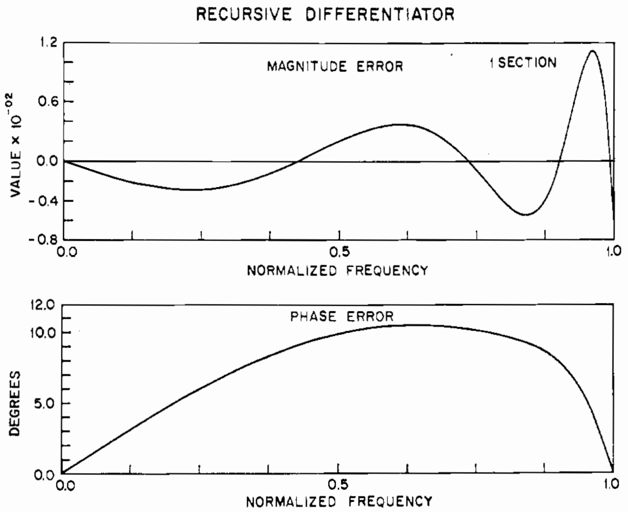
Fig. 2 shows that the peak magnitude error for the one-section design is about 1.1×10^{-2} and occurs near the Nyquist frequency. The peak phase error for this design is about 10.5 degrees and occurs at a normalized frequency of about 0.6. For the two-section design of Fig. 3, the peak magnitude error is about 6.3×10^{-3} occurring near the Nyquist frequency, and the peak phase error is again 10.5 degrees. It should be noted that the magnitude error is under 1.0×10^{-3} for about 95 percent of the bandwidth.

TABLE I

Poles and Zeros of Recursive Differentiators

One-Section		
Zeros:	1.00000000,	−0.67082621
Poles:	−0.14240300,	−0.71698670
A:	0.36637364	
Two-Section		
Zeros:	0.99999949,	−0.86810806
	0.32672838,	−0.44183252
Poles:	−0.10779165,	−0.87602073
	0.33494085,	−0.51312758
A:	0.36804011	
Three-Section		
Zeros:	0.99999956,	−0.87737870
	0.36692749,	−0.49648721
	$-0.15993072\times10^{-2}\pm j0.72664683\times10^{-1}$	
Poles:	−0.11127243,	−0.88411119
	0.35896158,	−0.55390515
	0.63811464×10^{-1}	-0.63137788×10^{-1}
A:	0.36789870	

Fig. 2. The magnitude and phase error curves for the optimum one-section recursive approximation to the ideal differentiator.



The errors of the three-section differentiator of Fig. 4 are quite similar to the two-section differentiator. The most notable improvement is a halving of the magnitude error in the low-frequency region. It seems clear that the approximation error reduction in going from two to three sections is small compared to the error reduction in going from one to two sections. Thus further increases in the number of sections would probably not change the results dramatically. Furthermore, since the design technique approximated only the magnitude characteristics of the differentiator, the phase error did not change appreciably as the number of sections increased, as can easily be seen from Figs. 2 through 4.

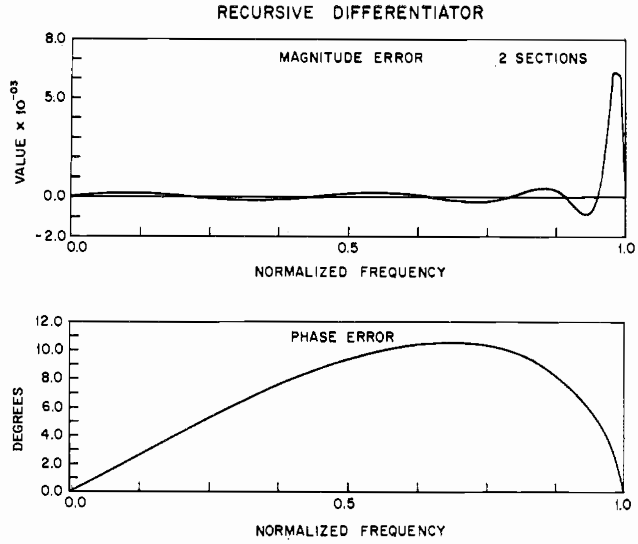
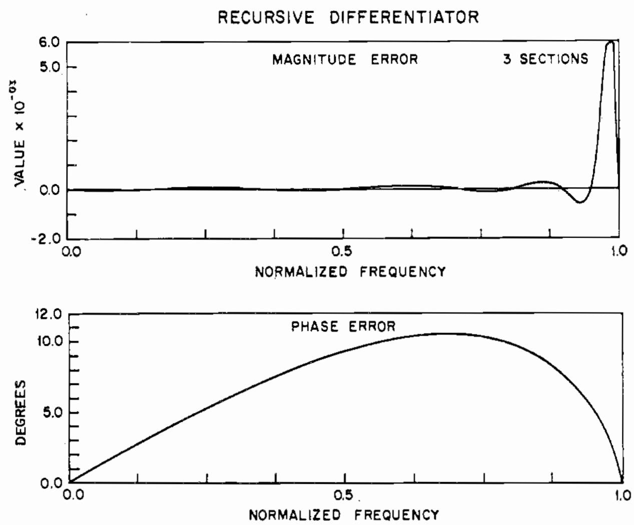


Fig. 3. The magnitude and phase error curves for the optimum two-section recursive approximation to the ideal differentiator.

Fig. 4. The magnitude and phase error curves for the optimum three-section recursive approximation to the ideal differentiator.



Nonrecursive Designs

An optimal design method for nonrecursive filters was recently presented by Gold and Jordan, and developed by Rabiner, Gold, and McGonegal. In this technique, linearly spaced samples of the frequency response of the desired filter were specified and the continuous frequency response was determined using the discrete Fourier transform. The interpolation formula obtained was

$$H(e^{j\omega T}) = \frac{\exp\left[-\frac{j\omega NT}{2}\left(1-\frac{1}{N}\right)\right]}{N} \cdot \sum_{k=0}^{N-1} \frac{H_k e^{-j\pi k/N} \sin\left(\frac{\omega NT}{2}\right)}{\sin\left(\frac{\omega T}{2}-\frac{\pi k}{N}\right)} \tag{2}$$

where

$$H_k = H(e^{j\omega T})\big|_{\omega=2\pi k/NT} \qquad k = 0, 1, \cdots, N-1; \tag{3}$$

i.e., $\{H_k\}$ are values of the continuous frequency response at equally spaced points around the unit circle; T is the sampling period; and N is the duration of the impulse response in samples. By making the substitution

$$H_k = jG_k e^{j\pi k/N}, \tag{4}$$

each of the terms inside the summation of (2) becomes imaginary, and thus the entire sum is imaginary. For N even, the complex factor outside the summation in (2) represents a pure delay of an integer ($e^{-j\omega T(N/2)}$) plus one-half ($e^{-j\omega T(1/2)}$) number of samples. Thus (2) suggests that a differentiator with exactly half a sample delay can be designed nonrecursively by setting

$$G_k = \begin{cases} k/(N/2), & k = 0, 1, \cdots, N/2 \\ (N-k)/(N/2), & k = \frac{N}{2}+1, \cdots, N-1 \end{cases} \tag{5}$$

and applying the substitution of (4) into (2). Equation (2) shows that in the resultant interpolated frequency response, the magnitude response approximates the differentiator magnitude response (with no approximation error at the frequencies where the exact values were specified), and the phase response is *exactly* the phase response shown at the bottom of Fig. 1, i.e., half a sample delay curve $\pm\pi$ radians. [It should be noted that the H_k are complex, as seen from (4).] By varying some of the G_k, or equivalently the H_k, the magnitude approximation error can be reduced without affecting the phase curve at all.

Examples of full-band nonrecursive differentiators for values of N from 16 to 256 in powers of 2 are given in Figs. 5 through 9, and the relevant data are tabulated in Table II. For each of these cases a single frequency sample (one of the G_k) was optimally chosen to minimize peak error. Table II lists the value of this variable frequency sample ($G_{N/2}$) as well as the peak magnitude error for the various values of N used. In each of Figs. 5 through 9 are shown the impulse response at the top of the figure, the magnitude response curve in the middle, and the magnitude error curve at the bottom. Since the phase curve is always identical to the phase curve at the bottom of Fig. 1, there was no need to plot it again.

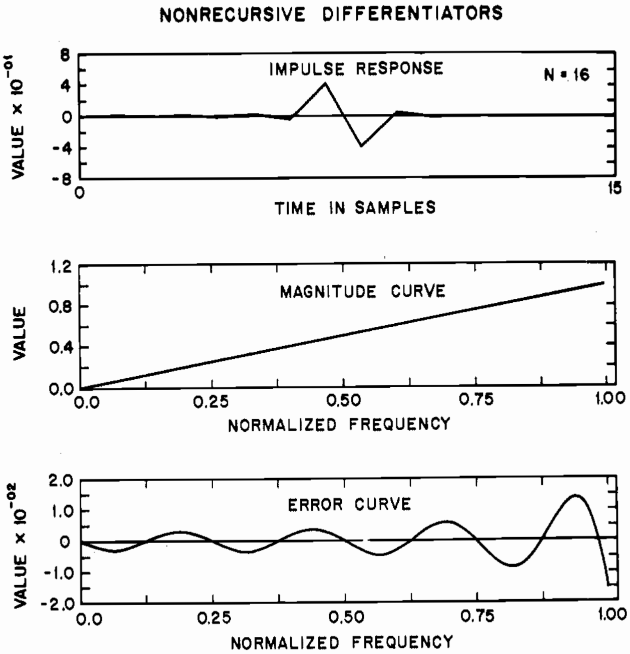
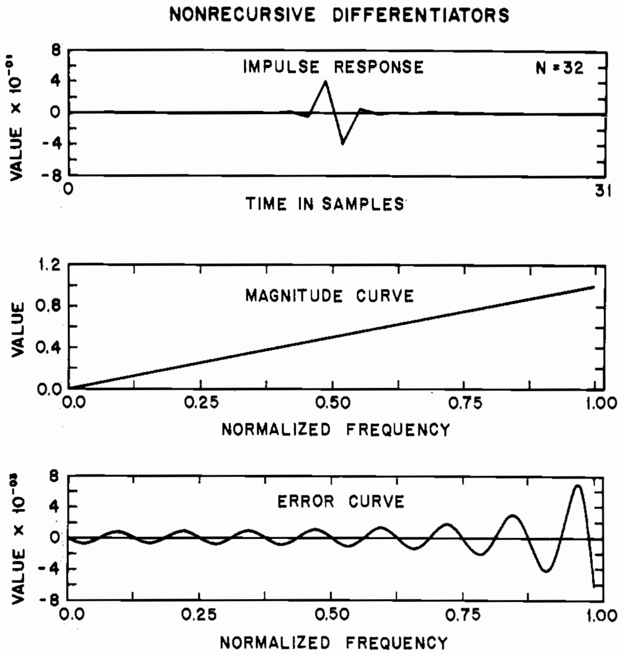


Fig. 5. The impulse response, magnitude curve, and magnitude error curve for the optimum nonrecursive approximation to the ideal differentiator with $N=16$.

Fig. 6. The impulse response, magnitude curve, and magnitude error curve for the optimum nonrecursive approximation to the ideal differentiator with $N=32$.



As seen in Fig. 5, the peak magnitude error for $N=16$ is about 1.4×10^{-2} and occurs at about 95 percent full band. For $N=32$ the peak magnitude error is 6.9×10^{-3} again occurring near the Nyquist frequency. Peak magnitude

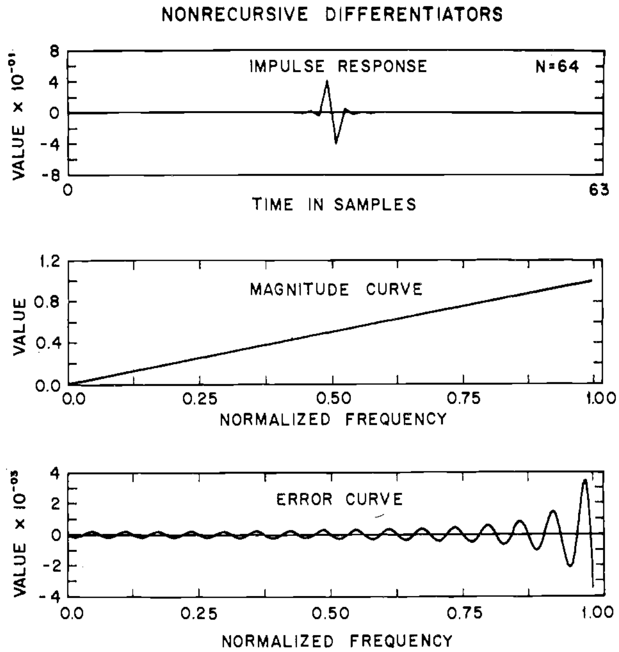
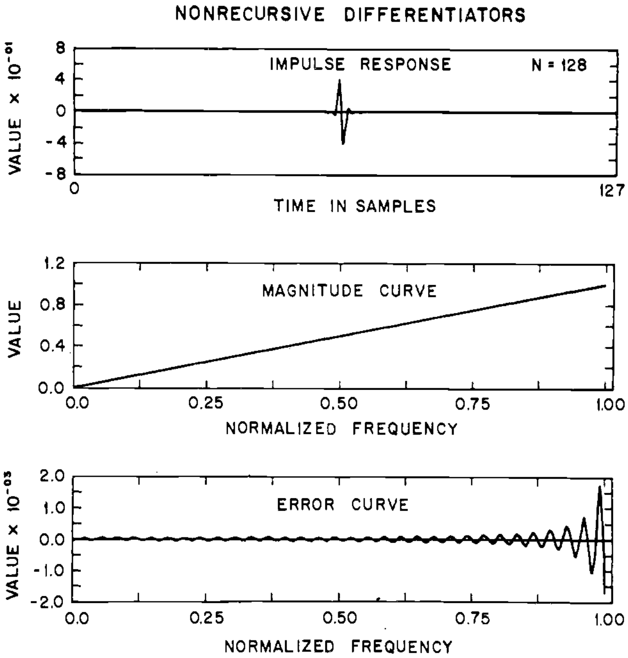


Fig. 7. The impulse response, magnitude curve, and magnitude error curve for the optimum nonrecursive approximation to the ideal differentiator with $N=64$.

Fig. 8. The impulse response, magnitude curve, and magnitude error curve for the optimum nonrecursive approximation to the ideal differentiator with $N=128$.



errors for $N=64$, 128, and 256 are 3.5×10^{-3}, 1.7×10^{-3}, and $9.\times10^{-4}$, respectively. Thus a doubling of N tends to halve the magnitude error. Therefore at the expense of increased N it would seem that the error can be made arbitrarily small.

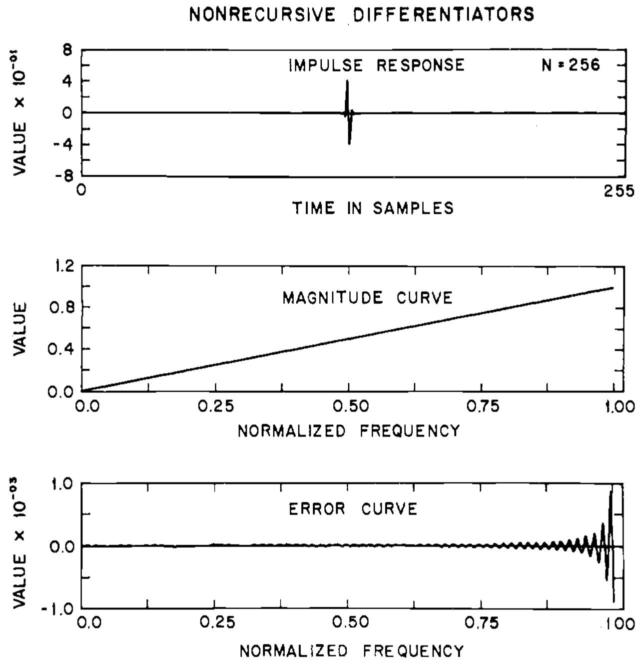


Fig. 9. The impulse response, magnitude curve, and magnitude error curve for the optimum nonrecursive approximation to the ideal differentiator with $N=256$.

TABLE II

Coefficients for Full-Band Nonrecursive Differentiators

N	$G_{N/2}$	Peak Magnitude Error
16	0.98609619	0.013909
32	0.99306030	0.006944
64	0.99653320	0.003472
128	0.99826661	0.001735
256	0.99913330	0.000868

Additional Nonrecursive Designs

It was of interest to consider designs of nonrecursive differentiators whose bandwidth was less than 100 percent full band for small values of $N (N=16, 32)$. For these cases, the three largest frequency samples ($G_{N/2}$, $G_{(N/2)-1}$, $G_{(N/2)-2}$) were optimally varied to minimize the peak error over the band of interest. The resulting designs are given in Table III. In this table the three coefficients are given, along with the peak error for several values of percentage bandwidth.

For $N=16$ the results in Table III show that the peak magnitude error drops very fast as percentage bandwidth decreases, going from 2.7×10^{-3} for 95 percent bandwidth, to 0.7×10^{-5} for 80 percent bandwidth. For $N=32$ the peak magnitude error drops even faster going from 3.8×10^{-4} for 95 percent bandwidth to 8×10^{-7} for 80 percent bandwidth. Thus merely by restricting the bandwidth of differentiation to a reasonable value (less than

TABLE III

Coefficients for Wide-Band Nonrecursive Differentiators

Percent Band-width	Peak Magnitude Error	$G_{N/2}$	$G_{(N/2)-1}$	$G_{(N/2)-2}$
		$N=16$		
95	0.00269	0.96256714	0.87614811	0.75000559
90	0.00072	0.94945069	0.87565707	0.74964216
85	0.00022	0.93826903	0.87324582	0.74978209
80	0.00007	0.92890015	0.86994255	0.75000000
		$N=32$		
95	0.00038	0.97510987	0.93785916	0.87484839
90	0.00003	0.96475830	0.93508185	0.87500000
85	0.000002	0.95614625	0.93098622	0.87483514
80	0.0000008	0.95259399	0.92893748	0.87453343

100 percent), the peak magnitude error can be made quite small even for small values of N.

Comparison of these data with earlier results [1], [4] on nonrecursive differentiator design shows considerable improvements. For a given N and a given bandwidth, the peak magnitude error of designs given in this paper is considerably less than the peak magnitude error resulting from earlier designs.

Comparison Between Recursive and Nonrecursive Designs

It is very difficult to compare the different designs for differentiators (or any other filter for that matter) because there are many issues which must be considered. For example, the recursive designs are easily realized with only a few multiplications per output sample; whereas for a large value of N a large number of multiplications per sample are required in the nonrecursive case (for realization by direct convolution). However, for realization by fast convolution, using the fast Fourier transform, the processing time is only weakly sensitive to the value of N used.

In terms of the accuracy of approximation, the one-section recursive differentiator has somewhat smaller peak magnitude error than the $N=16$ full-band nonrecursive case. However, there is a large phase error in the recursive design and none in the nonrecursive one. The two- and three-section recursive designs have peak magnitude errors comparable to the $N=32$ nonrecursive design.

Finally, recursive differentiators can be designed to have a small delay, whereas the delay for nonrecursive differentiators is generally about $N/2$ samples.

Conclusions

Designs for full-band recursive and nonrecursive differentiators have been presented and discussed. Additional designs for small values of N, where the differentiation bandwidth is less than 100 percent, have been presented for the nonrecursive case.

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Lawrence R. Rabiner (S'62–M'67), for a photograph and biography, please see this issue, page 106.

Kenneth Steiglitz (S'57–M'64), for a photograph and biography, please see this issue, page 129.