at an odd (namely third) multiple of the first formant [11]. The effect of weighting is to reduce the effect of high-order Walsh coefficients.

## Summary

The recursive relationship between arithmetic and logical autocorrelation functions of a wide sense stationary process is derived. The Fourier (Walsh) power spectra are computed from the weighted arithmetic (logical) autocorrelation function by means of the fast Fourier (Walsh) transforms. Examples from speech and imagery data shows that the discrete Fourier and Walsh spectra closely resemble the spectral representation of these processes in terms of eigenvalues and eigenvectors of the covariance matrix.

## References

[1] J. E. Gibbs, "Walsh spectrometry, a form of spectral analysis well suited to binary digital computation," Nat. Phys. Lab., Teddington, Middlesex, England, unpublished rep., 1967.
[2] J. E. Gibbs and F. O. Pichler, "Comments on transformation of Fourier power spectra into Walsh power spectra," in *Proc. 1971 Symp. Applications of Walsh Functions*, pp. 51–54.
[3] J. E. Gibbs and H. A. Gebbie, "Application of Walsh functions to transform spectroscopy," *Nature*, vol. 224, pp. 1012–1013, Dec. 6, 1969.
[4] F. R. Pichler, "Some aspects of a theory of correlation with respect to Walsh harmonic analysis," Dept. Elec. Eng., Univ. Maryland, College Park, Tech. Res. Rep. R-70-11, Aug. 1970.
[5] W. D. Ray and R. M. Driver, "Further decomposition of the Karhunen–Loève series representation of a stationary random process," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 663–668, Nov. 1970.
[6] A. E. Kahveci and E. L. Hall, "Frequency domain design of sequency filters," in *Proc. 1972 Symp. Applications of Walsh Functions*, pp. 198–209.
[7] D. A. Pitassi, "Fast convolution using the Walsh transform," in *Proc. 1971 Symp. Applications of Walsh Functions*, pp. 130–133.
[8] E. R. Kretzmer, "Statistics of television signals," *Bell Syst. Tech. J.*, vol. 31, pp. 761–763, July 1952.
[9] L. E. Franks, "A model for the random video television signals," *Bell Syst. Tech. J.*, vol. 45, pp. 609–630, Apr. 1966.
[10] S. J. Campanella and G. S. Robinson, "A comparison of orthogonal transformations for digital speech processing," *IEEE Trans. Commun. Technol.*, vol. COM-19, Pt. I, pp. 1045–1050, Dec. 1971.
[11] ——, "Digital sequency decomposition of voice signals," in *Proc. 1970 Symp. Applications of Walsh Functions*, pp. 230–237.

# Linear Program Design of Finite Impulse Response (FIR) Digital Filters

LAWRENCE R. RABINER

*Abstract*—The use of optimization techniques for designing digital filters has become widespread in recent years. Among the techniques that have been used include steepest descent methods, conjugate gradient techniques, penalty function techniques, and polynomial interpolation procedures. The theory of linear programming offers many advantages for designing digital filters. The programs are easy to implement and yield solutions that are guaranteed to converge. There are many areas of finite impulse response (FIR) filter design where linear programming can be used conveniently. These include design of the following: filters of the frequency sampling type; optimal filters where the passband and stopband edge frequencies of the filter may be specified exactly; and filters with simultaneous constraints on the time and frequency response. The design method is illustrated by examples from each of these areas.

Manuscript received February 10, 1972. This paper was presented at the IEEE Arden House Workshop on Digital Filtering, Harriman, N. Y., January 1972.
The author is with Bell Telephone Laboratories, Inc., Murray Hill, N. J. 07974.

## Introduction

Many techniques exist for designing digital filters using optimization procedures. For example, Herrmann and Schuessler have designed equiripple error approximations to finite impulse response (FIR), low-pass, and bandpass filters using nonlinear programming procedures [1], [2]. This work has been extended by Hofstetter *et al.* [3] and by Parks and McClellan [4] to solve for the deisred filters using polynomial interpolation techniques. Rabiner *et al.* [5] used a steepest descent technique to obtain FIR digital filters with minimax error in selected bands with the constraint that only a few of the filter coefficients were variable. Steiglitz [6] and Athanasopoulos and Kaiser [7] have used nonlinear optimization techniques to obtain recursive filter approximations to arbitrary frequency response specifications.

Recently, attention has been focused on the use of linear programming techniques for the design of digital filters [8]–[10]. Many digital filter design problems are inherently linear in the design parameters and hence, are in a sense, natural candidates for linear programming optimization. Furthermore, linear programs are easy to implement and are generally guaranteed to converge to a unique solution. The rate of convergence of the programs is moderately fast, thus making this technique practical for problems with on the order of 100 parameters.

There are many areas of an FIR filter design where linear programming can be used conveniently. These include the design of the following: frequency sampling filters; optimal (minimax) absolute or relative error approximations to arbitrary frequency response characteristics, where the passband and stopband edge frequencies of the filter may be specified exactly; two-dimensional filters of the frequency sampling type or with optimal error approximation; and filters with simultaneous constraints on characteristics of both the time and frequency response of the filter. Several of these design areas have been examined and examples showing how to apply linear programming techniques in specific cases will be presented. In the next section the general framework of linear programming is presented and several practical aspects of linear programs are discussed. The following sections show how the general FIR linear-phase filter-design problem is linear in either the filter impulse-response coefficients, or equivalently the DFT coefficients, and how this problem is solved in specific cases.

#### Linear Programming

The general linear programming problem can be mathematically stated in the following form: find $\{x_j\} j=1, 2, \cdots, N$ subject to the constraints

$$x_j \geq 0, \qquad j = 1, 2, \cdots, N \tag{1}$$

$$\sum_{j=1}^{N} c_{ij} x_j = b_i, \qquad i = 1, 2, \cdots, M (M < N) \tag{2}$$

such that

$$\sum_{j=1}^{N} a_j x_j \tag{3}$$

is minimized.

The above problem is referred to as the "primal problem" and, by a duality principle, can be shown to be mathematically equivalent to the "dual problem:" find $\{y_i\} i=1, 2, \cdots, M$ subject to the constraints

$$\sum_{i=1}^{M} c_{ij} y_i \leq a_j, \qquad j = 1, 2, \cdots, N \tag{4}$$

such that

$$\sum_{i=1}^{M} b_i y_i \tag{5}$$

is maximized.

The remainder of this paper refers to the dual problem as this is the most natural form for the digital filter design problems being considered.

One characteristic of linear programs is that given there is a solution, it is guaranteed to be a unique solution, and these are several well-defined procedures for arriving at this solution within $(M+N)$ iterations. There are also straightforward techniques for determining if the solution is unconstrained or poorly constrained.

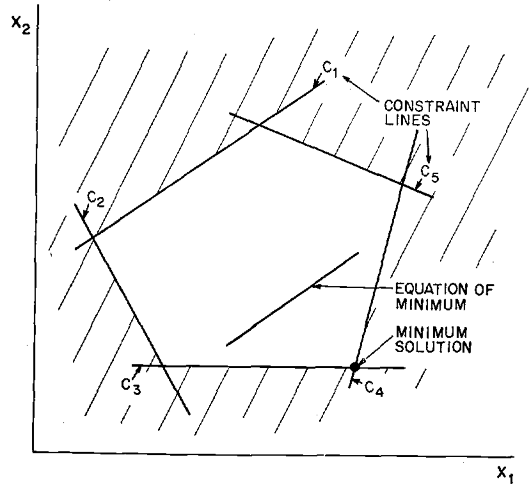


**Fig. 1.** Graphical explanation of the dual linear program.

A simple graphical interpretation of the linear program with two variables is shown in Fig. 1. Each constraint line ($C1$–$C5$) is a linear inequality in $x_1$ and $x_2$. Therefore, a straight line can be drawn representing the equivalent linear equality, and half of the solution space (shown by shaded lines) is eliminated as a possible solution. When all the constraint lines have been drawn, only a small region of the $x_1$, $x_2$ plane is still admissible as a solution to the minimization phase. The linear equation for finding the minimum is then plotted in the $x_1$, $x_2$ plane. By successively reducing the value for the minimum, the point where two of the constraint lines intersect becomes the solution, i.e., the absolute minimum consistent with the constraints. In the example of Fig. 1, the minimum occurs where constraint lines $C3$ and $C4$ interesect.

An important general property of the linear programming problem is that the constraint equations form a convex polyhedron, and that the minimum value of the desired linear function occurs at an extreme point of the polyhedron, i.e., where $M$ of the inequalities attain equality.

The next section shows that linear-phase FIR filters are linear in the design parameters, and hence, can be optimally designed using linear programming techniques.

#### Linear-Phase FIR Filters

Let $\{h_n\}, n=0, 1, \cdots, N-1$ be the impulse response of a causal FIR digital filter. The requirement of linear phase implies

$$h_n = h_{N-n-1}. \tag{6}$$

The filter frequency response can be determined in terms of the $\{h_n\}$ as

$$H(e^{j\omega T}) = \sum_{n=0}^{N-1} h_n e^{-j\omega T n}. \tag{7}$$

For the case where $N$ is odd, (7) can be combined with (6) to give

$$H(e^{j\omega T}) = \underbrace{e^{-j\omega(N/2)-1T}}_{\text{(linear-phase term)}} \left[ h_{(N-1/2} + \underbrace{\sum_{n=0}^{(N-3)/2} 2h_n \cos\left[((N-1)/2 - n)\omega T\right]}_{\text{(purely real, linear in } \{h_n\})} \right]. \tag{8}$$

Equation (8) shows $H(e^{j\omega T})$ to consist of a purely linear-phase term corresponding to a delay of $(N-1)/2$ samples and a term that is purely real and linear in the impulse-response coefficients. It is the second term in (8) that is used for approximating arbitrary magnitude response characteristics. For the case where $N$ is even, the result of (8) is modified to

$$H(e^{j\omega T}) = \underbrace{e^{-j\omega(N/2)-1T}}_{\text{(linear-phase term)}} \left[ \underbrace{\sum_{n=0}^{(N-1)/2} 2h_n \cos\left[((N-1)/2 - n)\omega T\right]}_{\text{(purely real, linear in } \{h_n\})} \right]. \tag{9}$$

Equation (9) shows that for $N$ even, the linear-phase term corresponds to a delay of an integer plus one half the number of samples. The center of symmetry of $\{h_n\}$ is midway between samples $(N/2)$ and $(N/2)-1$. The remainder of (9) is again a real term, which is linear in the impulse-response coefficients.

The DFT relation can be used to show that the filter frequency response is also a linear function of the DFT coefficients $\{H_k\}$. It is derived elsewhere [11] that the frequency response of linear-phase FIR filters can be written as

$$H(e^{j\omega T}) = e^{-j\omega T((N-1)/2)} \frac{\sin \frac{\omega NT}{2}}{N} \times \left[ \frac{H_0}{\sin \frac{\omega T}{2}} - \sum_{k=1}^{K} \frac{(-1)^k H_k 4 \cos \frac{\pi k}{N} \sin \frac{\omega T}{2}}{\left(\cos \omega T - \cos \frac{2\pi k}{N}\right)} \right] \tag{10}$$

where

$$K = \begin{cases} (N-1)/2, & N \text{ odd} \\ N/2 - 1, & N \text{ even.} \end{cases}$$

The significance of (10) is that the frequency response of a linear-phase FIR filter is linear in the $\{H_k\}$ as well as in the $\{h_n\}$; hence linear programming techniques can be used to optimize the values of all or a selected set of DFT or impulse-response coefficients. In the next sections we show how linear programming has been applied in several general cases.

### Design of Frequency Sampling Filters

Previously, design of frequency sampling filters was accomplished using a steepest descent minimization [5]. This technique was capable only of minimizing the peak out-of-band ripple when several DFT coefficients in a transition band between the passband and stopband were optimized. Another limitation of the technique was that the amount of computation it took to optimize the variable DFT coefficients grew exponentially with the number of unconstrained variables. Thus the largest problems attempted had four coefficients variable. This problem is readily solved in a much more general form using linear programming techniques. Furthermore, the computation time to get the more general solutions is considerably less than for the steepest descent algorithm used previously.

A typical specification for a low-pass filter to be approximated by a frequency sampling design is shown in Fig. 2. The heavy points show the DFT coefficients, and the solid curve shows the interpolated frequency response. The passband-edge frequency is $F_1$, and the stopband-edge frequency is $F_2$. Since the length of the filter impulse response is $N$ samples, there are $N$ DFT coefficients (called frequency samples) to be specified. Those DFT coefficients that are in the passband are arbitrarily assigned the value 1.0, and those that fall in the stopband are assigned the value 0.0. The DFT coefficients in the transition band are free variables and are labeled $T_1$, $T_2$ in Fig. 2. The approximation problem can be set up as a linear program in the following manner. Let $T_3$ equal the peak stopband ripple, then the design problem consists of finding values of $(T_1, T_2)$ to satisfy the following constraints.

1) The in-band ripple is less than or equal to some prescribed tolerance $\epsilon$.

2) The peak out-of-band ripple $T_3$ is to be minimized.

Mathematically this problem can be stated as follows: find $(T_1, T_2, T_3)$ subject to the constraints

$$1 - \epsilon \le F(\omega) + \sum_{i=1}^{2} T_i D(\omega, i) \le 1 + \epsilon, \qquad 0 \le \omega \le 2\pi F_1 \tag{11}$$

$$-T_3 \le F(\omega) + \sum_{i=1}^{2} T_i D(\omega, i) \le T_3, \qquad 2\pi F_2 \le \omega \le \pi F_s \tag{12}$$

where $F(\omega)$ is the contribution of the fixed DFT coefficients (the 1.0's in band), $D(\omega, i)$ is the contribution of the $i$th variable transition coefficient and is of the form shown in (10), and $F_s$ is the sampling frequency.

A suitable reshuffling of terms in (11) and (12) puts

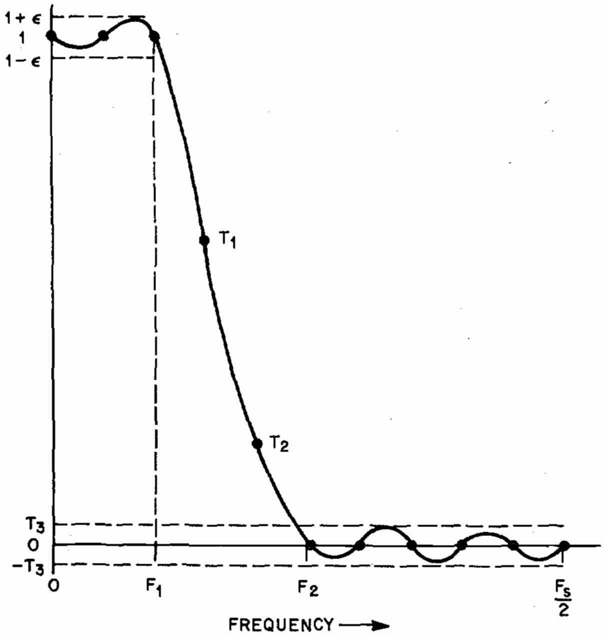


Fig. 2. Typical specification for a frequency sampling low-pass filter with transition coefficients $T_1$, $T_2$.

the set of equations in the form of the dual problem of linear programming. The final equations are of the following form: find $(T_1, T_2, T_3)$ subject to the constraints

$$\left.\begin{aligned} \sum_{i=1}^{2} T_i D(\omega, i) &\leq 1 + \epsilon - F(\omega) \\ -\sum_{i=1}^{2} T_i D(\omega, i) &\leq -1 + \epsilon + F(\omega) \end{aligned}\right\} 0 \leq \omega \leq 2\pi F_1 \quad (13)$$

$$\left.\begin{aligned} \sum_{i=1}^{2} T_i D(\omega, i) - T_3 &\leq -F(\omega) \\ -\sum_{i=1}^{2} T_i D(\omega, i) - T_3 &\leq F(\omega) \end{aligned}\right\} 2\pi F_2 \leq \omega \leq \pi F_s \quad (14)$$

where $(-T_3)$ is maximized.

The inequalities of (13) and (14) are evaluated at a suitably dense set of frequencies in the appropriate range of interest (an eight to one interpolation between DFT coefficients seems to be sufficient) to yield the necessary set of equations for the linear program.

## Results of Frequency Sampling Low-Pass Filters

A wide variety of frequency sampling low-pass filters have been designed using the results of (13) and (14). Previously, using the steepest descent algorithm, constraints on the in-band ripple $\epsilon$ could not be maintained [5]. With the linear programming design, tradeoff relations between in-band and out-of-band ripple can be obtained for fixed number of transition samples or equivalently fixed width of transition band. Typical tradeoff relations are illustrated in Fig. 3 for the case of three transition samples. In this figure the log of out-of-

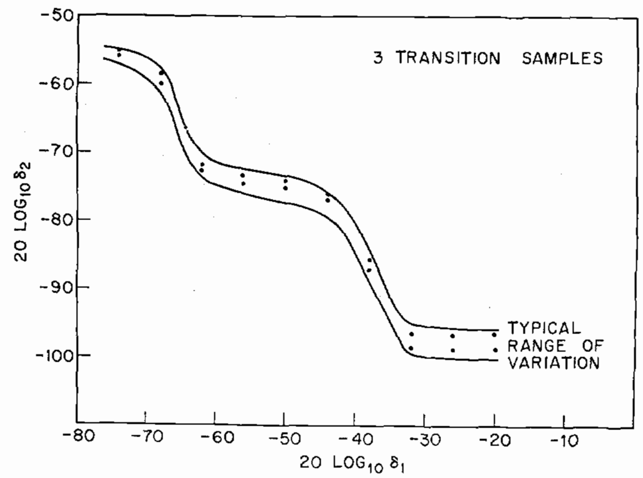


Fig. 3. Tradeoff relations between $\delta_1$ and $\delta_2$ for low-pass frequency sampling filters with three variable transition coefficients.

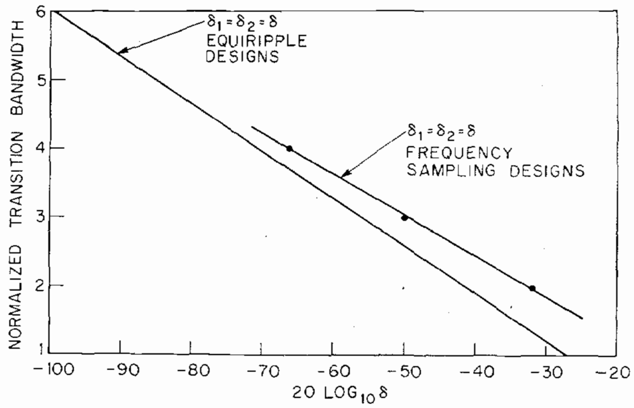


Fig. 4. Comparison between the curves of normalized transition bandwidth versus $\delta$ for equiripple filters and frequency sampling filters.

band ripple $\delta_2$ versus the log of in-band ripple $\delta_1$ is plotted.

The varying nature of the curves of Fig. 3 is due to the variance in the measured points (heavy dots) as a function of filter bandwidth. The solid curves in Fig. 3 show an estimated underbound and overbound on the typical behavior of the tradeoff relations. It is seen from Fig. 3 that the in-band-out-of-band ripple tradeoff is a highly nonlinear function for the frequency sampling filter.

Fig. 4 shows a comparison between equiripple filters and frequency sampling designs for the specialized case where the in-band ripple and out-of-band ripple are equal. In this figure the normalized width of the transition band[1] is plotted as a function of log $\delta$, where $\delta$ is the ripple. For the frequency sampling designs the data are at normalized transition widths of 4, 3, and 2, corresponding to 3, 2, and 1 transition samples. At these normalized transition bandwidths the ripple is −66,

[1] The normalized width of transition band is defined as $D = N \cdot (F_2 - F_1)/F_s$, where $N$ is the impulse-response duration in samples, $F_s$ is the sampling frequency and $F_1$ and $F_2$ are the passband and stopband edge frequencies.

$-50$, and $-32$ dB, respectively. The equiripple designs attain the same ripple values at transition bandwidths of approximately 3.7, 2.6, and 1.4. The percentage difference in bandwidth for the 3 cases is 8.1, 15.4, and 42.9. Thus, except for the 1 transition-point case, the transition bandwidths for frequency sampling designs are reasonably close to the bandwidths of equiripple filters.

### Design of Optimal Filters

Just as a few of the DFT coefficients in a transition band could be varied to design reasonably efficient frequency sampling filters all of the DFT coefficients, or equivalently, all the impulse-response coefficients could be varied to give an optimal[2] approximation to any desired frequency response. Such optimal approximations have been designed previously using nonlinear optimization procedures [1], [2] and by polynomial interpolation methods [3], [4]. However, the use of linear programming techniques, although significantly slower in running, offers many advantages over other existing design procedures. The design procedure is guaranteed to converge within a fixed number of iterations. Critical frequencies of the desired response can be specified exactly. The programs converge over a very wide range of parameter values. Finally, with the existence and increased understanding of integer linear programming techniques, one can combine the design problem with the coefficient quantization problem to design optimum filters with a prescribed word length.

To see how the design of optimal filters can be accomplished using linear programming techniques, it is simplest to use as an example the design of a low-pass filter. Consider the following set of specifications:

$$\left.\begin{array}{l}\text{stopband ripple } \pm\,\delta_2\\ \text{passband ripple } \pm\,\delta_1\end{array}\right\}\quad \text{minimized or specified}$$

$$\begin{array}{ll}\text{passband edge } F_1 & \text{specified}\\ \text{stopband edge } F_2 & \text{specified.}\end{array}$$

In this example either $\delta_1$, $\delta_2$ or some linear combination is minimized. One can also consider the situation where $\delta_1$ and $\delta_2$ are proportionally related, i.e., $\delta_1=k_1\delta$, $\delta_2=k_2\delta$ where $k_1$ and $k_2$ are constants, and $\delta$ is minimized. In this manner a constant ratio between passband and stopband ripple is maintained. By way of example we will consider the case where $\delta_1$ is specified, and $\delta_2$ is minimized. The linear program that realizes these specifications can be stated as follows: find $\{h_n\}$, $\delta_2$ subject to the constraints[3]

$$\left.\begin{array}{l} h_0+2\displaystyle\sum_{n=1}^{(N-1)/2} h_n\cos\omega nT\leq 1+\delta_1\\ -h_0-2\displaystyle\sum_{n=1}^{(N-1)/2} h_n\cos\omega nT\leq -1+\delta_1\end{array}\right\}\; 0\leq\omega\leq 2\pi F_1 \qquad (16)$$

$$\left.\begin{array}{l} h_0+2\displaystyle\sum_{n=1}^{(N-1)/2} h_n\cos\omega nT-\delta_2\leq 0\\ -h_0-2\displaystyle\sum_{n=1}^{(N-1)/2} h_n\cos\omega nT-\delta_2\leq 0\end{array}\right\}\; 2\pi F_2\leq\omega\leq \pi F_s \qquad (17)$$

where $(-\delta_2)$ is maximized.

Before proceeding to some typical designs, it is important to note some properties of linear programming problems and show how they affect the optimal filter design problem. The solution to a linear programming problem of the type shown above with $L$ variables and $M$ inequality constraints occurs when at least $L$ of the $M$ equations are solved with equality (instead of inequality), the remaining inequalities being met with inequality. For the optimal filter design problem this implies that there are at least $L$ frequencies at which the ripple obtains a maximum. The practical implications of the result are best illustrated with respect to Fig. 5, which shows the frequency response of an equiripple optimal filter with passband ripple $\delta_1$, stopband ripple $\delta_2$, passband-edge frequency $F_1$, and stopband-edge frequency $F_2$. The length of the filter impulse response is $N$ samples. If $N_p$ is the number of ripples in the passband and $N_s$ is the number of ripples in the stopband then

$$N_p+N_s\leq\frac{(N+1)}{2},\qquad N\text{ odd.} \qquad (18)$$

Equation (18) is due to the fact that an $N$th degree polynomial (the $z$ transform of the filter impulse response) has at most $(N+1)/2$ points of zero derivative in the frequency range from 0 to $F_s/2$ Hz. In addition to attaining a maximum value at each of the ripple frequencies, the error attains a maximum value at $f=F_1$ and at $f=F_2$, i.e., at the edges of the transition band. (In fact, this is how we define the transition-band edges.) Thus the number of error maxima $N_e$ satisfied the inequality

$$N_e\leq\frac{(N+1)}{2}+2. \qquad (19)$$

[2] The filters being discussed in this section are optimal in the sense of the theory of Chebyshev approximation on compact sets, i.e., the error of approximation exhibits at least $(N+1)/2+1$ alternations (of equal amplitude) over the frequency ranges of interest. In most cases, all the peaks of the error function are of the same amplitude. Therefore, these filters are often referred to as equiripple filters.

[3] From this point or for convenience we are assuming $\{h_n\}$ is defined from $-(N-1)/2\leq n\leq (N-1)/2$, and is symmetric around $n=0$. Since $N$ is odd (8) can be simplified to the form

$$H(e^{j\omega T})=h_0+\sum_{n=1}^{(N-1)/2} 2h_n\cos(\omega nT).$$

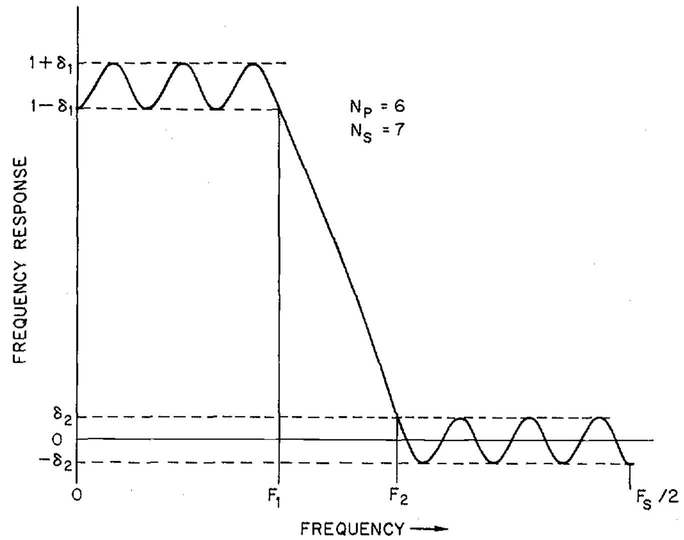


**Fig. 5.** **Typical frequency response for an optimal filter, defining $N_p$ as the number of passband maxima and $N_s$ as the number of stopband maxima.**

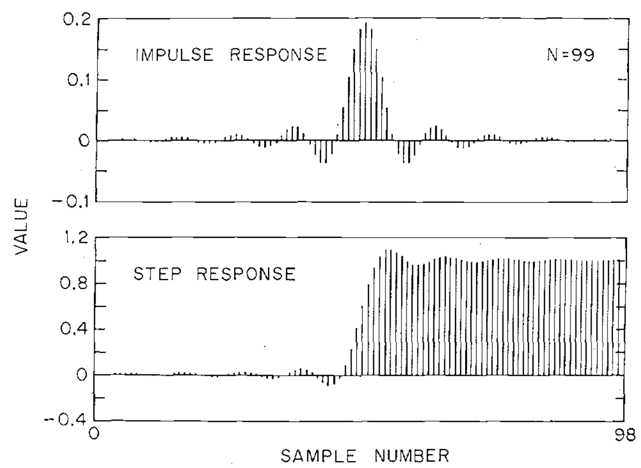


**Fig. 6.** **Impulse and step response for an optimal digital filter with 99-point impulse response.**

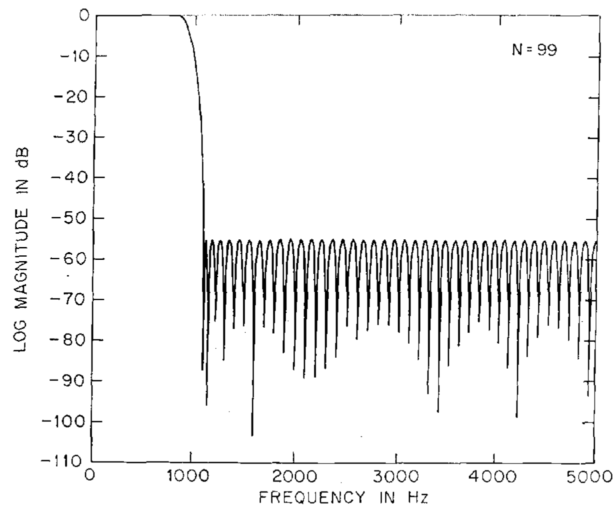


**Fig. 7.** **Frequency response of an optimal digital filter with 99-point impulse response.**

The number of variables $N_v$ in the linear program of (16) and (17) is

$$N_v = \frac{(N+1)}{2} + 1 \tag{20}$$

where $(N+1)/2$ coefficients of the impulse response are variable, and one ripple coefficient is variable. Thus (20) shows that the minimum number of error maxima from the linear program solution, although optimal, is one less than the maximum number of error maxima obtainable.[4] A discussion of the effects of the extra ripple peak on the width of the transition band is given by Hofstetter *et al.* [12]. For all practical purposes the loss of the extra ripple is negligible in terms of normalized transition bandwidth, etc. At this point it is worthwhile showing some results of the design procedure.

[4] Parks and McClellan [4] have labeled the cases where all the ripples are present as "extra ripple" designs.

## Optimal Filter Designs—Low-Pass Filter Examples

Using the linear program of (16) and (17) we have designed filters with impulse response durations of up to 99 samples. Figs. 6 and 7 show plots of impulse and step responses and the log magnitude response of a low-pass filter designed from the following specifications:

in-band ripple $\delta$,
out-of-band ripple $\delta$,

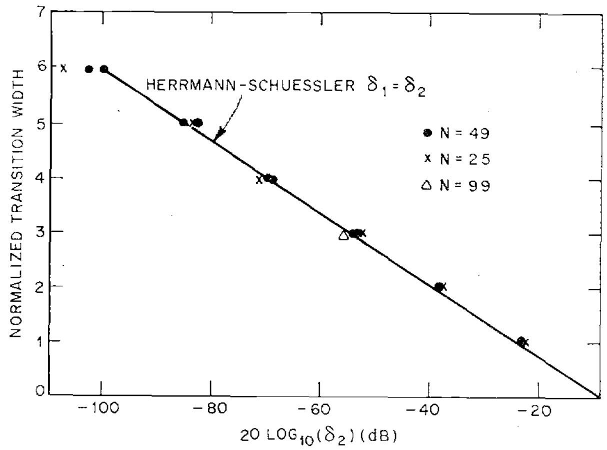


Fig. 8. Comparison between the curves of normalized transition bandwidth versus $\delta$ for equiripple filters with maximum number of ripples and optimal filters with one ripple omitted. Normalized bandwidth is defined as $D = N \cdot (F_2 - F_1)/F_s$.

passband-edge frequency 808 Hz,
stopband-edge frequency 1111 Hz,
sampling frequency 10 000 Hz.

The minimum value of $\delta$, as chosen by the linear program was $\delta = 0.001724$ or $-55.3$ dB.

Fig. 8 shows a comparison of the normalized transition bandwidth versus log $\delta_2$ for Herrmann–Schuessler equiripple filters with the maximum number of ripples and the optimal linear program filters with one ripple omitted. The solid line in this figure shows the Herrmann–Schuessler data for $\delta_1 = \delta_2$, and the data points show the linear program data for several values of $N$, the impulse-response duration. Clearly, the differences between the data are insignificant as stated earlier. (The data points that fall below the solid line in Fig. 8 are due to the error in representing the equiripple data by a straight line on these coordinates.)

## Optimal Filter Designs—Other Examples

As stated earlier, the linear programming technique can design optimal approximations to any desired frequency response. To illustrate this feature we have designed several full-band differentiators [13].

To design a full-band differentiator we require $H(e^{j\omega T})$ to approximate the normalized response

$$\hat{H}(e^{j\omega T}) = j \frac{\omega}{(\omega_s/2)} \tag{22}$$

where $(\omega_s/2)$ is half the radian sampling frequency. To get an optimal error approximation we require

$$-\delta \leq \left| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \right| \leq \delta \tag{23}$$

where $\delta$ is minimized. To get a purely imaginary frequency response as in (22), we require the impulse response to satisfy the symmetry condition

$$h_n = -h_{N-n-1}, \qquad n = 0, 1, \cdots, N/2 - 1 \tag{24}$$

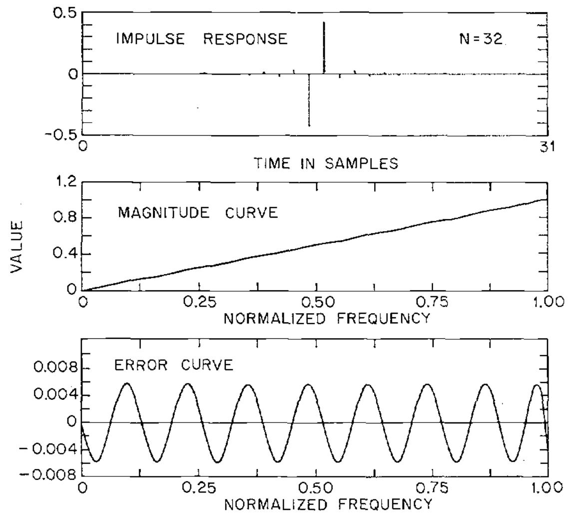


Fig. 9. Impulse response, frequency response, and error curve for a 32-point differentiator with optimal equiripple error.

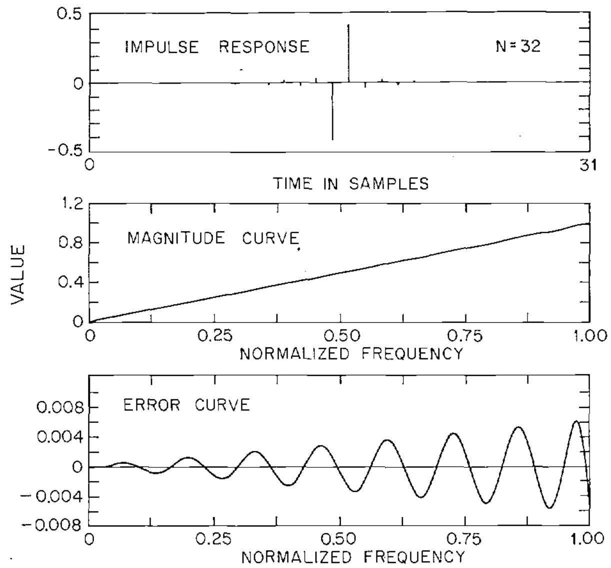


Fig. 10. Impulse response, frequency response, and error curve for a 32-point differentiator with optimal equiripple relative error.

where $N$ is even to take advantage of the half-sample delay [13]. An illustrative example of an $N = 32$-sample differentiator is shown in Fig. 9. In this figure is shown the impulse response, magnitude response, and the error curve. The peak error $\delta$ is approximately 0.0057.

One could also consider designing optimal relative error filters by changing the design equations slightly. For example, to design an optimal relative-error differentiator we require

$$-\delta\omega \leq \left| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \right| \leq \delta\omega, \tag{25}$$

i.e., the envelope of the error in approximation is linear with frequency because the desired frequency response is linear in frequency. An example of an $N = 32$-point differentiator designed in this manner is shown in Fig. 10. The peak error 6 is now 0.0062, only slightly higher than $\delta$ in the optimal solution. The linearity of the error envelope is evident in Fig. 10.

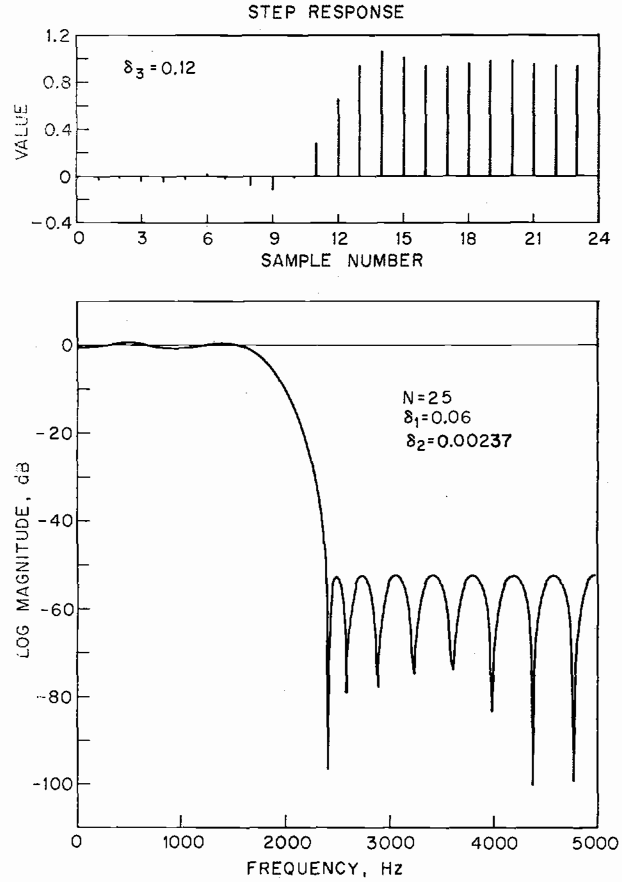


**Fig. 11.** Plot of step response and frequency response of an optimal equiripple low-pass filter.

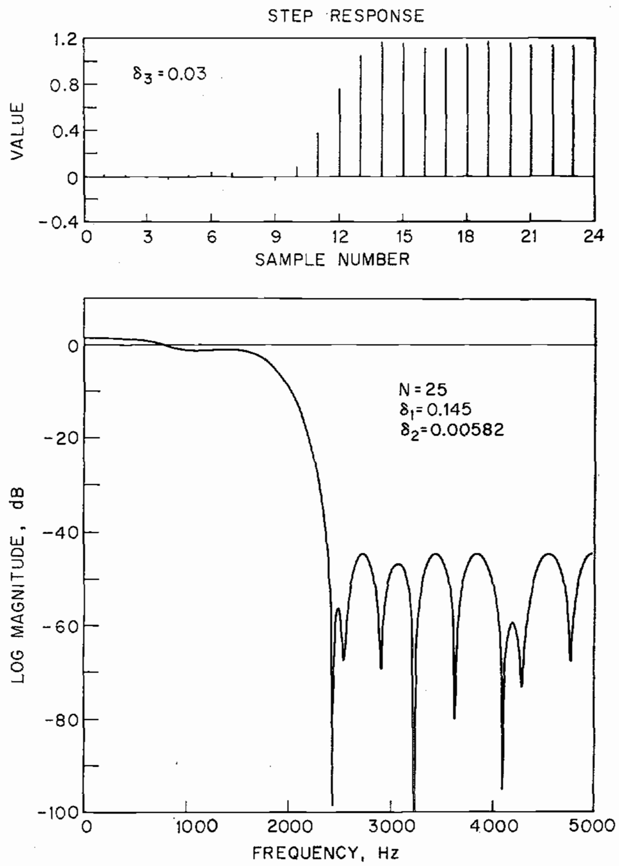


**Fig. 12.** Plot of the step response and frequency response for a low-pass filter with simultaneous constraints on both time and frequency response.

### Design of Filters with Simultaneous Constraints on the Time and Frequency Response

We have discussed design of digital filters that approximate characteristics of a specified frequency response only. Quite often one would like to impose simultaneous restrictions on both the time and frequency response of the filter. For example, in the design of low-pass filters, one would often like to limit the step response overshoot or ripple; at the same time maintaining some reasonable control over the frequency response of the filter. Since the step response is a linear function of the impulse-response coefficients, a linear program is capable of setting up constraints of the type discussed above. By way of example, we consider the design of a low-pass filter with the following specifications.

*Passband:*

$$1 - \delta_1 \leq h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos \omega nT \leq 1 + \delta_1. \quad (26)$$

*Stopband:*

$$-\delta_2 \leq h_0 + \sum_{n=1}^{(N-1)/2} 2h_n \cos \omega nT \leq \delta_2. \quad (27)$$

*Step Response:*

$$-\delta_3 \leq g_n \leq \delta_3$$
$$n = -(N-1)/2, \cdots, -(N-1)/2 + N \quad (28)$$

where $h_n$ is the symmetric impulse response of the filter ($h_n = h_{-n}$, where $n = 0, 1, \cdots, (N-1)/2$), $g_n$ is the filter step response defined by

$$g_n = \begin{cases} \sum_{m=-(N-1)/2}^{n} h_m, & -(N-1)/2 \leq n \leq \infty \\ 0, & n < -(N-1)/2 \end{cases}$$

and $N_1$ is the number of samples of the step response being constrained. For optimization there are several alternatives that are possible. One could fix any one or two of the parameters $\delta_1$, $\delta_2$, or $\delta_3$ and minimize the other(s). Alternatively one could set $\delta_1 = \alpha_1\delta$, $\delta_2 = \alpha_2\delta$, and $\delta_3 = \alpha_3\delta$, where $\alpha_1$, $\alpha_2$, and $\alpha_3$ are constants, and simultaneously minimize all three deltas.

To demonstrate this technique we have designed a low-pass filter with $N = 25$ and no constraint on $\delta_3$. This design is an optimal equiripple filter as discussed earlier and is shown in Fig. 11. In this case we have set $\delta_1 = 25\ \delta_2$ and we obtain $\delta_3 = 0.12$, $\delta_1 = 0.06$, and $\delta_2 = 0.00237$. The results of setting $\delta_3 = 0.03$ and then minimizing the frequency ripple are shown in Fig. 12. The equiripple character of the frequency response has been sacrificed in order to constrain the peak step response ripple. The ripple values for this new design are $\delta_1=0.145$ and $\delta_2=0.00582$. Using this linear programming technique one can obtain tradeoffs between any of the deltas to get a design best suited to the particular application. The filter of Fig. 12 was designed for smoothing characteristic speech parameters where step-response overshoot is a very important perceptual parameter.

## Design of Two-Dimensional FIR Filters

The techniques of FIR filter design using linear programming are readily extendable to two or more dimensions [14]. Both frequency sampling and optimal filters have been designed in this manner.

## Computational Considerations

Since one of the major aspects of digital filter design by optimization procedures is the amount of computation necessary to produce a desired result, it is worthwhile discussing some of the details of our simultations.

The programs we have used throughout this study are APMM [15], an IBM scientific subroutine that computes a Chebyshev approximation of a given real function over a discrete range, and MINLIN, a program written at Bell Laboratories by Mrs. W. Mammel. The running time of these programs is highly dependent on the number of variables $L$, the number of inequalities $P$, and the "complexity" of the results, which determines the number of iterations required to attain a solution. We have found that the time per iteration is proportional to $L^2P$. Our typical experience is that it takes on the order of 10 s to design the frequency sampling filters discussed earlier (i.e., $L\leq 3$, $P$ on the order of 1000). The total range of times to design optimal filters using APMM is shown below.

| $N$ | Number of Iterations | Total Time(s) |
|---|---|---|
| 25 | 27–58 | 14–47 |
| 49 | 53–82 | 117–194 |
| 99 | 128 | 1200 |

Thus, although the computation time is reasonably high, it is not impractical to design high-order filters with this technique. The argument can also be made that the most important application of these techniques is in the designs of FIR filters with small values of $N$ (i.e., $N\leq 50$) in which case the computation time starts becoming more reasonable.

## Conclusions

The design of linear-phase FIR digital filters is shown to be a linear programming problem, and many useful problems can be solved using this technique. Examples have illustrated several design areas that are reasonable candidates for linear program designs.

## Acknowledgment

The author would like to acknowledge the help given by Mrs. W. Mammel of Bell Telephone Laboratories, Holmdel, N. J., in converting her linear programming subroutine MINLIN to a form that was more useful for the work presented in this paper. The author would also like to acknowledge the useful and informative discussions on this topic with H. Helms of Bell Laboratories.

## References

[1] O. Herrmann and H. W. Schuessler, "On the design of selective nonrecursive digital filters," presented at the 1970 Arden House Workshop, Harriman, N. Y., Jan. 1970.
[2] O. Herrmann, "On the design of nonrecursive digital filters with linear phase," *Electron. Lett.*, vol. 6, no. 11, pp. 328–329, 1970.
[3] E. Hofstetter, A. Oppenheim, and J. Siegel, "A new technique for the design of nonrecursive digital filters," in *Proc. 5th Annu. Princeton Conf. Inform. Sci. and Syst.*, 1971, pp. 64–72.
[4] T. W. Parks and J. H. McClellan, "Chebyshev approximation for nonrecursive digital filters with linear phase," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 189–194, Mar. 1972.
[5] L. R. Rabiner, B. Gold, and C. A. McGonegal, "An approach to the approximation problem for nonrecursive digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 83–106, June 1970.
[6] K. Steiglitz, "Computer-aided design of recursive digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 123–129, June 1970.
[7] J. Athanasopoulos and J. F. Kaiser, "Constrained optimization problems in digital filter design," presented at the 1970 IEEE Arden House Workshop, Harriman, N. Y., Jan. 1970.
[8] H. D. Helms, "Digital filters with equiripple or minimax responses," *IEEE Trans. Audio Electroacoust.*, vol. AU-19, pp. 87–93, Mar. 1971.
[9] D. W. Tufts, D. W. Rorabacher, and W. E. Mosier, "Designing simple, effective digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 142–158, June 1970.
[10] R. K. Cavin, III, C. H. Ray, and V. T. Rhyne, "The design of optimal convolutional filters via linear programming," *IEEE Trans. Geosci. Electron.*, vol. GE-7, pp. 142–145, July 1969.
[11] L. R. Rabiner and R. W. Schafer, "Recursive and nonrecursive realizations of digital filters designed by frequency sampling techniques," *IEEE Trans. Audio Electroacoust.*, vol. AU-19, pp. 200–207, Sept. 1971.
[12] E. Hofstetter and A. Oppenheim, "On optimum nonrecursive digital filters," presented at the 9th Allerton Conf. Circuit and Syst. Theory, Monticello, Ill., Oct. 1971.
[13] L. R. Rabiner and K. Steiglitz, "The design of wide-band recursive and nonrecursive digital differentiators," *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 204–209, June 1970.
[14] J. V. Hu, "Frequency domain design of two-dimensional finite impulse response digital filters," in *Proc. Two-Dimensional Digital Signal Processing Conf.*, Univ. Missouri, Columbia, Oct. 1971.
[15] *IBM System/360 Programming Manual*, Sci. Subroutine Package (360 A-CM-03 X), Fortran Prog. APMM, pp. 283–288.