On the Design of Optimum FIR Low-Pass Filters with Even Impulse Response Duration

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Abstract—A great deal of attention has been given recently to the theory of designing optimum, linear phase, finite impulse response (FIR) low-pass digital filters where N, the filter impulse response duration, was odd. In this paper it is shown that the inclusion of optimal filters with even values of N gives additional flexibility to the general filter design problem. In particular, it will be shown that for certain ranges of filter cutoff frequencies, length N filters (N may be either even or odd) have smaller ripple than length $(N+1)$ filters. Finally, the general properties of optimal filters with even values of N are discussed. These include: filter types, scaling procedures, Chebyshev solutions, and symmetry of the basic design curves. The necessary modifications to existing design programs for filters with odd values of N to give filters with even values of N are also discussed.

Introduction

There have been a number of papers recently [1]-[8] that discuss the theory of optimal (in the minimax sense), linear phase, finite impulse response (FIR) low-pass digital filters. These papers have, for the most part, concentrated on filters with impulse response durations (N) that are odd. There is one good reason for this choice of odd values of N, and that is that linear phase filters with odd values of N have a delay that is an integral number of samples. Thus, one can create an equivalent unrealizable impulse response with zero delay by advancing the impulse response the appropriate number of samples. Linear phase filters with even values of N have a delay that is not an integral number of samples, and hence cannot be simply shifted to create a zero-phase sequence. The consequences of this nonintegral number of samples delay for N even are increased computational complexity in designing the actual filters, as well as difficulty in understanding the basic properties of the optimal solutions. For these reasons most investigators have chosen to neglect the case of even impulse response durations, while gaining a better understanding of the odd impulse response duration cases. In this paper we attempt to discuss theoretical and computational considerations in the design of optimal low-pass filters with even values of N. We show that, in many ways, the optimal solutions for even and odd values of N are similar; however, they do differ considerably in some ways.

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Before discussing the characteristics of optimal filters with even values of N, it is worthwhile summarizing the four possible cases of FIR filters with "linear phase." Let the impulse response of the filter be $\{h(n),\ n = 0, 1, 2, \cdots, N-1\}$ where N may be either odd or even. The frequency response of the filter is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}. \quad (1)$$

To achieve a linear phase response, $H(e^{j\omega})$ is required to be of the form

$$H(e^{j\omega}) = \pm \mid H(e^{j\omega}) \mid e^{-j\alpha\omega} \quad (2)$$

where α is a real positive constant with the physical significance of delay in samples. The factor $\pm$ in (2) is necessary, since $H(e^{j\omega})$ is actually of the form

$$H(e^{j\omega}) = H^*(e^{j\omega})\, e^{-j\alpha\omega} \quad (3)$$

where $H^*(e^{j\omega})$ is a real function taking on both positive and negative values. To achieve the constraints of (2) or (3) it is necessary and sufficient [9] that

$$h(n) = h(N-1-n), \qquad 0 \leqslant n \leqslant N-1 \quad (4)$$

in which case it is easily shown that

$$\alpha = \left(\frac{N-1}{2}\right). \quad (5)$$

When N is odd, the delay is always an integral number of samples. However, when N is even, the delay is not an integral number of samples, but instead has an extra delay of half a sample. In both cases, however, the frequency response is entirely real, to within a linear phase term.

The requirements of (2) are that the filter has both constant group delay and constant phase delay. In many cases we are content with only constant group delay—in which case we can define [10] a second case of a "linear phase" filter in which the phase of $H(e^{j\omega})$ is a piecewise linear function of ω, i.e.,

$$H(e^{j\omega}) = \pm \mid H(e^{j\omega}) \mid e^{j(\beta-\alpha\omega)}. \quad (6)$$

It is easy to show that the only possible solutions for $\beta \in [-\pi, \pi]$ are $\beta = \pm k\pi/2$, $k = 0, 1, 2$. If $\beta = 0, \pm\pi$ (6) is identical to (2). Thus the only new cases are when $\beta = \pm \pi/2$. In these cases, it is readily shown that the impulse response satisfies the condition

$$h(n) = -h(N-1-n), \qquad 0 \leqslant n \leqslant N-1 \quad (7)$$

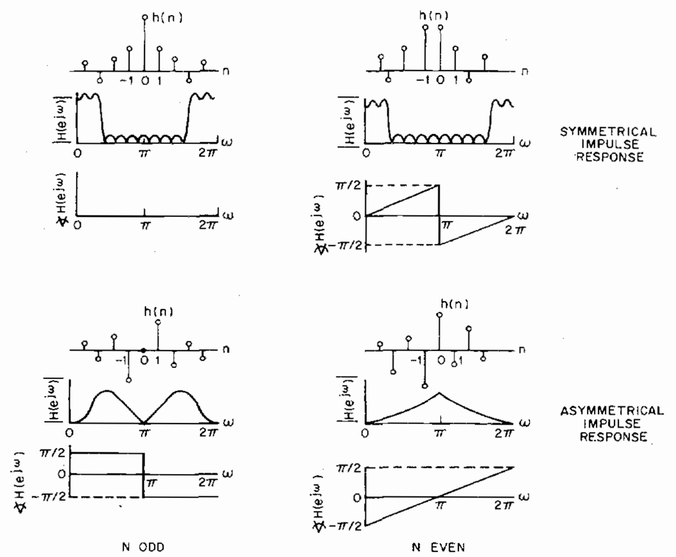

Fig. 1. Summary of the impulse and frequency responses of the four possible cases of a linear phase filter.

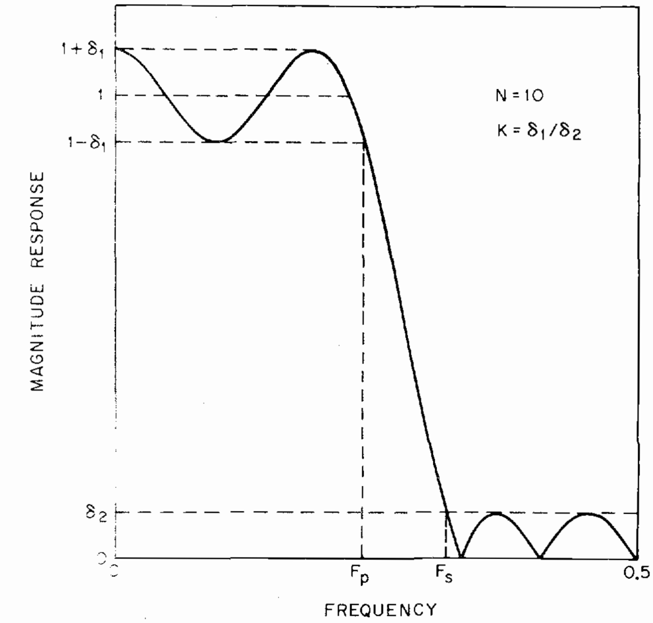

Fig. 2. Definition of optimal filter parameters.

and α again is $(N - 1)/2$. Thus the frequency response for these filters is seen to be purely imaginary, to within a linear phase, corresponding to a delay of $\alpha = (N - 1)/2$ samples. Application of the conditions of (6) and (7) has been made in the design of wide-band differentiators [11] and Hilbert transformers [12].

In summary, depending on the value of N (odd or even), and the symmetry of the impulse response sequence (symmetric or asymmetric) there are four possibilities for the frequency response. Fig. 1 summarizes these four cases by showing the four types of impulse response, and the corresponding frequency responses (both magnitude and phase). For the design of low-pass filters, researchers have concentrated on the case of N odd, and a symmetric impulse response. In this paper we discuss the case of N even, and a symmetric impulse response. The remaining two cases are not applicable to low-pass filters, since for these cases the frequency response is constrained to be zero at $\omega = 0$—and this is clearly unacceptable for a low-pass filter.

Frequency Response for *N* Even

If we impose the symmetry condition of (4) into (1), and shift $h(n)$ to the left by $N/2$ samples, we get a new sequence $\tilde{h}(n)$ defined from $-(N/2)$ to $N/2 - 1$ with the symmetry

$$\tilde{h}(n) = \tilde{h}(-n-1), \qquad n = 0, 1, \cdots, \frac{N}{2} - 1. \qquad (8)$$

The resulting frequency response is therefore

$$\tilde{H}(e^{j\omega}) = e^{j\omega/2} \sum_{n=0}^{(N/2)-1} 2\tilde{h}(n) \cos\left[\omega\left(n + \tfrac{1}{2}\right)\right] \qquad (9)$$

which consists of a linear phase term $(e^{j\omega/2})$ equivalent to one-half a sample advance, and a term which is purely real. If we define $\overline{H}(e^{j\omega})$ as

$$\overline{H}(e^{j\omega}) = \tilde{H}(e^{j\omega})e^{-j\omega/2} = \sum_{n=0}^{(N/2)-1} 2\tilde{h}(n) \cos\left[\omega\left(n + \tfrac{1}{2}\right)\right] \qquad (10)$$

we see that $\overline{H}(e^{j\omega})$ is purely real and can be used for design purposes since the term $e^{j\omega/2}$ does not affect the magnitude response of the filter. It should be noted that one important characteristic of linear phase filters with even values of N is that

$$H(e^{j\pi}) = 0 \qquad (11)$$

independent of $\{h(n)\}$, i.e., the filter has a zero at $z = -1$. We will see later that the constraint of (11) limits the range of low-pass filters that can be designed.

The optimal low-pass filter problem can be stated as a weighted Chebyshev approximation problem by defining B_p and stopband B_s as

$$B_p = \{f, \quad 0 \leqslant f \leqslant F_p\}$$
$$B_s = \{f, \quad F_s \leqslant f \leqslant 0.5\}. \qquad (12)$$

The desired frequency response of the low-pass filter is

$$D(e^{j\pi f}) = \begin{cases} 1, & f \in B_p \\ 0, & f \in B_s. \end{cases} \qquad (13)$$

The weighting function

$$W(e^{j2\pi f}) = \begin{cases} 1/K = \delta_2/\delta_1, & f \in B_p \\ 1, & f \in B_s \end{cases} \qquad (14)$$

allows the designer freedom to specify the relative magnitude of the error in the passband and stopband. Fig. 2 shows the frequency response of a typical low-pass filter.

Defining F as the union of the passband and stopband, i.e.,

$$F = B_p \cup B_s \tag{15}$$

the optimal filter design problem becomes one of finding the set of $\tilde{h}(n)$ which minimizes

$$\max_{f \in F} W(e^{j2\pi f}) \cdot |D(e^{j2\pi f}) - \bar{H}(e^{j2\pi f})|. \tag{16}$$

The problem of (16) can be solved directly, using linear programming techniques [5], or using a Remez multiple exchange algorithm [4]. For the linear programming solution, the set of equations implied by (16) is stated explicitly as

$$1 - K\delta_2 \leqslant \sum_{n=0}^{(N/2)-1} 2\tilde{h}(n) \cos [\omega(n + \tfrac{1}{2})] \leqslant 1 + K\delta_2, \quad 0 \leqslant \omega \leqslant 2\pi F_p \tag{17}$$

$$-\delta_2 \leqslant \sum_{n=0}^{(N/2)-1} 2\tilde{h}(n) \cos [\omega(n + \tfrac{1}{2})] \leqslant \delta_2, \quad 2\pi F_s \leqslant \omega \leqslant \pi$$

$$\delta_2 \text{ minimized} \tag{18}$$

where δ_2 is the maximum stopband deviation, and $\delta_1 = K\delta_2$ is the maximum passband deviation (see Fig. 2).

For solution of (16) by the Remez technique, the trigonometric polynomial of (10) is generally converted to a real polynomial by the substitution [3], [4]

$$x = \cos \omega \tag{19}$$

which, after considerable manipulation (see the Appendix), gives

$$\tilde{H}(x) = \tilde{H}(e^{j\omega}) \big|_{x=\cos\omega} = \sqrt{1+x} \sum_{n=0}^{(N/2)-1} b(n)x^n. \tag{20}$$

The sequence $b(n)$ is straightforwardly related to the sequence $\tilde{h}(n)$ through trigonometric identities. Equation (20) shows $\tilde{H}(x)$ to be the product of a polynomial in x, weighted by the function $\sqrt{1+x}$—hence $\tilde{H}(x)$ is *not* a polynomial in x. Therefore, many of the simple properties of the solution by Remez techniques cannot be used. There is a straightforward way of handling this difficulty. If we define $P(x)$, the polynomial part of (20), as

$$P(x) = \sum_{n=0}^{(N/2)-1} b(n)x^n \tag{21}$$

we can express the design constraints [(16) converted to x] as

$$\begin{aligned} 1 - K\delta_2 \leqslant P(x)\sqrt{1+x} \leqslant 1 + K\delta_2, &\quad X_p \leqslant x \leqslant 1 \\ -\delta_2 \leqslant P(x)\sqrt{1+x} \leqslant \delta_2, &\quad -1 \leqslant x \leqslant X_s \end{aligned} \tag{22}$$

where $X_p = \cos(2\pi F_p)$, and $X_s = \cos(2\pi F_s)$. If we divide all parts of the inequalities of (22) by $\sqrt{1+x}$ we get the revised equations

$$\begin{aligned} \frac{1 - K\delta_2}{\sqrt{1+x}} \leqslant P(x) \leqslant \frac{1 + K\delta_2}{\sqrt{1+x}}, &\quad X_p \leqslant x \leqslant 1 \\ \frac{-\delta_2}{\sqrt{1+x}} \leqslant P(x) \leqslant \frac{\delta_2}{\sqrt{1+x}}, &\quad -1 \leqslant x \leqslant X_s. \end{aligned} \tag{23}$$

This revised set of equations may be thought of as having changed the desired frequency response (13) to

$$D(e^{j2\pi f}) = \begin{cases} \dfrac{1}{\sqrt{1+\cos 2\pi f}}, & f \in B_p \\ 0, & f \in B_s \end{cases} \tag{24}$$

and the desired weighting function [(14)] to

$$W(e^{j2\pi f}) = \begin{cases} \dfrac{\sqrt{1+\cos 2\pi f}}{\mathrm{K}}, & f \in B_p \\ \sqrt{1+\cos 2\pi f}, & f \in B_s. \end{cases} \tag{25}$$

The polynomial constraint equations (23) can be solved by Remez procedures in much the same way as they were in the case of odd values of N [3], [4]. The procedure of Parks and McClellan [4] is presently being modified for these cases. The results presented in the following sections were derived using the linear programming approach.

Optimality Criteria

The basic optimality criteria given by Parks and McClellan [4] must be modified to include even values of N. In this case (N even) their first theorem can be stated as follows.

Theorem 1: Let F be any closed subset of $[0, \frac{1}{2} - \epsilon]$ where ϵ can be made arbitrarily small.[1] In order that

$$\bar{H}(e^{j2\pi f}) = \sum_{n=0}^{(N/2)-1} 2\tilde{h}(n) \cos [2\pi f(n + \tfrac{1}{2})]$$

be the unique best approximation on F to $D(e^{j2\pi f})$, it is necessary and sufficient that the error function

$$E(e^{j2\pi f}) = W(e^{j2\pi f}) \cdot [D(e^{j2\pi f}) - H(e^{j2\pi f})]$$

[1] The closed subset cannot include the point $f = 0.5$ because at this point the set of vectors $\{\cos[2\pi f(n + \frac{1}{2}]\}$, $0 \leqslant n \leqslant (N/2) - 1$ is not independent and thus does not satisfy the Haar condition.

exhibit on F at least $(N/2 + 1)$ alternations. Thus

$$E(e^{j2\pi F_i}) = -E(e^{j2\pi F_{i+1}}) = \pm \| E \|$$

with

$$F_0 < F_1 \cdots < F_{N/2}$$

and

$$F_i \in F.$$

Here

$$\| E \| = \max_{f \in F} \left| E(e^{j2\pi f}) \right|.$$

For N even, the second theorem of Parks and McClellan can be stated as follows.

Theorem 2: For a filter of length N samples (the approximation is being done with $N/2$ functions) the resulting error curve must exhibit $(N/2 + 1)$ or $(N/2 + 2)$ peaks. Two of the peaks are located at F_p and F_s. The point $f = 0$ is guaranteed to be a peak if there are $(N/2 + 2)$ peaks; otherwise it may or may not be a peak. The point $f = 0.5$ is never a peak.

The proofs of the above theorems are given by trivial extensions of those of Parks and McClellan, and hence will not be reiterated. Instead, the implications of these theorems will be discussed. An important result of the above theorems is that the optimal filter for N even may be an extraripple filter with $(N/2 + 2)$ peaks in the error curve), or an equiripple filter with $N/2 + 1$ equal magnitude peaks in the error function. We will see later that a restricted form of scaled extraripple filters also exists as an optimal solution. Fig. 3(a) shows the magnitude response of a typical extraripple filter ($N = 10$) with $K = 1$, $F_p = 0.3426$, $F_s = 0.41623$. As predicted by the theorems, there are $(10/2 + 2) = 7$ points at which the error function is a maximum. In contrast, Fig. 3(b) shows the magnitude response of an equiripple filter ($N = 10$, $K = 1$, $F_p = 0.155$, $F_s = 0.2445$) where there are six points at which the error function is a maximum.

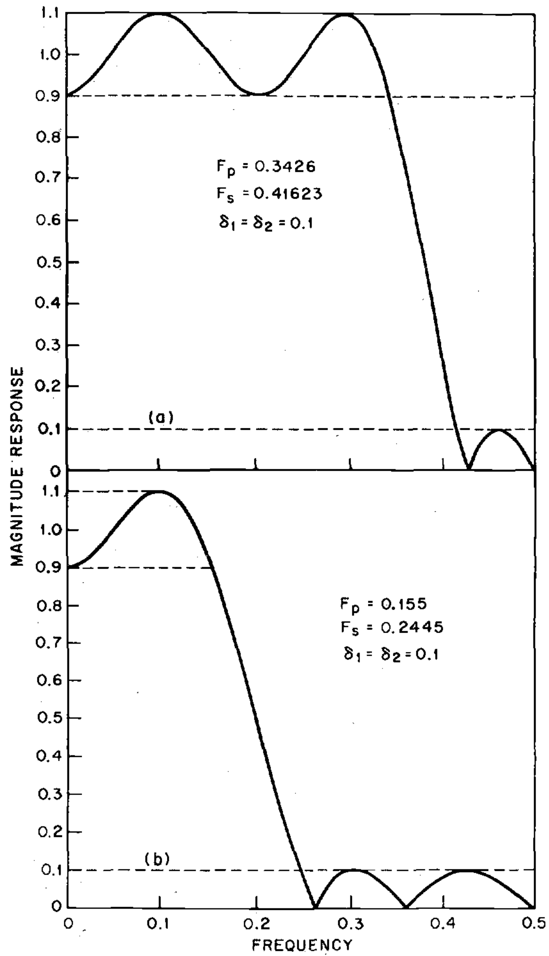

Fig. 3. (a) The magnitude response of an extraripple filter ($N = 10$). (b) The magnitude response of an equiripple filter with one less than the maximum number of ripples.

Comparisons Between Even and Odd N

It is relatively easy to show that, if we restrict ourselves to either odd or even values of N, an optimal filter with impulse response duration of $(N - 2)$ samples *cannot* achieve better specifications (i.e., smaller peak error) than an optimal filter with impulse response duration of N samples. This is clear since the set of filters with impulse response duration of $(N - 2)$ samples is a subset of the set of filters with impulse response duration of N samples. Thus an optimal member of the subset of a larger set cannot be better than the optimal member of the larger set. However, the above argument is not valid when one compares optimal filters with impulse response duration of N samples, with filters with impulse response durations of $(N - 1)$ samples. *A priori* one *cannot predict* which filter can achieve better specifications.

To illustrate the above argument, Fig. 4 shows a plot of the curves of transition width ($\Delta F = F_s - F_p$) versus F_p for lengths 9, 10, and 11 filters where $\delta_1 = \delta_2 = 0.1$ in all cases. From this figure several observations can be made.

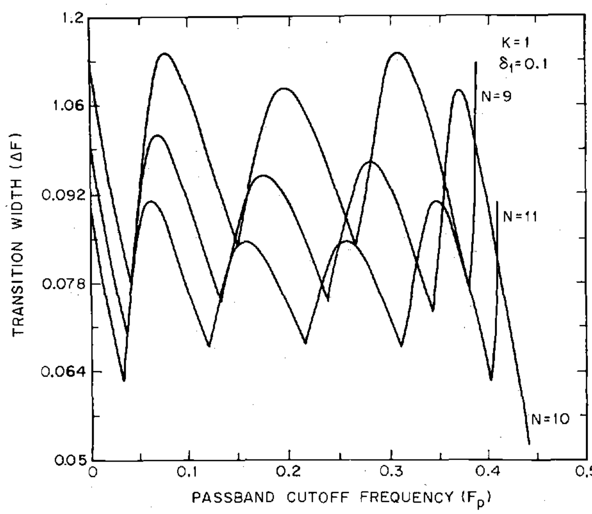

Fig. 4. Transition width versus passband cutoff frequency for even and odd values of N for optimal filters with $K = 1$.

Observation 1: The transition width for $N = 10$ filters is sometimes smaller than for $N = 11$ filters, and

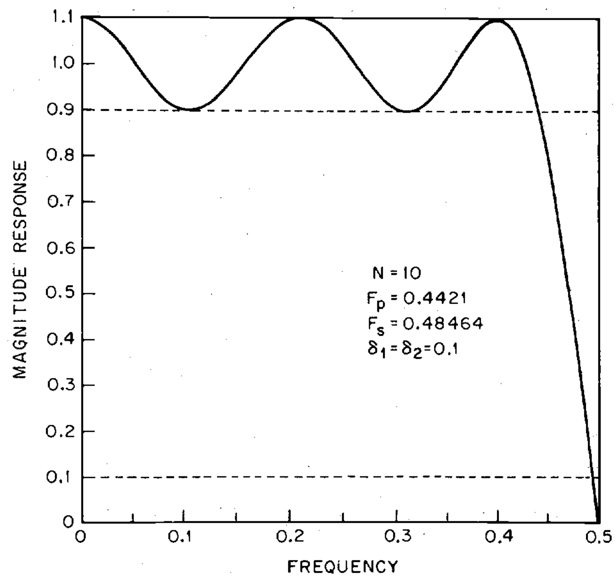

Fig. 5. The magnitude response of an extraripple filter with one ripple in the stopband.

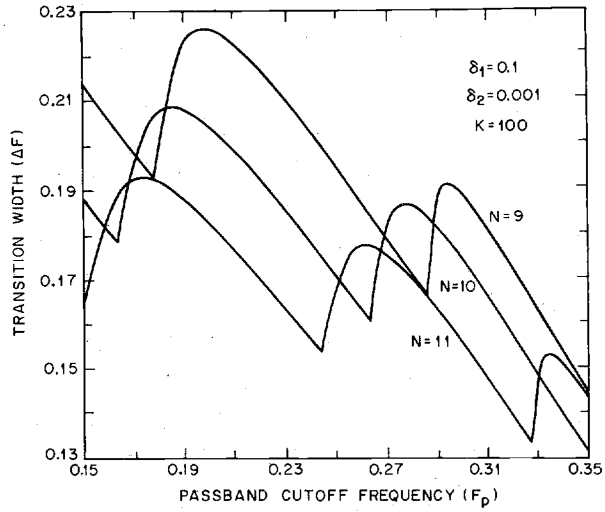

Fig. 6. Transition width versus passband cutoff frequency for even and odd values of N for optimal filters with K = 100.

is sometimes larger than for $N = 9$ filters with the same values of F_p.

Observation 2: There is no symmetry in the curve of ΔF versus F_p for $N = 10$, although the curves for $N = 9$ and $N = 11$ are symmetrical in that for every point on the curve of F versus F_p with coordinates $(\tilde{F}_p, \Delta\tilde{F} = \tilde{F}_s - \tilde{F}_p)$ there is a symmetrical point with coordinates $(0.5 - \tilde{F}_s, \Delta\tilde{F})$.

Observation 3: The curve of ΔF versus F_p for $N = 10$ ends at an extraripple solution.

The explanation for Observations 2 and 3 is related to the behavior of the optimal solutions at $f = 0.5$ for N even. At this point $\tilde{H}(e^{j\pi}) = 0$, i.e., the error function does *not* have a peak as it does in the case of N odd. Thus a simple transformation of variables [6] does not yield an optimal filter from an optimal filter, as for N odd. Therefore, no simple symmetry in the curve of ΔF versus F_p is maintained. The explanation for Observation 3 is given in Fig. 5, which shows the magnitude response of the last extraripple filter. Because of the constraint of a zero at $f = 0.5$, one cannot obtain filters with values of F_s arbitrarily close to 0.5, as for N odd.

The importance of Observation 1 should not be underestimated. It is an unexpected and surprising result that a filter with $N = 10$ (i.e., approximation using five functions) can achieve given ripple specifications with a smaller transition width than a filter with $N = 11$ (i.e., approximation using six functions). Stated in a slightly different way, given fixed values of F_p, F_s, and K, a length 10 filter can achieve smaller ripple than a length 11 filter. For example, for the case $F_p = 0.3426$, $F_s = 0.41623$, $K = 1$, the length 11 filter achieves $\delta_1 = \delta_2 = 0.128215$, whereas the length 10 filter achieves $\delta_1 = \delta_2 = 0.1$. Expressed as a logarithm, the stopband attenuation of the length 10 filter is approximately 2.2 dB better than for the equivalent length 11 filter.

Fig. 6 shows a plot of ΔF versus F_p for $N = 9, 10$, and 11 with $K = 100$ ($\delta_1 = 0.1$, $\delta_2 = 0.001$). This figure shows F_p in the range $0.15 \leq F_p \leq 0.35$. The behavior of the curve of transition width versus F_p is similar to that of Fig. 4 in that the $N = 10$ solutions sometimes have smaller transition widths than the $N = 11$ solutions, and sometimes have larger transition widths than the $N = 9$ solutions.

Scaled Extraripple Filters

It has been shown previously [7] that one can use a simple scaling procedure to obtain certain optimal solutions from the extraripple solutions when N is odd. These scaling procedures only partially extend to the N even case. Consider the filter response $\tilde{H}(x)$ of (20), which was defined as

$$\tilde{H}(x) = \sqrt{1+x}\, P(x), \qquad -1 \leq x \leq 1 \tag{26}$$

where $P(x)$ is a polynomial in x of the form

$$\sum_{n=0}^{(N/2)-1} b(n)x^n.$$

If we make the substitution

$$x = \alpha x' + \beta \tag{27}$$

into (26) we get

$$\tilde{H}(x') = \sqrt{1 + \alpha x' + \beta} \sum_{n=0}^{(N/2)-1} b(n)(\alpha x' + \beta)^n \tag{28}$$

$$= \sqrt{1+\beta} \sqrt{1 + \frac{\alpha x'}{1+\beta}} \sum_{n=0}^{(N/2)-1} b(n)(\alpha x' + \beta)^n. \tag{29}$$

Since we require the resulting filter to be of the correct form, i.e.,

$$\tilde{H}(x') = \sqrt{1 + x'} \sum_{n=0}^{(N/2)-1} c(n)\,(x')^n, \qquad (30)$$

this imposes the constraint

$$\alpha = 1 + \beta. \qquad (31)$$

Using this result either α, or equivalently β, is as yet unspecified. The constraint of (31) guarantees that at $x' = -1$, $\tilde{H}(x') = 0$. One is now free to map the arbitrary point $x = X_1$ to $x' = +1$. If X_1 is between X_L (the point nearest to $x = 1$ where $\tilde{H}(x) = 1 \pm \delta_1$) and + 1, the resulting filter is an optimal filter. If we impose the constraint that the point $x = X_1$ (arbitrary) is mapped to $x' = 1$ then

$$\beta = \frac{X_1 - 1}{2}. \qquad (32)$$

Optimal filters obtained by the mapping of (27) have been shown to have an error peak at $f = 0$ that (except in the extremes of scaling) is not equal to the error peaks at all other error extrema. These filters have been called scaled extraripple filters [7]. Additional details on these solutions are available in [7].

Although (27) shows how to scale in the vicinity of $f = 0$, there is no readily obvious way to scale in the vicinity of $f = 0.5$ since $\tilde{H}(e^{j\pi})$ is constrained to be 0 at this point. Thus a simple linear scaling cannot be applied. The experimental evidence is that there is some form of scaling which is occurring, but no insight into its exact form has been attained.

The sequence of filters in Fig. 7(a)–(e) shows the magnitude response of five of the filters of Fig. 4. Fig. 7(a) shows a scaled extraripple filter response where the error at $f = 0$ is about 0.02, whereas the error at all other peaks is 0.1. Fig. 7(b) shows the extraripple filter response from which the response of Fig. 7(a) was obtained. Fig. 7(c) shows an optimal filter response where the error in the last extremum is much smaller than the other error extrema. A scaling procedure to account for how this type of filter response is obtained is not yet known. Fig. 7(d) shows a filter response where the error curve at $f = 0.5$ has a triple zero due to the unexplained behavior noted above. Fig. 7(e) shows the magnitude response of a filter with a larger value of F_p than the one of Fig. 7(d). This filter is an equiripple filter with $(N/2 + 1)$ peaks in the error function.

Chevyshev Solutions

As noted previously [8], an analytical solution for the optimal filter coefficients can be obtained in the special case of one ripple in either the passband, or stopband when N is odd. When N is even, analytical expressions can be obtained *only* for the case of one

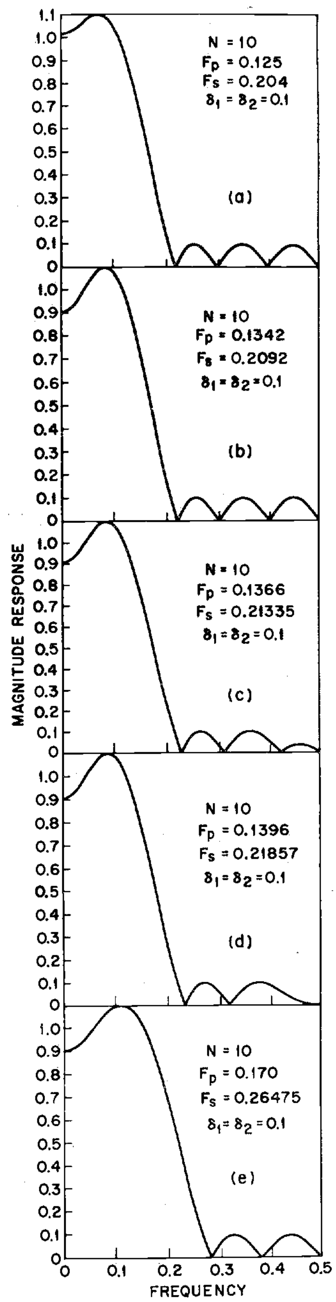

Fig. 7. The magnitude responses of five optimal filters showing various amounts of scaling.

passband ripple. In this section we outline the procedure necessary to obtain this solution.

Consider the scaled Chebyshev polynomial

$$\tilde{H}(x) = \delta_2 T_M(x), \qquad -X_0 \leq x \leq X_0,\ M \text{ odd} \qquad (33)$$

where $T_M(x)$ is a standard Chebyshev polynomial, defined by

$$T_M(x) = \begin{cases} \cos\,[M \cos^{-1} x], & -1 \leq x \leq 1 \\ \cosh\,[M \cosh^{-1} x], & |x| > 1 \end{cases} \qquad (34)$$

and X_0 is the point where $T_M(X_0) = 1 + \delta_1/\delta_2$. When M is odd, the Chebyshev polynomials are asymmetric around $x = 0$ (see Fig. 8) and thus may be written in

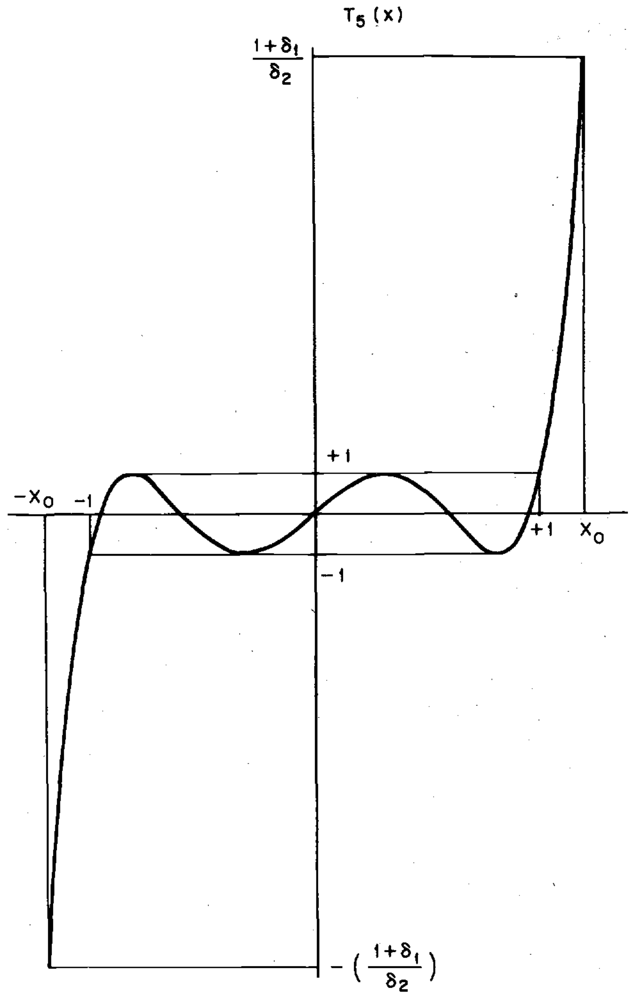

Fig. 8. The Chebyshev polynomial as a function of x.

the form

$$T_M(x) = \sum_{m=0}^{M} b(m)x^m \tag{35}$$

where $b(m) = 0$, $m = 0, 2, 4, \cdots, M - 1$. The values of $b(m)$ for m odd specify the exact polynomial. If we use the substitution

$$x = X_0 \cos(\omega/2) \tag{36}$$

in (35), it is readily shown (see Appendix) that

$$\tilde{H}(e^{j\omega}) = \delta_2 T_M(x) \,\Big|_{x=X_0 \cos(\omega/2)}$$

$$= \sum_{m=0}^{(M-1)/2} d(m) \cos(\omega(m + \tfrac{1}{2})) \tag{37}$$

which is essentially the frequency response of a linear phase filter with N even (10) if we set $N = M + 1$. Since the interval $0 \leqslant x \leqslant X_0$ is mapped via (36) to the interval $\pi \geqslant \omega \geqslant 0$, the resulting filter is an extraripple filter that has one passband ripple, and all the remaining ripples are in the stopband. The deviation of the magnitude response is δ_1 in the passband and δ_2 in the stopband. Such filters have been called the Chebyshev solution to the optimal filter design problem [8].

It should be noted that no equivalent procedure has been found for obtaining optimum filters with only one stopband ripple, as in the case of odd values of N. This difficulty is again related to the lack of symmetry in the points $\omega = 0$ and $\omega = \pi$.

Summary

This paper has presented a discussion of several of the properties of optimal, linear phase, FIR, low-pass filters designed with the constraint that the impulse duration N was even. This constraint on N led to optimality criteria on the approximation error function, similar to those in the case of odd values of N. However, the properties of the various types of solution for even N differed, in many cases, from those of odd N. One striking and unexpected result was that, in certain cases, optimal filters of duration N samples could achieve better approximations than the equivalent optimal filters of duration $(N + 1)$ samples. Another interesting property of even length filters was that the curve of transition width versus passband cutoff frequency was not symmetrical. Furthermore, scaling procedures on the extraripple solutions were obtained only for values of passband cutoff frequency below the passband cutoff frequency of the extraripple solutions—but no simple scaling procedure could be found for values above the passband cutoff frequency. Finally, analytical solutions for the Chebyshev case were obtained, only in the case of one passband ripple.

Appendix

We wish to show that the trigonometric polynomial

$$\tilde{H}(e^{j\omega}) = \sum_{n=0}^{(N/2)-1} 2h(n) \cos(\omega(n + \tfrac{1}{2})) \tag{A-1}$$

is obtained from the ordinary polynomial

$$H(x) = \sum_{n=0}^{N-1} b(n)x^n \tag{A-2}$$

(where $b(n) = 0$, n even) by the substitution

$$x = X_0 \cos(\omega/2). \tag{A-3}$$

This is easiest to show by expanding (A-1) into the form

$$\tilde{H}(e^{j\omega}) = \sum_{n=0}^{(N/2)-1} 2h(n) [\cos(\omega n) \cos(\omega/2) - \sin(\omega n) \sin(\omega/2)] \tag{A-4}$$

and substituting a function of x for each trigonometric function of ω. For simplicity we let $X_0 = 1$. (In the end result we can replace x by x/X_0 if desired.) From (A-3) we can show that

$$\cos\omega = x^2 - 1 \quad \text{(A-5)}$$

$$\sin(\omega/2) = \sqrt{1 - x^2} \quad \text{(A-6)}$$

$$\sin\omega = 2x\sqrt{1 - x^2} \quad \text{(A-7)}$$

$$\cos n\omega = \sum_{i=0}^{n} \alpha_i(\cos\omega)^i = \sum_{i=0}^{n} \alpha_i(x^2 - 1)^i \quad \text{(A-8)}$$

$$\sin n\omega = \sin\omega \sum_{i=0}^{n-1} \beta_i(\cos\omega)^i$$

$$= 2x\sqrt{1 - x^2} \sum_{i=0}^{n-1} \beta_i(x^2 - 1)^i. \quad \text{(A-9)}$$

Substituting (A-3) (with $X_0 = 1$), and (A-5) to (A-9) into (A-4) gives

$$H(x) = \tilde{H}(e^{j\omega})\big|_{x=\cos(\omega/2)}$$

$$= \sum_{n=0}^{(N/2)-1} \left[x \sum_{i=0}^{n} \alpha_i(x^2 - 1)^i - 2x(1 - x^2) \sum_{i=0}^{n-1} \beta_i(x^2 - 1)^i \right]$$

$$= x \sum_{n=0}^{(N/2)-1} a(n)(x^2 - 1)^n$$

$$= \sum_{n=0}^{(N/2)-1} c(n)x^{2n+1}. \quad \text{(A-10)}$$

By substituting $m = 2n + 1$ into (A-10) we get

$$\tilde{H}(x) = \sum_{\substack{m=0 \\ m \text{ odd}}}^{N-1} d(m)x^m. \quad \text{(A-11)}$$

If we define $b(m)$ as

$$b(m) = \begin{cases} 0 & m \text{ even} \\ d(m) & m \text{ odd} \end{cases} \quad \text{(A-12)}$$

then (A-11) can be written as

$$\tilde{H}(x) = \sum_{m=0}^{N-1} b(m)x^m \quad \text{(A-13)}$$

which is the desired result.

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