Recursive and Nonrecursive Realizations of Digital Filters Designed by Frequency Sampling Techniques

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Abstract

A great deal of work has been done recently on techniques for optimally designing finite duration impulse response (FIR) filters. One of these techniques, called frequency sampling, is a method for designing a digital filter from a set of samples of the desired filter frequency response. In this paper we present a unified discussion of the various types of frequency sampling designs and show how to realize them, both recursively and nonrecursively.

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Introduction

Several techniques for designing finite duration impulse response (FIR) digital filters have been proposed [1]–[5]. One of these techniques, called frequency sampling, allows the designer to specify values of the filter's frequency response[1] at equispaced frequencies, and hence derive an approximation to the desired continuous frequency response. For certain types of standard filter designs, such as lowpass, bandpass, and highpass filters, computer programs have been written to optimally choose several of the frequency samples in transition regions between passband and stopband to optimize the filter design [3], [4].

Several types of frequency sampling designs can be used. One of the relevant design parameters is whether N, the number of samples in the impulse response, is even or odd. Another important parameter is the frequency of the first sample of the frequency response. When the initial sample is at 0 frequency, the filters are called Type 1 designs. When the initial sample is offset by an angle of π/N rad, i.e., half a sampling interval, the filters are called Type 2 designs [4].

An FIR filter can be realized either recursively or nonrecursively. Recursive realizations of frequency sampling designs are simple to program and use and can be very efficient. Nonrecursive realizations include direct convolution and fast convolution. Depending on whether N is even or odd, and on whether the design is Type 1 or Type 2, the realizations of these filters can be vastly different. In this paper we present a unified treatment of all the cases of frequency sampling designs and give explicit formulas for their realizations.

Type 1 Frequency Sampling Designs

Fig. 1 shows an arbitrary frequency response (solid curves) that one wants to approximate and a sequence of N frequency samples H_k that can be represented as

$$H_k = |H_k| e^{j\theta_k}, \qquad k = 0, 1, \cdots, N-1. \tag{1}$$

Using the discrete Fourier transform (DFT), a finite duration impulse response can be determined from the H_k's as

$$h_n = \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{(j2\pi/N)kn}, \qquad n = 0, 1, \cdots, N-1. \tag{2}$$

The z transform of h_n, which is of the form

$$H(z) = \sum_{n=0}^{N-1} h_n z^{-n}, \tag{3a}$$

has the property that

$$H(z)\Big|_{z = e^{j(2\pi/N)k}} = H_k. \tag{3b}$$

Thus, the continuous frequency response of the filter has the desired values at the frequency samples. However, between

[1] These values are called "frequency samples."

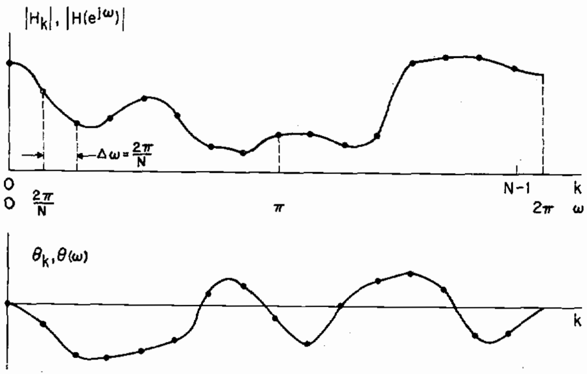

Fig. 1. The frequency samples (heavy points) and the desired continuous frequency response (solid curves) for a Type 1 FIR filter. The upper curve shows the magnitude response and the lower curve shows the phase response.

frequency samples, the continuous frequency response may differ significantly from the desired frequency response. Using (2) and (3), we can solve for $H(z)$ in terms of the samples H_k as

$$H(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} H_k e^{j(2\pi/N)nk} \right] z^{-n}, \tag{4}$$

which, by interchanging the order of summation and performing the summation over the n index, can be put in the form

$$H(z) = \frac{(1 - z^{-N})}{N} \sum_{k=0}^{N-1} \frac{H_k}{1 - z^{-1} e^{j(2\pi/N)k}} \cdot \tag{5a}$$

Real Impulse Response Coefficients

In most applications, the desired filter impulse response is purely real. This constraint introduces the following symmetry on the frequency samples:

$$|H_k| = |H_{N-k}|, \tag{6}$$

$$\theta_k = -\theta_{N-k}, \tag{7}$$

and if $H_0 > 0$,

$$\theta_0 = 0. \tag{8}$$

That is, the magnitude of the frequency samples is a real, symmetric sequence and the phase is a real, anti-symmetric sequence.

Assuming that the value of N is even (we will discuss the differences when N is odd later), we can apply the constraints of (8) to (5) to give

$$H(z) = \frac{(1 - z^{-N})}{N} \left[\sum_{k=1}^{(N/2)-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1} e^{j(2\pi/N)k}} + \sum_{k=(N/2)+1}^{N-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1} e^{j(2\pi/N)k}} + \frac{H_0}{1 - z^{-1}} + \frac{H_{N/2}}{1 + z^{-1}} \right]. \tag{9}$$

Making the substitution $k' = N - k$ in the second summation and using (6) and (7), we obtain

$$H(z) = \frac{(1 - z^{-N})}{N} \left[\sum_{k=1}^{(N/2)-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1} e^{j(2\pi/N)k}} + \sum_{k'=1}^{(N/2)-1} \frac{|H_k| e^{-j\theta_{k'}}}{1 - z^{-1} e^{-j(2\pi/N)k'}} + \frac{H_0}{1 - z^{-1}} + \frac{H_{N/2}}{1 + z^{-1}} \right]. \tag{10}$$

Combining complex conjugate terms in the first and second summation gives

$$H(z) = \frac{(1 - z^{-N})}{N} \left[\sum_{k=1}^{(N/2)-1} \frac{2|H_k| \left[\cos \theta_k - z^{-1} \cos (\theta_k - 2\pi k/N)\right]}{1 - 2z^{-1} \cos ((2\pi/N)k) + z^{-2}} + \frac{H_0}{1 - z^{-1}} + \frac{H_{N/2}}{1 + z^{-1}} \right]. \tag{11}$$

In the most general case, both the frequency samples H_k and the filter impulse response coefficients h_n are complex. Equation (5a) shows that the filter can be realized recursively as a cascade of a comb filter $(1 - z^{-N})$ which has N zeros at

$$z_k = e^{j(2\pi/N)k}, \qquad k = 0, 1, \cdots, N - 1, \tag{5b}$$

and a parallel connection of N complex resonators[2] whose poles occur at the zeros of the comb filter. The filter can also be realized nonrecursively by solving for the N impulse response coefficients using (2) and then performing discrete convolution.

Equation (11) can be realized recursively as a cascade of a comb filter and a parallel connection of $(N/2) - 1$ resonators and two networks with real axis poles.

It has been noted [1] that it is wise to move both the zeros and poles of the recursive realization slightly inside the unit circle to avoid inexact cancellation of a pole by a zero, and hence avoid instability problems that might arise. The suggested transformation is to replace z^{-1} by rz^{-1} where r is in the order of $1 - 2^{-L}$ and L is the number of bits used to represent the coefficients. Thus (11) becomes

$$H(z) = \frac{1 - r^N z^{-N}}{N} \left[\sum_{k=1}^{(N/2)-1} \frac{2|H_k| \left[\cos \theta_k - z^{-1} r \cos (\theta_k - 2\pi k/N)\right]}{1 - 2r \cos (2\pi k/N) z^{-1} + z^{-2} r^2} + \frac{H_0}{1 - rz^{-1}} + \frac{H_{N/2}}{1 + rz^{-1}} \right]. \tag{12}$$

The realization of the kth resonator is shown in Fig. 2(A), and the difference equation relating the samples of the resonator output v_n to its input samples u_n is

$$v_n = 2 \cos (\theta_k) u_n - 2r \cos (\theta_k - 2\pi k/N) u_{n-1} + 2r \cos (2\pi k/N) v_{n-1} - r^2 v_{n-2} \tag{13}$$

[2] The term "resonator" is used in this paper to denote a system having either a single complex pole or a complex conjugate pole pair inside or on the unit circle in the z plane.

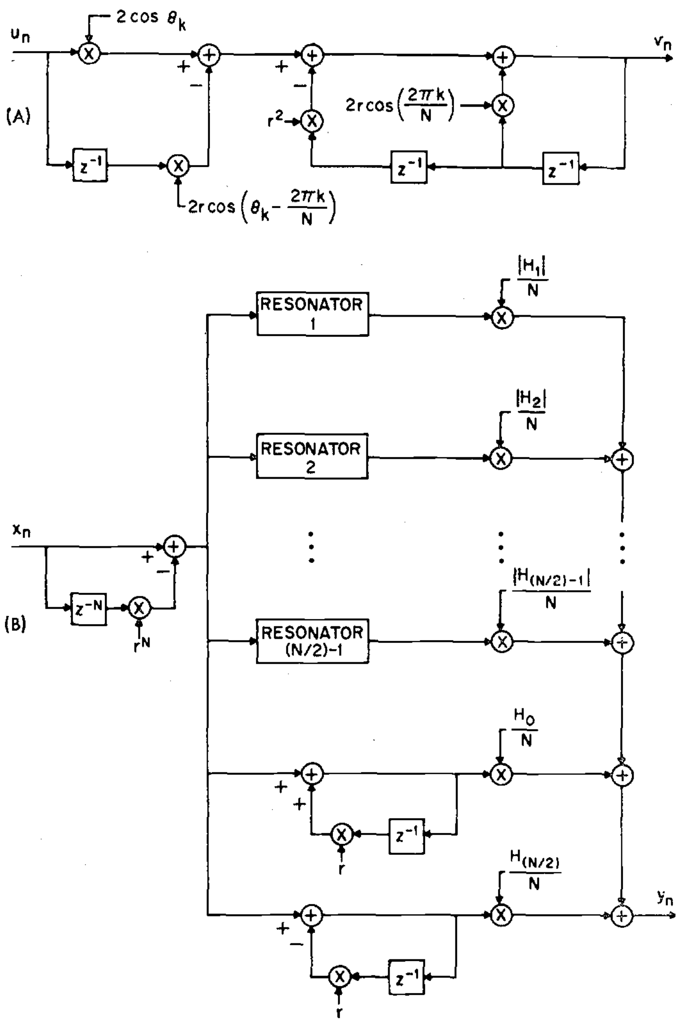

Fig. 2. The recursive realization of a Type 1 FIR filter with real impulse response coefficients. (A) Realization of the kth resonator section. (B) Cascade of the comb filter and the parallel bank of resonators.

where $n=0, 1, \cdots$ is the sample number and it is assumed that all initial conditions are zero. When $r<1$, each of the resonators requires four multiplications per output sample. The entire realization of (11) is shown in Fig. 2(B). All z-transform equations presented in the remainder of this paper assume that the zeros and poles lie on the unit circle (i.e., $r=1$). Figs. 2, 5, and 9, depicting recursive realizations of these filters, however, are drawn assuming that the above stabilizing transformation has been performed (i.e., $r<1$).

The filter of (11) can be realized by direct convolution after solving for the impulse response coefficients h_n using (2). Because of the symmetry constraints on the H_k's [(6)–(8)], (2) can be reduced to

$$h_n = \frac{H_0}{N} + \frac{(-1)^n}{N} H_{N/2} + \sum_{k=1}^{(N/2)-1} 2\,|H_k| \cos\left(\theta_k + (2\pi/N)nk\right). \quad (14)$$

For the case where N is odd, (11), (13), and (14) are modified slightly. The upper index in the summation becomes $(N-1)/2$ instead of $(N/2)-1$, and the term involving $H_{N/2}$ is missing because, when N is odd, there is no frequency sample at half the sampling frequency.

Linear Phase Type 1 Filters

One of the reasons FIR filters are of importance is that they can be designed to have an exactly linear phase, i.e., they can approximate an arbitrary magnitude frequency response with no phase error. It is this aspect of FIR filters that makes them useful for designing standard lowpass, bandpass, and highpass filters. It can be shown [7] that if the impulse response is of length $N=2\tau+1$ samples, then the filter will have linear phase with a delay of τ samples (not necessarily an integer) if and only if the impulse response is symmetric (i.e., $h_n=h_{N-1-n}$). Depending on whether N is even or odd, there are two distinct impulse response shapes that can satisfy this symmetry constraint. These are illustrated in Fig. 3. If N is even, the impulse response must be as shown in Fig. 3(A) to obtain linear phase. The important properties of this impulse response include the following.

1) There is no unique peak in the impulse response.
2) The impulse response is symmetric, i.e.,

$$h_n = h_{N-1-n}, \qquad n = 0, 1, \cdots, (N/2) - 1. \quad (15)$$

3) The center of symmetry lies midway between sample $(N/2)$ and $(N/2)-1$, i.e., the delay of the filter is $\tau=(N-1)/2$ samples, a noninteger delay.

For the case where N is odd, as is usually implicitly assumed in the literature on FIR filters, the shape of the impulse response of a realizable filter must be as shown in Fig. 3(B) to obtain linear phase. In contrast to Fig. 3(A), the following is noted.

1) There is a unique peak in the impulse response at sample $(N-1)/2$.
2) The impulse response is symmetric, i.e.,

$$h_n = h_{N-n-1}, \qquad n = 0, 1, \cdots, (N-1)/2. \quad (16)$$

3) The center of symmetry occurs at sample $(N-1)/2$, i.e., the delay of the filter is $\tau=(N-1)/2$ samples, an integer.

In the next sections we derive expressions for the z transforms of Type 1 design, linear phase filters for the cases of even and odd values of N.

N Even: An example of the frequency response specifications for a linear phase filter is shown in Fig. 4. The additional constraints on the frequency samples are

$$\theta_k = \begin{cases} (-2\pi/N)k\tau, & k=0, 1, \cdots, (N/2)-1 \\ (2\pi/N)(N-k)\tau, & k=(N/2)+1, \cdots, N-1 \end{cases} \quad (17)$$

and

$$H_{N/2} = 0 \quad (18)$$

where τ is the filter delay. The constraint of (18) can be shown valid by examining the effect of the frequency sample $H_{N/2}$ on the impulse response. From (14) we see that it contributes a term of the form

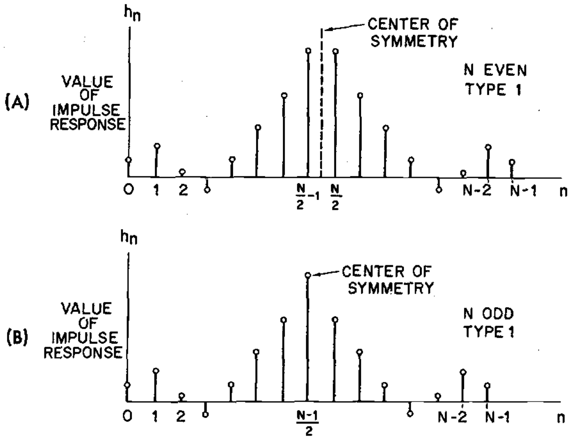

Fig. 3. The shape of the impulse response for Type 1, linear phase FIR filters. (A) Resulting impulse response when N is even. (B) Impulse response when N is odd.

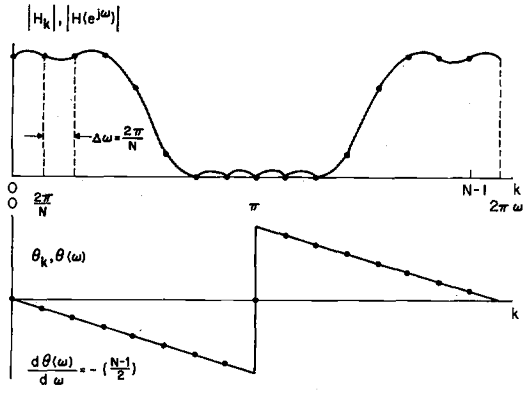

Fig. 4. The frequency samples (heavy points) and the desired continuous frequency response (solid curves) for a Type 1, linear phase FIR filter. Magnitude and phase curves are shown.

$$h_n = (-1)^n H_{N/2}, \tag{19}$$

which is not a symmetric sequence. Thus $H_{N/2}$ must be zero to satisfy the linear phase constraint.[3] Furthermore, from Fig. 4, one can see that at half the sampling frequency ($z=-1$) there is a phase discontinuity of $2\pi\tau$ rad. However, since there is a zero at half the sampling frequency ($H(z)|_{z=-1}=H_{N/2}=0$), the phase discontinuity must be ($\pi+2\pi m_0$) rad, where m_0 is an integer. Hence, τ the filter delay is ($m_0+1/2$) samples. This fact is consistent with the observation from Fig. 3(A) that the delay of the filter is not an integer number of samples. It is also clear from Fig. 3(A) that m_0, the integer part of the delay, is $(N/2)-1$ samples and the total delay τ is $(N-1)/2$ samples.

Substituting the constraints discussed above with those of (17) and (18) into (11) gives the z transform as

$$H(z) = \frac{(1-z^{-N})}{N}\left[\sum_{k=1}^{(N/2)-1} \frac{2\,|H_k|\left[\cos\left(\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)\right) - z^{-1}\cos\left[-\frac{2\pi k}{N}\left(\frac{N-1}{2}\right)-\frac{2\pi k}{N}\right]\right]}{1-2z^{-1}\cos\left(\frac{2\pi k}{N}\right)+z^{-2}} + \frac{H_0}{1-z^{-1}}\right], \tag{20}$$

which can be put in the form

$$H(z) = \frac{(1-z^{-N})}{N}\left[\sum_{k=1}^{(N/2)-1} \frac{(-1)^k\,|H_k|\,2\cos\left(\frac{\pi k}{N}\right)(1-z^{-1})}{1-2z^{-1}\cos\left(\frac{2\pi k}{N}\right)+z^{-2}} + \frac{H_0}{1-z^{-1}}\right]. \tag{21}$$

The recursive realization corresponding to (21) is shown in Fig. 5. Fig. 5(A) shows how the kth resonator section is realized. The difference equation relating the output of the resonator v_n to its input u_n is of the form

$$v_n = u_n + 2r\cos\left(\frac{2\pi k}{N}\right)v_{n-1} - r^2 v_{n-2}.$$

This resonator requires two multiplications per output point. Fig. 5(B) shows the recursive realization of (21) where each of the individual resonators is of the form shown at the top of the figure.

For realization of the filter of (21) by direct convolution, (2) is used to give the impulse response coefficients h_n. Because of the symmetry of the impulse response of (15), the direct convolution realization is of the form

$$y_n = \sum_{m=0}^{(N/2)-1} h_m[x_{n-m} + x_{n-N+1+m}]. \tag{22}$$

Thus, because of symmetry constraints, only $N/2$ multiplications are required to compute each sample of the output sequence.

N Odd: For the case when N is odd, the main design difference is that there is no frequency sample at half the sampling frequency. As seen from Fig. 3(B), the filter delay is the same as for N even, that is, $(N-1)/2$ samples, which in this case is an integer number of samples delay. Thus the derivation leading up to (21) is identical for N odd as for N even, with

[3] It should be noted that in this case and the corresponding Type 2 design, this constraint imposes a restriction on the design of highpass filters.

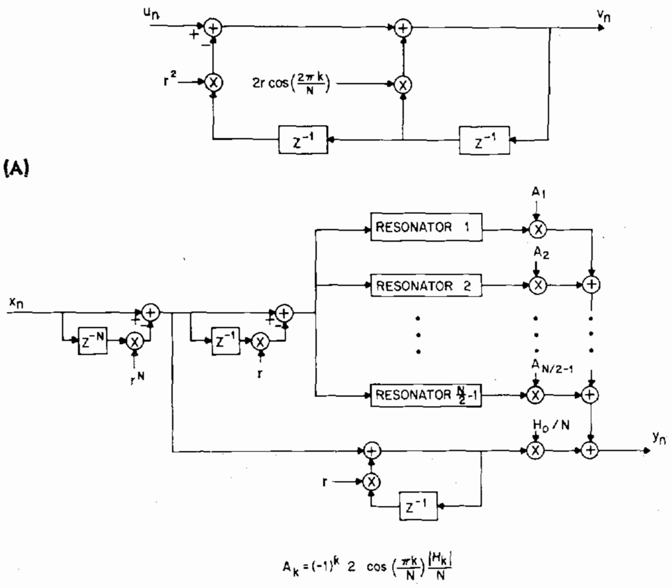

Fig. 5. The recursive realization of a Type 1, linear phase FIR filter. (A) kth resonator section. (B) Total realization.

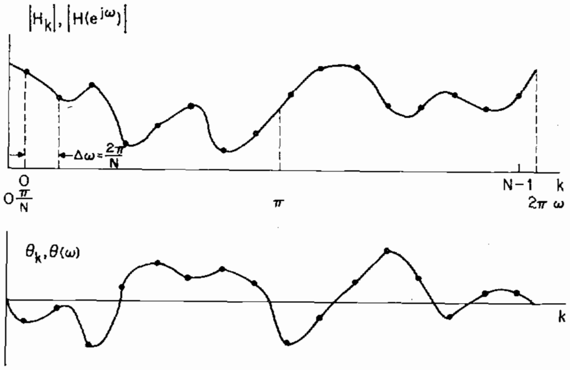

Fig. 6. The frequency samples (heavy points) and the desired continuous frequency response (solid curves) for a Type 2 FIR filter. Magnitude and phase curves are shown.

the exception that the upper index in the summation becomes $(N-1)/2$ instead of $(N/2)-1$. All comments about realization of the N even case apply to the N odd case. The realization by direct convolution is given by

$$y_n = h_{(N-1)/2} x_{n-(N-1)/2} + \sum_{k=0}^{(N-3)/2} h_k (x_{n-k} + x_{n-N+1+k}). \qquad (23)$$

Type 2 Frequency Sampling Designs

For the filters discussed above, the basic idea in design is that N zeros are spaced uniformly around the unit circle (the comb filter does this), beginning at the point $z=1$, and each of these zeros may be cancelled by a pole. The amplitude of the frequency response at each of the uniformly spaced frequencies is determined by the value of the frequency sample H_k. A second type of frequency sampling filter can be designed where the frequency response is specified at uniformly spaced points around the unit circle, beginning at the point $z = e^{j(\pi/N)}$. This corresponds to a frequency of $1/(2NT)$ Hz, where T is the sampling period. These designs were labeled as Type 2 designs, in contrast to the Type 1 designs discussed earlier [4].

Fig. 6 shows an arbitrary set of frequency response specifications for Type 2 designs. The solid curves show the desired continuous frequency response (magnitude and phase) and the circles show the set of frequency samples H_k which can be represented as

$$H_k = |H_k| e^{j\theta_k} = H(z)\Big|_{z = e^{j(2\pi/N)(k+1/2)}}, \quad k = 0, 1, \cdots, N-1 \qquad (24)$$

where $H(z)$ is the z transform of the FIR filter. Since the frequency samples do not align with the frequencies required by the IDFT to give the impulse response, we must rotate the array of frequency samples by an angle of $-\pi n/N$ to use the IDFT. This frequency rotation is equivalent to multiplying the impulse response coefficients by $(e^{-j\pi n/N})$. Thus the IDFT give the modulated impulse responses as

$$h_n e^{(-j\pi n/N)} = \sum_{k=0}^{N-1} H_k e^{j(2\pi/N)nk}, \quad n = 0, 1, \cdots, N-1. \qquad (25)$$

Multiplication of both sides of (25) by $e^{j(\pi n/N)}$ yields the impulse response coefficients as

$$h_n = \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{j(2\pi/N)n(k+1/2)}, \quad n = 0, 1, \cdots, N-1. \qquad (26)$$

Using the definition of the z transform (3a), we get

$$H(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} H_k e^{j(2\pi/N)n(k+1/2)} \right] z^{-n}, \qquad (27)$$

which, by interchanging summations and performing the summation over the n index, can be put in the form

$$H(z) = \frac{(1 + z^{-N})}{N} \sum_{k=0}^{N-1} \frac{H_k}{1 - z^{-1} e^{j(2\pi/N)(k+1/2)}}. \qquad (28)$$

Without making additional constraints on the frequency samples, or equivalently on the impulse response, (28) represents the most general case of the z transform for Type 2 designs. Realization techniques are similar to those discussed earlier for Type 1 designs. The only unusual point to keep in mind is that for direct convolution realization of the filter of

(28), (26) must be used to obtain the filter impulse response coefficients.

Real Impulse Response Coefficients

The necessary and sufficient conditions on the frequency samples required to obtain a real impulse response for Type 2 designs are

$$|H_k| = |H_{N-1-k}| \tag{29}$$

$$\theta_k = -\theta_{N-1-k}. \tag{30}$$

Assuming that the value of N is even (we will discuss the changes when N is odd later), and applying the constraints of (29) and (30), (28) becomes

$$H(z) = \frac{(1+z^{-N})}{N}\left[\sum_{k=0}^{(N/2)-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1}e^{j(2\pi/N)(k+1/2)}} + \sum_{k=N/2}^{N-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1}e^{j(2\pi/N)(k+1/2)}}\right]. \tag{31}$$

Making the substitution $k' = N-1-k$ in the second summation, and using (29) and (30), we obtain

$$H(z) = \frac{(1+z^{-N})}{N}\left[\sum_{k=0}^{(N/2)-1} \frac{|H_k| e^{j\theta_k}}{1 - z^{-1}e^{j(2\pi/N)(k+1/2)}} + \sum_{k'=0}^{(N/2)-1} \frac{|H_{k'}| e^{-j\theta_{k'}}}{1 - z^{-1}e^{-j(2\pi/N)(k'+1/2)}}\right]. \tag{32}$$

Combining terms in the first and second summations gives

$$H(z) = \frac{(1+z^{-N})}{N}\left[\sum_{k=0}^{(N/2)-1} \frac{2|H_k|\left[\cos(\theta_k) - z^{-1}\cos\left(\theta_k - \frac{2\pi}{N}\left(k+\frac{1}{2}\right)\right)\right]}{1 - 2z^{-1}\cos\left[\frac{2\pi}{N}\left(k+\frac{1}{2}\right)\right] + z^{-2}}\right]. \tag{33}$$

The filter described by (33) can be realized recursively as before, or by convolutional techniques after obtaining the filter impulse response coefficients. Because of the symmetry constraints on the frequency response, the impulse response coefficients can be solved for directly as

$$h_n = \frac{1}{N}\sum_{k=0}^{(N/2)-1} 2|H_k| \cos\left(\theta_k + \frac{2\pi}{N}n\left(k+\frac{1}{2}\right)\right). \tag{34}$$

For the case where N is odd, (33) is modified slightly. The upper index in the summation becomes $(N-3)/2$ instead of $(N/2)-1$, and the term

$$\frac{H_{(N-1)/2}}{1+z^{-1}}, \tag{35}$$

representing a frequency sample at $(z=-1)$, is included inside the brackets. The impulse response coefficient expression (34) is also modified by changing the upper summation limit to $(N-3)/2$, and by the addition of the term

$$\frac{1}{N}(-1)^n H_{(N-1)/2}, \qquad n = 0, 1, \cdots, N-1, \tag{36}$$

which represents the contribution of the frequency sample at half the sampling frequency to the impulse response.

Linear Phase Type 2 Filters

As discussed earlier for Type 1 designs, the additional constraints of linear phase on the frequency response implies that the filter impulse response be a real, symmetric sequence. However, the impulse responses of Fig. 3 for Type 1 designs for N even or odd cannot be obtained for Type 2 designs. This is because for N even, Type 1 designs, there was a frequency sample at half the sampling frequency that was always zero. This meant a phase discontinuity of π rad was necessary at half the sampling frequency, implying a noninteger delay in the impulse response. For Type 2 designs with N even, there is no frequency sample at half the sampling frequency. Thus for linear phase response, only integer delays are allowable for this case. Similarly for N odd, Type 1 designs, there was no frequency sample at half the sampling frequency, which implied an integer number of samples delay in the impulse response. However for N odd, Type 2 designs, the presence of a frequency sample at half the sampling frequency implies a half-integer number of samples delay in the impulse response. From (36) it is clear that when N is odd, $H_{(N-1)/2}=0$ to maintain symmetry. This means that the number of nonzero frequency samples will be even regardless of whether N is even or odd. This is opposite to the case of Type 1 designs where the number of nonzero frequency samples was always odd. By substitution of an appropriate linear phase in the right side of (26), it can be shown that the above constraint (i.e., there is always an even number of frequency samples) implies that one value of the impulse response h_n will be zero. Specifically, if the filter delay is set to $\tau=N/2$, then the first impulse response sample h_0 is zero, whereas if $\tau=(N/2)-1$, then the last impulse response sample h_{N-1} is zero.

Fig. 7 summarizes the possible impulse response shapes for Type 2 designs. When N is even, as shown in Fig. 7(A), the impulse response has the following properties.

1) There are $(N-1)$ nonzero values of the impulse response, i.e., $h_{N-1}=0$.
2) The impulse response is symmetric, i.e.,

$$h_n = h_{N-n-2}, \qquad n = 0, 1, \cdots, (N/2)-1. \tag{37}$$

3) The delay of the filter is $(N/2)-1$ samples, an integer number of samples delay.

When N is odd, as shown in Fig. 7(B), the impulse response has the following properties.

1) There are $(N-1)$ nonzero values of the impulse response, i.e., $h_{N-1}=0$.

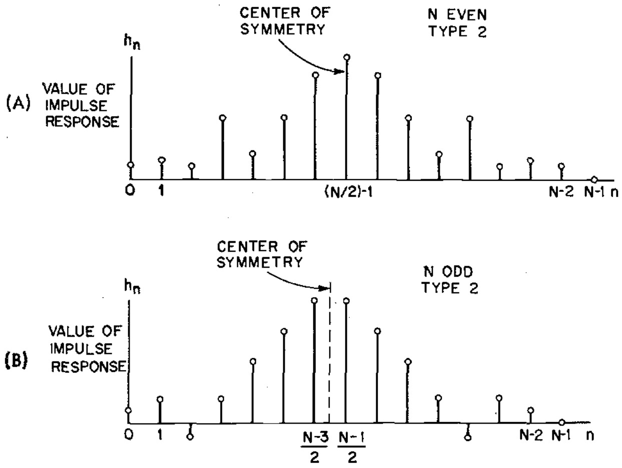

Fig. 7. The shape of the impulse response for Type 2, linear phase FIR filters. (A) Resulting impulse response when N is even. (B) Impulse response when N is odd.

2) The impulse response is symmetric, i.e.,

$$h_n = h_{N-n-2}, \qquad n = 0, 1, \cdots, (N-3)/2. \tag{38}$$

3) The delay of the filter is $(N/2)-1$ samples, a noninteger number of samples delay.

Next we derive expressions for the z transforms of linear phase filters for Type 2 designs for even and odd values of N.

N Even, Type 2 Designs: Fig. 8 shows the frequency response specifications for a linear phase filter Type 2 design when N is even. The constraints on phase are of the form

$$\theta_k = \begin{cases} \dfrac{-2\pi}{N}\left(k+\dfrac{1}{2}\right)\left(\dfrac{N}{2}-1\right), & k = 0, 1, \cdots, \dfrac{N}{2}-1 \\ \dfrac{2\pi}{N}\left(N-k-\dfrac{1}{2}\right)\left(\dfrac{N}{2}-1\right), & k = \dfrac{N}{2}, \cdots, N-1. \end{cases} \tag{39}$$

Substitution of the constraint of (39) into (33) gives the z transform as

$$H(z) = \frac{(1+z^{-N})}{N}\left[\sum_{k=0}^{(N/2)-1} \frac{2\,|H_k|\left[\cos\left[-\dfrac{2\pi}{N}\left(k+\dfrac{1}{2}\right)\left(\dfrac{N}{2}-1\right)\right] - z^{-1}\cos\left[-\pi(k+1/2)\right]\right]}{1-2z^{-1}\cos\left(\dfrac{2\pi}{N}(k+1/2)\right)+z^{-2}}\right], \tag{40}$$

which can be put in the form

$$H(z) = \frac{(1+z^{-N})}{N}\left[\sum_{k=0}^{(N/2)-1} \frac{2\,|H_k|\,(-1)^k \sin\left[\dfrac{2\pi}{N}(k+1/2)\right]}{1-2z^{-1}\cos\left(\dfrac{2\pi}{N}(k+1/2)\right)+z^{-2}}\right]. \tag{41}$$

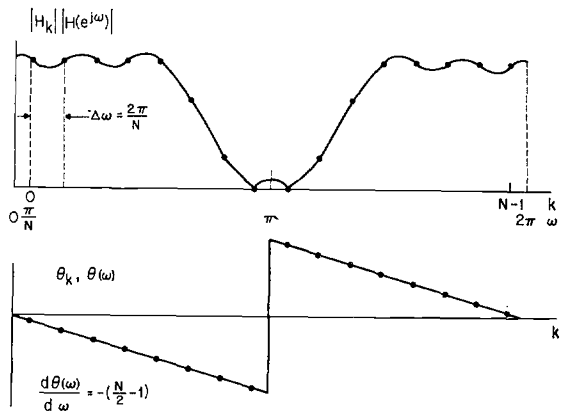

Fig. 8. The frequency samples (heavy points) and the desired continuous frequency response (solid curves) for a Type 2, linear phase FIR filter. Magnitude and phase curves are shown.

The recursive realization corresponding to (41) is shown in Fig. 9. Fig. 9(A) shows how the kth resonator section is realized. The difference equation of the resonator is of the form

$$v_n = u_n + 2r\cos\left[\frac{2\pi}{N}(k+1/2)\right]v_{n-1} - r^2 v_{n-2} \tag{42}$$

and requires only two multiplications per iteration. The full realization of (41) is shown in Fig. 9(B).

For direct convolution realization of the filter of (41), the realization is of the form

$$y_n = \sum_{m=0}^{(N/2)-1} h_m[x_{n-m} + x_{n-N+m+2}] + h_{N/2-1}x_{n-N/2-1} \tag{43}$$

where the impulse response coefficients are

$$h_n = \frac{1}{N}\sum_{k=0}^{(N/2)-1} 2\,|H_k|\,(-1)^k \cdot \sin\left[\frac{2\pi}{N}(n-1)(k+1/2)\right], \tag{44}$$

as seen by using the phase function of (39) in (34). It can be seen from (44) that h_{N-1} is indeed zero.

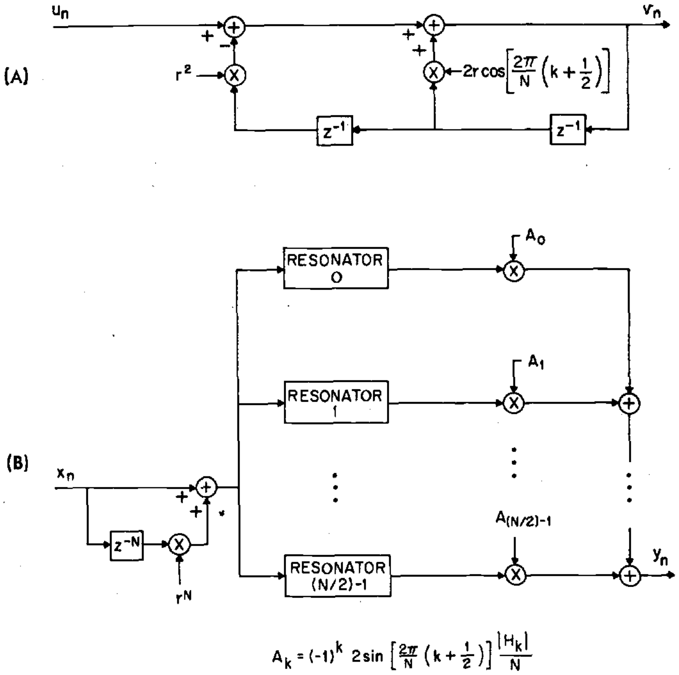

Fig. 9. The recursive realization of a Type 2, linear phase FIR filter. (A) *k*th resonator section. (B) Total realization.

N Odd, Type 2 Designs: The derivation leading up to (41)–(44) is identical for N odd as for N even, except that the upper index in the summations is $(N-3)/2$ in (41) and (44) and $(N-1)/2$ in (43), and the second term in (43) is absent. All comments about realization of the N even case apply equally well to the N odd case.

Conclusion

We have discussed recursive realizations of finite duration impulse response digital filters, focusing primarily on linear phase filters. Such filters can also be realized by direct convolution or by fast convolution [8], [9] techniques. The choice of which realization to use depends on many factors; however, the basic concern is generally computation speed. Since computation speed is determined largely by the number of multiplications and additions required in the realization, it is useful to compare the different realizations on this basis. Restricting ourselves to linear phase filters, (22), (23) and (43) indicate that direct convolution requires approximately N additions and $N/2$ multiplications for each output sample. In the recursive realization, if there are K nonzero complex conjugate pairs of frequency samples, then approximately $3K$ additions and $3K$ multiplications are required per output sample. Assuming that the multiplications require the most time, we see that the recursive realization will be faster as long as $K<N/6$.

This analysis, although not precise, indicates the circumstances in which the recursive realization may be preferred over direct convolution. Clearly, if N is large and K is small, as would be the case for narrowband lowpass, bandpass, or highpass filters designed by the frequency sampling technique, the recursive realization can be faster than direct convolution.

A third method of realizing FIR filters utilizes a fast Fourier transform (FFT) algorithm to achieve increased efficiency over direct convolution [8], [9]. Stockham [8] states that this technique is faster than direct convolution for values of N greater than 32. This implies that recursive realizations will be more efficient than the FFT method for values of K less than approximately 5 or 6.

In the high-speed convolution technique, the impulse response is augmented with a number of zero samples to obtain a sequence of length L which is a highly composite number. The DFT of this sequence is then computed with an appropriate FFT algorithm. It is interesting to note that when the impulse response has a unique peak, as for N odd, Type 1 design or N even, Type 2 design, then the augmented impulse response sequence can be rotated so that its discrete Fourier transform will be purely real. This results in considerable saving when the DFT of the impulse response is multiplied by the DFT of a segment of the input sequence. This comment, of course, applies to any FIR filter.

It is clear from the preceding discussion that there are numerous details to be concerned with in realizing finite impulse response digital filters. We have presented in considerable detail the case of linear phase frequency sampling designs; however, much of what was discussed applies to FIR filters designed by windowing or other optimization techniques.

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