Letters to the Editor

On the Properties of Frequency Transformations for Variable Cutoff Linear Phase Digital Filters

R. E. CROCHIERE AND L. R. RABINER

Abstract—**In a recent paper by Oppenheim, Mecklenbrauker, and Mersereau [1] a class of variable cutoff linear phase digital filters has been proposed. The implementation of this class of filters is achieved by replacing a subnetwork in a prototype network such that it performs a frequency mapping of the prototype filter response. By varying a small number of coefficients in the subnetwork the frequency transformation can be varied. In this letter we examine the characteristics of this transformation in greater detail and discuss the range of parameters over which it can be useful.**

I. Introduction

It has been shown in [1] that a variable cutoff linear phase digital filter can be implemented by replacing appropriate subnetworks in a Taylor structure [2] by a subnetwork which performs a frequency transformation of the original (prototype) network. This frequency transformation can be varied by varying a small number of coefficients in the subnetwork. The order of this transformation can be of any degree desired, however, only the first-order transformation leads to a structure which is canonical and which monotonically maps a region of the original frequency domain into the entire transformed frequency domain. For these reasons the first-order transformation is of greatest interest. In this letter we examine the characteristics of this first-order transformation and discuss the range of parameters over which it is useful.

II. The General First-Order Transformation

As shown in [1] the first-order transformation takes the form of a straight line in the $\cos-\omega$, $\cos-\Omega$ plane and can be expressed as

$$\cos\Omega = \frac{-A_0}{A_1} + \frac{1}{A_1}\cos\omega \tag{1}$$

where ω corresponds to the frequency of the prototype filter. Ω corresponds to the frequency of the transformed filter and A_1 and A_0 are parameters of the transformation which can be varied. Since the transformation maps part of the ω domain to all of the Ω domain the magnitude of the slope, $|d(\cos\Omega)/d(\cos\omega)| = |1/A_1|$, must be greater than one. If this slope is positive the transformation will be defined as a *forward transformation* (i.e., increasing ω maps to increasing Ω) and if it is negative the transformation will be referred to as an *inverted transformation*

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The authors are with Bell Laboratories, Murray Hill, NJ 07974.

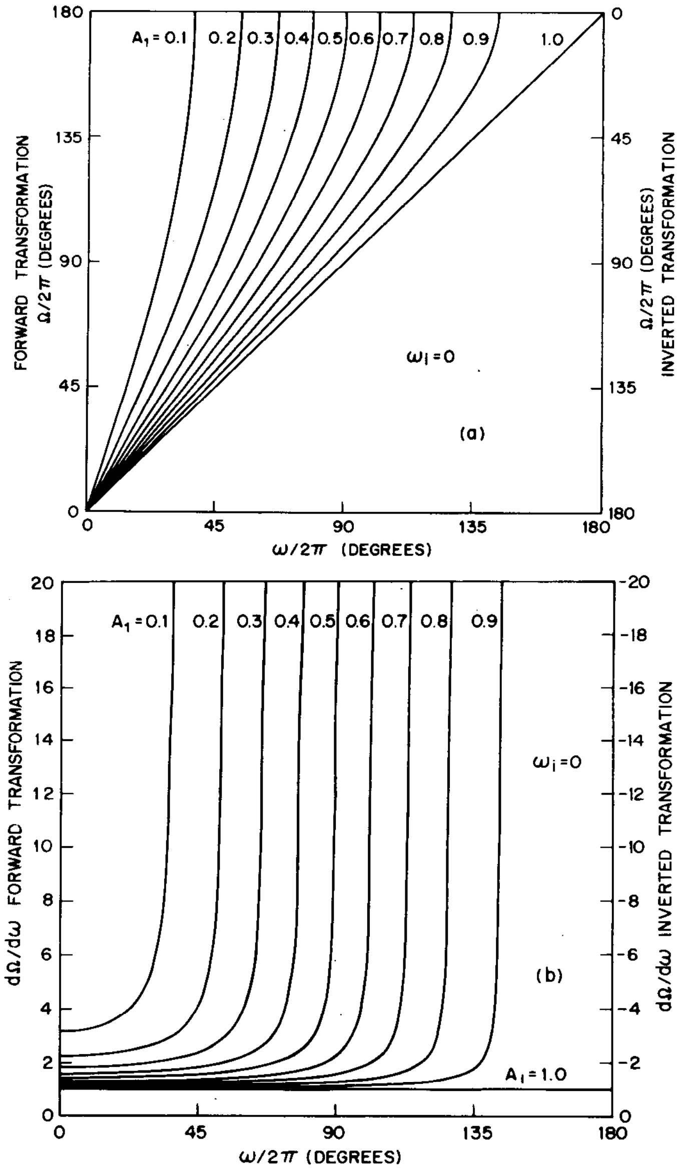

Fig. 1. Plots of Ω versus $d\Omega/d\omega$ for values of A, from 0.1 to 1.0 and $\omega_i = 0$.

(i.e., increasing ω maps to decreasing Ω). Therefore,

$$0 \leqslant A_1 \leqslant 1, \quad \text{forward transformation} \tag{2a}$$

$$-1 \leqslant A_1 \leqslant 0, \quad \text{inverted transformation.} \tag{2b}$$

A_0 in combination with A_1 determines the frequency of invariance of the transformation. That is, the frequency ω_i which maps to Ω_i such that $\omega_i = \Omega_i$ for the forward transformation or $\omega_i = \pi - \Omega_i$ for the inverted transformation. Given a desired frequency of

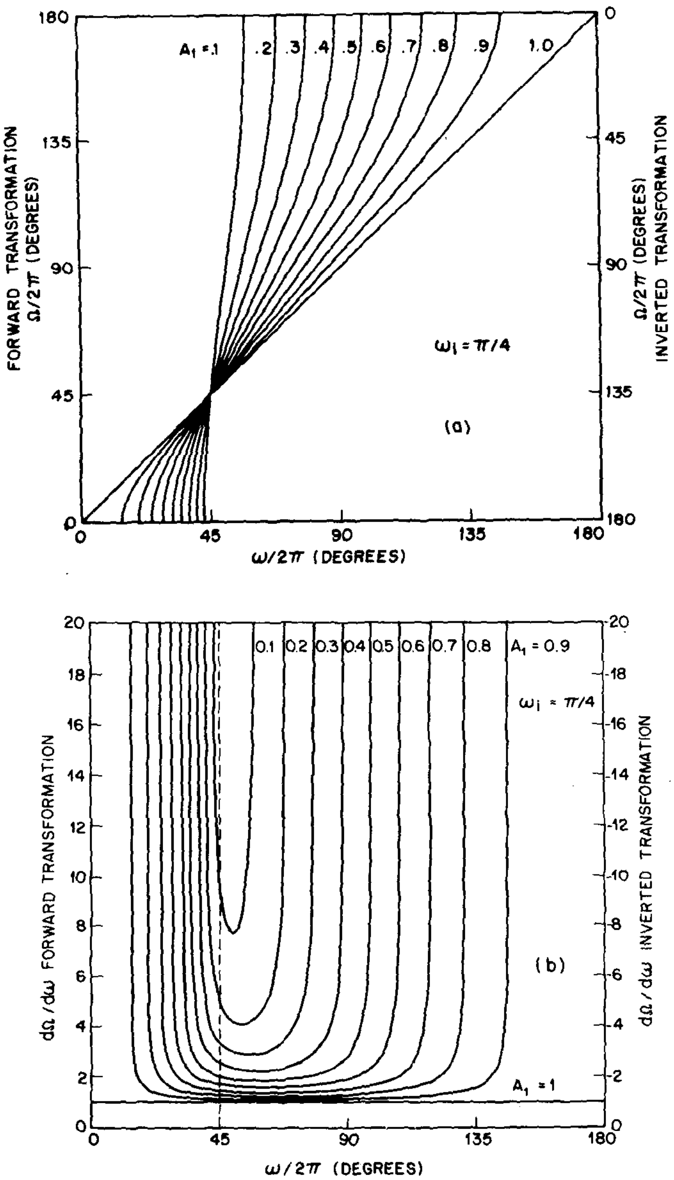

Fig. 2. Plots of Ω versus $d\Omega/d\omega$ for $\omega_i = \pi/4$.

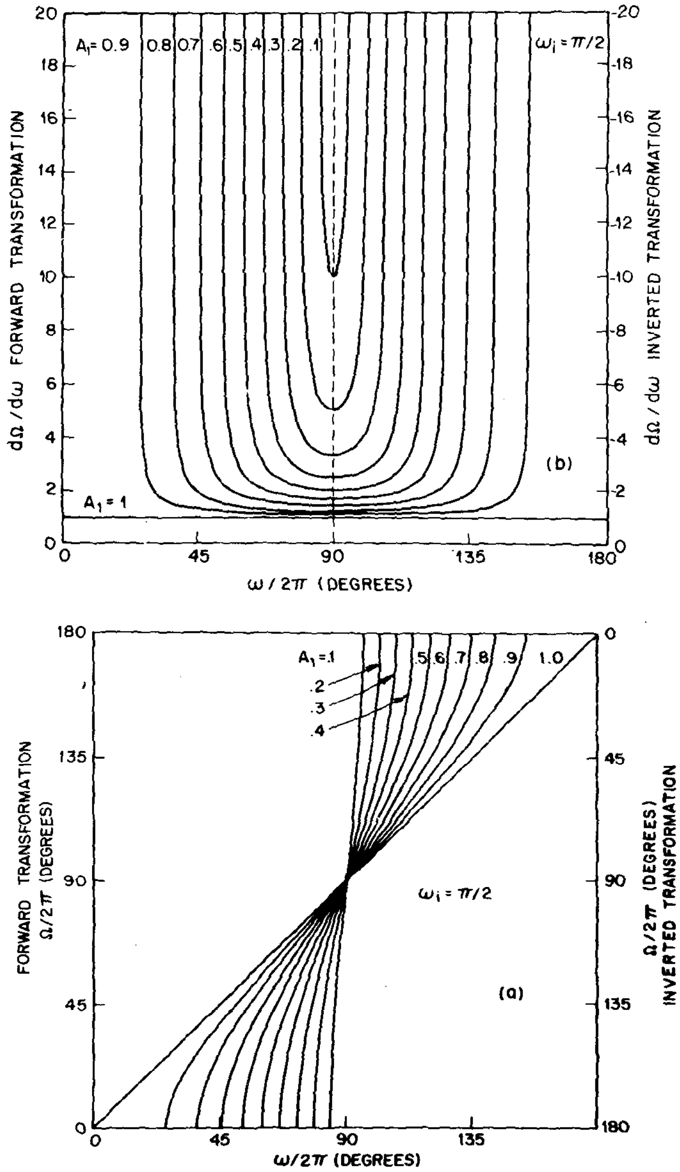

Fig. 3. Plots of Ω versus $d\Omega/d\omega$ for $\omega_i = \pi/2$.

invariance, ω_i, and a desired slope $1/A_1$, A_0 becomes

$$A_0 = (1 - |A_1|)\cos\omega_i. \tag{3}$$

Thus ω_i and $d(\cos\Omega)/d(\cos\omega)$ can be chosen in accordance with a two term Tayler expansion about the frequency of invariance.

Another important consideration in the transformation is the unwarped frequency slope $d\Omega/d\omega$. This function is useful in characterizing the spread of the transformation at a given frequency. It is helpful, for example, in determining the spread of the transition band in a low-pass to low-pass mapping or the spread of the passband (and transition bands) in a variable bandpass mapping.

An interesting anamoly occurs in this slope at ω_i when ω_i approaches 0 or π. It can be shown that at the frequency of invariance, $d\Omega/d\omega|_{\omega_i} = d(\cos\Omega)/d(\cos\omega) = 1/A_1$ for $\omega_i \neq 0$, π, but when $\omega_i = 0$, π the slope abruptly changes to $d\Omega/d\omega|_{\omega_i} = 1/\sqrt{|A_1|}$. While this anomaly has no practical consequence in the transformation it suggests that one must be careful about the notion of using the slope at ω_i as a measure of the spread of the transformation.

III. Plots of the Transformation and its Slope

Fig. 1 shows a series of plots of both Ω versus ω and $d\Omega/d\omega$ versus ω for several values of A_1 and ω_i. In particular, Fig. 1(a) and 1b show these functions for ω_i equal to 0 and for A_1 varying from 1.0 to 0.1 in steps of 0.1. Similarly, Figs. 2 and 3 show pairs of plots for $\omega_i = \pi/4$, and $\pi/2$, respectively. In each of the plots the left vertical scale applies to forward transformations and the right vertical scale applies to inverted transformations (with appropriate negative values of A_1). Several interesting properties of the transformation can be seen in these plots. First it can be seen that as the invariance frequency, ω_i, increases from 0 to $\pi/2$, the set of frequencies in the ω domain which maps into the region $0 \leqslant \Omega \leqslant \pi$ decreases for a fixed value of A_1. This means that as ω_i increases, the transformation spreads out the frequency response from a small region of the ω scale to the entire frequency region of the Ω scale. As ω_i increases beyond $\omega_i=\pi/2$, this effect reverses itself.

Another observation that can be made is that the derivative function $(d\Omega/d\omega)$ is approximately constant over some range of ω and then sharply increases as ω comes close to the point (s) which map to $\Omega=0$ and/or $\Omega=\pi$. This property of the derivative is an interesting one in that the function $d\Omega/d\omega$ can be interpreted as the ratio between the transition region of the transformed filter $(\Delta F')$ to the transition region of the original filter (ΔF), i.e.,

$$\frac{\Delta F'}{\Delta F}\approx\frac{d\Omega}{d\omega}. \tag{4}$$

This leads to the observation that for the transformation to be of some practical use, the derivative $d\Omega/d\omega$ should be as close to 1 as possible. For all practical purposes the transformation is reasonably good for the range $1\leqslant d\Omega/d\omega\leqslant 2$ and may be acceptable for many applications with considerably larger values of $d\Omega/d\omega$. The curves of Figs. 1(b), 2(b), and 3(b) are especially useful in showing the regions where $d\Omega/d\omega$ falls below some desired value. It can also be observed from these plots that the point of minimum $d\Omega/d\omega$ does not occur at frequency ω_i except when $\omega_i=0$, $\pi/2$ and π.

In summary we have attempted to characterize the transformation proposed in [1] and show the useful parameter range and frequencies over which it can be applied and also the limitations which are encountered.

References

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[2] H. W. Schuessler, "On structures for nonrecursive digital filters," *AEU*, vol. 26, pp. 255–258, 1972.

Synthesis of Recursive Digital Filters with Prescribed Attenuation and Group-Delay Characteristics

RAMAMURTY NEDUNURI

***Abstract*—A method of synthesizing recursive digital filters with prescribed attenuation and group-delay characteristics is presented. The method depends on the fact that the introduction of complex transmission zeros into the transfer function of the filter network allows independent control of both attenuation and group-delay responses of the filter. An example is presented which clearly demonstrates the improvement in attenuation response obtained when complex transmission zeros are introduced into the transfer function of the filter.**

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The author is at 6 Dunkirk Rise, Rochdale, Lancashire OL12 6UH, England.

I. Introduction

In the design of electronic systems (both analog and digital), we often require filters with prescribed attenuation and group-delay characteristics. One method of solution to this problem is to design a filter which satisfies the prescribed attenuation requirements only and then equalize the group-delay response of the filter by means of separate all-pass networks. However, in the case of analog filters, it is found that a more economical solution is obtained by synthesizing filters with complex transmission zeros [1], [2]. The introduction of complex transmission zeros into the transfer function of the filter allows independent control of both attenuation and group-delay responses of the filter and thus enables the designer to satisfy simultaneously prescribed constraints on attenuation and group delay.

In this paper we shall show that the above method can also be used in the synthesis of digital filters. In Section II we first describe a procedure for the synthesis of recursive digital filters with prescribed transmission zeros; the procedure is similar to that used in the synthesis of analog filters [3]. In Section III we give an example which clearly demonstrates the advantages of introducing complex transmission zeros into the transfer function of the filter network.

II. Synthesis Procedure

Once the transmission zeros of the filter network are specified, the problem is to determine the poles of the transfer function such that the attenuation response has an equal-ripple behavior in the passband. This is done through the transformation

$$s=\sqrt{\frac{z^2-2z\cos\omega_2T+1}{z^2-2z\cos\omega_1T+1}}, \qquad \operatorname{Re} s\geqslant 0$$

where

$$\begin{aligned}s&=\alpha+j\beta\\ z&=x+jy=e^{pT}\\ p&=\sigma+j\omega\\ T&=\text{sampling period}=1/f_s\\ f_s&=\text{sampling frequency}\\ \omega_1,\omega_2&=\text{cutoff frequencies.}\end{aligned}$$

It is not hard to see that this transformation maps the passband $\omega_1\leqslant\omega\leqslant\omega_2$ onto the imaginary axis $\infty\geqslant\beta\geqslant 0$ in the s plane. The procedure is as follows.

1) Transform all the p-plane transmission zeros into the s plane. If $p_i=\sigma_i+j\omega_i$ is a transmission zero, we have

$$z_i=e^{(\sigma_i+j\omega_i)T}$$

and

$$s_i=\alpha_i+j\beta_i=\sqrt{A_i+jB_i}$$

where

$$\begin{aligned}A_i&=\frac{u_iv_i+\mu_i^2}{v_i^2+\mu_i^2}\\ B_i&=\frac{(v_i-u_i)\,\mu_i}{v_i^2+\mu_i^2}\\ u_i&=\cosh\sigma_iT\cos\omega_iT-\cos\omega_2T\\ v_i&=\cosh\sigma_iT\cos\omega_iT-\cos\omega_1T\\ \mu_i&=\sinh\sigma_iT\sin\omega_iT.\end{aligned}$$