the equations being solved, which is also true of the reflection coefficients. Further, if the covariance matrix is Toeplitz, and the vector of dependent variables consists of autocorrelation terms, the Cholesky K-parameters are equivalent to the usual reflection coefficients.

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A Study of Techniques for Finding the Zeros of Linear Phase FIR Digital Filters

C. E. SCHMIDT AND L. R. RABINER

Abstract—**Since the majority of the zeros of standard finite-duration impulse-response (FIR) linear phase digital filters (e.g., low pass, bandpass, etc.) are located on the unit circle in the z-plane, it is possible to exploit this information in devising an efficient method for accurately solving for the locations of these zeros. To perform this task, three standard algorithms for finding the roots of a polynomial were evaluated. These methods were the bisection method, the modified false position method, and the Newton-Raphson method. Using a convergence criterion based on the value of the function (rather than the uncertainty in the position of the root), it was experimentally found that the Newton-Raphson method was the most accurate in determining the location of the roots, as well as being the most computationally efficient of the three methods. A second study was made to compare methods for determining all the zeros of the filter (i.e., the zeros off the unit circle, as well as those on the unit circle). Two sophisticated algorithms (the Jenkins-Traub Three-Stage Algorithm, and the Madsen-Reid Algorithm based on Newton's method) were used in this study as well as a deflation method in which the unit circle roots were first located, and then used to form a deflated polynomial which was used to find the roots which occurred off the unit circle. It was found that for polynomials of degree greater than about 100, both the Jenkins-Traub and Madsen-Reid methods were far superior (in terms of accuracy in locating the roots off the unit circle) to the deflation method because of inaccuracies incurred in deflating high-order polynomials. It was also found that the accuracies of both the Jenkins-Traub and Madsen-Reid methods were comparable; however, the Madsen-Reid method was between two and four times faster than the Jenkins-Traub method for the examples tested.[1]**

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I. INTRODUCTION

The problem of accurately locating the roots of high-order polynomials (of degree greater than 100) is an extremely difficult one. Such high degree polynomials often occur when working with linear phase finite-duration impulse-response (FIR) digital filters. It is the purpose of this correspondence to report on the results of an experimental study of several methods for accurately and efficiently solving for the locations of the roots of high-order FIR digital filter polynomials.

II. UNIT CIRCLE ROOTS

Since the majority of the zeros of standard FIR linear phase digital filters are located on the unit circle in the z-plane, the problem of finding these zeros can be trivially reduced to a problem of finding the real zeros of a real polynomial in the following way. The filter polynomial can be evaluated on the unit circle, giving the frequency response

$$H(e^{j\omega}) = h(0) + \sum_{n=1}^{(N-1)/2} 2h(n)\cos(\omega n), \tag{1}$$

where N is the filter impulse-response duration, $h(n)$, $n = -(N-1)/2, \cdots, (N-1)/2$ is the impulse response, and (because of the linear phase)

$$h(n) = h(-n), \qquad n = 0, 1, \cdots, \frac{N-1}{2}. \tag{2}$$

Equation (1) shows that the problem of finding the roots of the filter occurring on the unit circle is readily converted to a problem of finding real zeros of a real valued function (i.e., values of ω such that $H(e^{j\omega}) = 0$).

A wide variety of techniques are available for solving this problem. Three such methods were chosen for investigation. These were the bisection method, the modified false position method, and the Newton-Raphson method [1]. All these methods use iterative techniques to bound the locations of the roots as tightly as possible. Thus an initial estimate as to the location of a root was required, as well as a criterion for deciding when to terminate the search for the root. The initial estimates of root locations were obtained by evaluating $H(e^{j\omega})$ on a reasonably large grid of points using a standard fast Fourier transform (FFT). The convergence criterion was to terminate the search when the function magnitude fell below a specified value (epsilon) at either the estimate of the root, or at both endpoints of the interval in which the root was bound.

Table I presents a comparison of both the accuracy [Table I(a)] and efficiency [Table I(b)] of these three methods in locating the unit circle roots of three polynomials of degrees 48, 96, and 198, respectively.[2] The Newton-Raphson method was the most accurate of the methods across all three polynomials, and for both values of epsilon, the convergence criterion. The Newton-Raphson method was also uniformly

[1] The authors have been made aware of the fact that a newer implementation of the Jenkins-Traub method has been published in *Assoc. Comput. Mach. Trans. Math. Software*, vol. 1, no. 2, June 1975. This implementation should run somewhat faster than the earlier method cited in this report.

[2] A more complete set of comparisons has been made and is available on request from C. E. Schmidt, Bell Laboratories, Rm. 2D-525, Murray Hill, NJ 07974.

TABLE I

Epsilon	Bisection Method	Modified False Position Method	Newton-Raphson Method	Polynomial Degree
$.378\times10^{-5}$	$.18\times10^{-4}$	$.18\times10^{-4}$	$.11\times10^{-4}$	48
$.378\times10^{-7}$	$.15\times10^{-6}$	$.21\times10^{-6}$	$.25\times10^{-7}$	48
$.527\times10^{-5}$	$.10\times10^{-4}$	$.70\times10^{-5}$	$.48\times10^{-5}$	96
$.527\times10^{-7}$	$.98\times10^{-7}$	$.14\times10^{-6}$	$.92\times10^{-8}$	96
$.430\times10^{-5}$	$.45\times10^{-5}$	$.50\times10^{-5}$	$.28\times10^{-5}$	198
$.430\times10^{-7}$	$.50\times10^{-7}$	$.80\times10^{-7}$	$.51\times10^{-8}$	198

(a)

Average Error (radians) of Unit Circle Roots

Epsilon	Bisection Method	Modified False Position Method	Newton-Raphson Method	Polynomial Degree
$.378\times10^{-5}$	.94	.49	.42	48
$.378\times10^{-7}$	1.57	1.09	.59	48
$.527\times10^{-5}$	3.04	1.58	1.39	96
$.527\times10^{-7}$	6.11	3.82	1.99	96
$.430\times10^{-5}$	14.11	6.29	5.92	198
$.430\times10^{-7}$	25.81	15.98	9.13	198

(b)

Computation Time (Seconds) for Finding Unit Circle Roots

the most efficient of the three methods tested. The reason the Newton–Raphson method converged more rapidly to the actual root location was because it used the function derivative to make successive estimates of the root location, whereas the other two methods relied entirely on the function value. The increased computation for evaluating the derivative was more than compensated by the faster convergence of this method.

The results in Table I show the average error in the root location (in radians) across all the unit circle roots. Fig. 1 shows a plot of the actual error in each of the unit circle roots for the 198th degree polynomial using the Newton–Raphson method. Although the average error for this case was 0.5×10^{-8}, the peak error was about 10^{-7}.

III. Roots Off the Unit Circle

Although the majority of the roots of the FIR filter polynomials were on the unit circle, it was also important to be able to locate the roots which occurred off the unit circle. Three methods were investigated which could determine all the roots of the polynomial. Two of the methods were sophisticated algorithms for solving for complex roots of a real function. These were the Jenkins–Traub Three-Stage Algorithm [2], and the Madsen–Reid Algorithm based on Newton's method [3]. Neither of these methods made special use of the property that the majority of the roots of the polynomial were on the unit circle in the z-plane. The third method used the Newton–Raphson method to find the unit circle roots, and then deflated the polynomial to obtain a fairly low-order polynomial from which the roots found off the unit circle were obtained using either of the above methods.

Table II shows a comparison of the accuracy in determining both the unit circle roots and the roots found off the unit circle (in terms of average magnitude error), and it also provides a comparison of the computation times of the three methods for five different filter polynomials. Due to inaccuracies occurring in deflating high-order polynomials, the deflation method provided entirely inadequate root locations for the polynomials of degree greater than 48. However, both the Jenkins–Traub and Madsen–Reid methods provided extremely accurate results, even for a 248th degree polynomial.

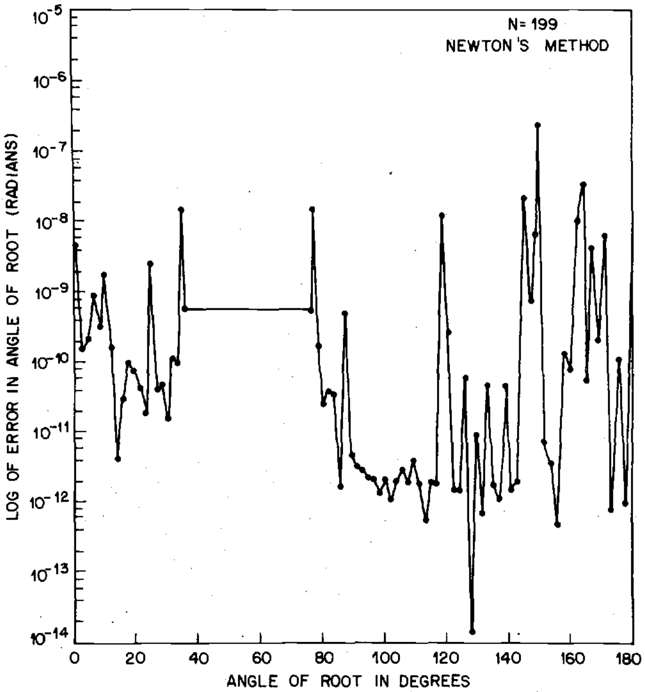

Fig. 1.

In comparing computation times, it was found that the Madsen–Reid method was from 2–4 times more efficient than the Jenkins–Traub method, as seen in Table II.

TABLE II

Method of finding Roots on *Unit Circle*	*Degree 48*	*Degree 96*	*Degree 148*	*Degree 198*	*Degree 248*
Deflation Method	2.4×10^{-8}	9.1×10^{-9}	5.6×10^{-9}	5.0×10^{-9}	8.3×10^{-9}
Jenkins-Traub Method	1.7×10^{-10}	6.7×10^{-11}	8.3×10^{-11}	1.1×10^{-8}	1.8×10^{10}
Madsen-Reid Method	1.7×10^{-10}	6.7×10^{-11}	7.2×10^{11}	9.6×10^{-10}	1.1×10^{-10}

Average Error (Radians) for Unit Circle Roots

Method of finding Roots off *Unit Circle*	*Degree 48*	*Degree 96*	*Degree 148*	*Degree 198*	*Degree 248*
Deflation Method	1×10^{-6}	2×10^{-4}	-	-	-
Jenkins-Traub alone	1×10^{-8}	5×10^{-9}	5×10^{-9}	1×10^{-7}	1×10^{-7}
Madsen-Reid Method	5×10^{-9}	5×10^{-9}	5×10^{-9}	1×10^{-7}	1×10^{-7}

Average Magnitude Error for Roots off the Unit Circle

Method of *finding Roots*	*Degree 48*	*Degree 96*	*Degree 148*	*Degree 198*	*Degree 248*
Deflation Method	.76 sec.	2.7 sec.	6.5 sec.	11.8 sec.	16.1 sec.
Jenkins-Traub alone	1.92 sec.	10.6 sec.	27.5 sec.	41.8 sec.	97.4 sec.
Madsen-Reid Method	.97 sec.	3.8 sec.	8.7 sec.	15.7 sec.	25.1 sec.

Computation Time for Finding all the Polynomial Roots

IV. SUMMARY

This study has compared three simple methods for finding unit circle roots of an FIR filter polynomial and shown the Newton–Raphson method to be more accurate and efficient than the bisection method or the modified false position method. However, when one is interested in all the roots of the polynomial, the gains in speed obtained by knowledge that most of the roots are on the unit circle are offset by inaccuracies in deflating the polynomial to solve for the remaining roots. The study has also shown that two readily available, sophisticated root-finding methods were both capable of accurately determining the locations of the roots of polynomials up to the 248th degree, the largest one tested. Finally, it was found that the Madsen–Reid method was significantly more efficient than the Jenkins–Traub method for the examples tested.

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On the Sensitivity of Charge-Coupled Device Transversal Filter Response to Charge-Transfer Inefficiency

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Abstract—**The design of charge-coupled device (CCD) transversal discrete-time filters with approximately linear phase and satisfying general magnitude constraints is considered. The charge-transfer inefficiency of the CCD is included in the problem formulation and an effective sensitivity analysis procedure is given for the charge-transfer inefficiency parameter.**

I. INTRODUCTION

The charge-coupled device (CCD) functions as a shift capable of storing and transferring analog data in the form of charge packets from one stage in the chain to the following stage. The capability of the CCD to store, and upon command, transfer information to the following series element allows it

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