Digital Speech Processing—Lecture 15

Speech Coding Methods Based on Speech Waveform Representations and Speech Models—Uniform and Non-Uniform Coding Methods
Analog-to-Digital Conversion (Sampling and Quantization)

Class of “waveform coders” can be represented in this manner
Information Rate

- Waveform coder information rate, $I_w$, of the digital representation of the signal, $x_a(t)$, defined as:

$$I_w = B \cdot F_s = B / T$$

where $B$ is the number of bits used to represent each sample and $F_s = \frac{1}{T}$, is the number of samples/second.
Speech Information Rates

• **Production level:**
  – 10-15 phonemes/second for continuous speech
  – 32-64 phonemes per language => 6 bits/phoneme
  – *Information Rate*=60-90 bps at the source

• **Waveform level**
  – speech bandwidth from 4 – 10 kHz => sampling rate from 8 – 20 kHz
  – need 12-16 bit quantization for high quality digital coding
  – *Information Rate*=96-320 Kbps => **more than 3 orders of magnitude** difference in Information Rates between the production and waveform levels
Speech Analysis/Synthesis Systems

• Second class of digital speech coding systems:
  o analysis/synthesis systems
  o model-based systems
  o hybrid coders
  o vocoder (voice coder) systems

• Detailed waveform properties generally not preserved
  o coder estimates parameters of a model for speech production
  o coder tries to preserve intelligibility and quality of reproduction from the digital representation
Speech Coder Comparisons

• Speech parameters (the chosen representation) are encoded for transmission or storage
  • analysis and encoding gives a data parameter vector
  • data parameter vector computed at a sampling rate much lower than the signal sampling rate
  • denote the “frame rate” of the analysis as $F_{fr}$
  • total information rate for model-based coders is:
    \[ I_m = B_c \cdot F_{fr} \]
  • where $B_c$ is the total number of bits required to represent the parameter vector
Speech Coder Comparisons

- **Waveform Coders** characterized by:
  - high bit rates (24 Kbps – 1 Mbps)
  - low complexity
  - low flexibility
- **Analysis/Synthesis Systems** characterized by:
  - low bit rates (10 Kbps – 600 bps)
  - high complexity
  - great flexibility (e.g., time expansion/compression)
Introduction to Waveform Coding

\[ x(n) = x_a(nT) \]

\[ X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j\Omega + \frac{j2\pi k}{T}) \]

\[ \omega = \Omega T \Rightarrow X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(\frac{j\omega}{T} + \frac{j2\pi k}{T}\right) \]
Introduction to Waveform Coding

\[ T = \frac{1}{F_s} \]

- To **perfectly recover** \( x_a(t) \) (or equivalently a lowpass filtered version of it) from the set of digital samples (as yet unquantized) we require that \( F_s = \frac{1}{T} > \) twice the highest frequency in the input signal.

- This implies that \( x_a(t) \) must first be lowpass filtered since speech is not inherently lowpass.

  - For telephone bandwidth the frequency range of interest is 200-3200 Hz (filtering range) \( \Rightarrow F_s = 6400 \text{ Hz, 8000 Hz} \)

  - For wideband speech the frequency range of interest is 100-7000 Hz (filtering range) \( \Rightarrow F_s = 16000 \text{ Hz} \)
Sampling Speech Sounds

- notice high frequency components of vowels and fricatives (up to 10 kHz) => need $F_s > 20$ kHz without lowpass filtering
  - need only about 4 kHz to estimate formant frequencies
  - need only about 3.2 kHz for telephone speech coding
it is clear that 4 kHz bandwidth is sufficient for most applications using telephone speech because of inherent channel band limitations from the transmission path
Statistical Model of Speech

• assume $x_a(t)$ is a sample function of a continuous-time random process
• then $x(n)$ (derived from $x_a(t)$ by sampling) is also a sample sequence of a discrete-time random process
• $x_a(t)$ has first order probability density, $p(x)$, with autocorrelation and power spectrum of the form

$$\phi_a(\tau) = E\left[ x_a(t)x_a(t + \tau) \right]$$

$$\Phi_a(\Omega) = \int_{-\infty}^{\infty} \phi_a(\tau) e^{-j\Omega\tau} d\tau$$
Statistical Model of Speech

• $x(n)$ has autocorrelation and power spectrum of the form:

$$ \phi(m) = E[x(n)x(n+m)] $$
$$ = E[x_a(nT)x_a(nT + mT)] = \phi_a(mT) $$

$$ \Phi(e^{j\Omega T}) = \sum_{m=-\infty}^{\infty} \phi(m)e^{-j\Omega T m} $$

$$ = \frac{1}{T} \sum_{k=-\infty}^{\infty} \Phi_a(\Omega + \frac{2\pi k}{T}) $$

• since $\phi(m)$ is a sampled version of $\phi_a(\tau)$, its transform is an infinite sum of the power spectrum, shifted by $\frac{2\pi k}{T} \Rightarrow$ aliased version of the power spectrum of the original analog signal
Speech Probability Density Function

- probability density function for $x(n)$ is the same as for $x_a(t)$ since $x(n) = x_a(nT) \Rightarrow$ the mean and variance are the same for both $x(n)$ and $x_a(t)$

- need to estimate probability density and power spectrum from speech waveforms
  - probability density estimated from long term histogram of amplitudes
  - good approximation is a gamma distribution, of the form:
    $$p(x) = \left[ \frac{\sqrt{3}}{8\pi\sigma_x} \right]^{1/2} e^{-\frac{\sqrt{3}|x|}{2\sigma_x}} \quad p(0) = \infty$$
  - simpler approximation is Laplacian density, of the form:
    $$p(x) = \frac{1}{\sqrt{2\sigma_x}} e^{-\frac{\sqrt{2}|x|}{\sigma_x}} \quad p(0) = \frac{1}{\sqrt{2\sigma_x}}$$
Measured Speech Densities

- Distribution normalized so mean is 0 and variance is 1 ($\mu=0$, $\sigma^2=1$)
- Gamma density more closely approximates measured distribution for speech than Laplacian
- Laplacian is still a good model and is used in analytical studies
- Small amplitudes much more likely than large amplitudes—by 100:1 ratio
Long-Time Autocorrelation and Power Spectrum

- analog signal
  \[ \phi_a(\tau) = E\{x_a(t)x_a(t + \tau)\} \iff \Phi_a(\Omega) = \int_{-\infty}^{\infty} \phi_a(\tau)e^{-j\Omega \tau} d\tau \]

- discrete-time signal
  \[ \phi(m) = E\{x(n)x(n + m)\} = E\{x_a(nT)x_a(nT + mT)\} = \phi_a(mT) \]
  \[ \iff \Phi(e^{j\Omega T}) = \sum_{m=-\infty}^{\infty} \phi(m)e^{-j\Omega Tm} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \Phi_a(\Omega + \frac{2\pi k}{T}) \]

- estimated correlation and power spectrum
  \[ \hat{\phi}(m) = \frac{1}{L} \sum_{n=0}^{L-1-m} x(n)x(n + m), \quad 0 \leq |m| \leq L - 1, \]
  \[ L \text{ is large integer} \]
  \[ \hat{\Phi}(e^{j\Omega T}) = \sum_{m=-M}^{M} w(m)\hat{\phi}(m)e^{-j\Omega Tm} \]

Estimates ≠ Expectations
- tapering with m due to finite windows
- need window on estimated correlation for smoothing because of discontinuity at ±M
Speech AC and Power Spectrum

- can estimate long term autocorrelation and power spectrum using time-series analysis methods

\[-1 \leq \hat{\rho}[m] = \frac{\hat{\phi}[m]}{\hat{\phi}[0]} \leq 1,\]

\[
= \frac{1}{L} \sum_{n=0}^{L-1-m} x(n)x(n+m)
\leq \frac{1}{L} \sum_{n=0}^{L-1-m} x(n)^2
\]

\[0 \leq m \leq L - 1, \text{ } L \text{ is window length} \]

• 8 kHz sampled speech for several speakers
• high correlation between adjacent samples
• lowpass speech more highly correlated than bandpass speech
Speech Power Spectrum

- power spectrum estimate derived from one minute of speech
- peaks at 250-500 Hz (region of maximal spectral information)
- spectrum falls at about 8-10 db/octave
- computed from set of bandpass filters
Alternative Power Spectrum Estimate

- estimate long term correlation, $\hat{\phi}(m)$, using sampled speech, then weight and transform, giving:
  \[
  \hat{\Phi}(e^{j2\pi k/N}) = \sum_{m=-M}^{M} w(m)\hat{\phi}(m)e^{-j2\pi km/N}
  \]
- this lets us use $\hat{\phi}(m)$ to get a power spectrum estimate $\hat{\Phi}(e^{j2\pi k/N})$ via the weighting window, $w(m)$

Contrast linear versus logarithmic scale for power spectrum plots
Estimating Power Spectrum via Method of Averaged Periodograms

- Periodogram defined as:
  \[
P(e^{j\omega}) = \frac{1}{LU} \left| \sum_{n=0}^{L-1} x(n) w(n) e^{-j\omega n} \right|^2
  \]
  where \( w[n] \) is an \( L \)-point window (e.g., Hamming window), and
  \[
  U = \frac{1}{L} \sum_{n=0}^{L-1} (w(n))^2
  \]
  \( U \) is a normalizing constant that compensates for window tapering

- use the DFT to compute the periodogram as:
  \[
P(e^{j(2\pi k/N)}) = \frac{1}{LU} \left| \sum_{n=0}^{L-1} x(n) w(n) e^{-j(2\pi k/N)n} \right|^2, \quad 0 \leq k \leq N - 1,
  \]
  \( N \) is size of DFT
Averaging (Short) Periodograms

- variability of spectral estimates can be reduced by averaging short periodograms, computed over a long segment of speech
- using an $L$-point window, a short periodogram is the same as the STFT of the weighted speech interval, namely:

$$X_r[k] = X_r(e^{j2\pi k/N}) = \sum_{m=rR}^{rR+L-1} x[m]w[rR-m]e^{-j2\pi km/N}, \quad 0 \leq k \leq N-1$$

- where $N$ is the DFT transform size (number of frequency estimates)
- Consider using $N_S$ samples of speech (where $N_S$ is large); the averaged periodogram is defined as:

$$\tilde{\Phi}(e^{j2\pi k/N}) = \frac{1}{KLU} \sum_{r=0}^{K-1} \left| X_r(e^{j2\pi k/N}) \right|^2, \quad 0 \leq k \leq N-1$$

- where $K$ is the number of windowed segments in $N_S$ samples, $L$ is the window length, and $U$ is the window normalization factor
- use $R = L/2$ (shift window by half the window duration between adjacent periodogram estimates)
88,000-Point FFT – Female Speaker

Single spectral estimate using all signal samples
Long Time Average Spectrum

Spectrum Estimates by Averaging Short Periodograms

- Log magnitude in dB
- Frequency in Hz

- Female speaker
- Male speaker
Instantaneous Quantization

• separating the processes of sampling and quantization
• assume $x(n)$ obtained by sampling a bandlimited signal at a rate at or above the Nyquist rate
• assume $x(n)$ is known to infinite precision in amplitude
• need to quantize $x(n)$ in some suitable manner
Quantization and Coding

Coding is a two-stage process
1. quantization process:
   \[ x(n) \rightarrow \hat{x}(n) \]
2. encoding process:
   \[ \hat{x}(n) \rightarrow c(n) \]
   - where \( \Delta \) is the (assumed fixed) quantization step size

Decoding is a single-stage process
1. decoding process:
   \[ c'(n) \rightarrow \hat{x}'(n) \]
   - if \( c'(n) = c(n) \), (no errors in transmission) then \( \hat{x}'(n) = \hat{x}(n) \)
\[ \hat{x}'(n) \neq x(n) \Rightarrow \text{coding and quantization loses information} \]

Assume \( \Delta' = \Delta \)
B-bit Quantization

• use B-bit binary numbers to represent the quantized samples => $2^B$ quantization levels

• **Information Rate of Coder**: $I = B F_S$ = total bit rate in bits/second
  
  – $B=16, F_S = 8$ kHz => $I=128$ Kbps
  
  – $B=8, F_S = 8$ kHz => $I=64$ Kbps
  
  – $B=4, F_S = 8$ kHz => $I=32$ Kbps

• goal of waveform coding is to get the highest quality at a fixed value of $I$ (Kbps), or equivalently to get the lowest value of $I$ for a fixed quality

• since $F_S$ is fixed, need most efficient quantization methods to minimize $I$
Quantization Basics

- assume $|x(n)| \leq X_{max}$ (possibly $\infty$)
  - for Laplacian density (where $X_{max} = \infty$),
    can show that 0.35% of the samples fall outside the range $-4\sigma_x \leq x(n) \leq 4\sigma_x$ => large quantization errors for 0.35% of the samples
  - can safely assume that $X_{max}$ is proportional to $\sigma_x$
Quantization Process

• quantization => dividing amplitude range into a finite set of ranges, and assigning the same bin to all samples in a given range

\[
\begin{align*}
0 &= x_0 < x(n) \leq x_1 \Rightarrow \hat{x}_1 \ (100) \\
x_1 < x(n) \leq x_2 \Rightarrow \hat{x}_2 \ (101) \\
x_2 < x(n) \leq x_3 \Rightarrow \hat{x}_3 \ (110) \\
x_3 < x(n) < \infty \Rightarrow \hat{x}_4 \ (111)
\end{align*}
\]

\[
\begin{align*}
x_{-1} < x(n) \leq x_0 &= 0 \Rightarrow \hat{x}_{-1} \ (011) \\
x_{-2} < x(n) \leq x_{-1} \Rightarrow \hat{x}_{-2} \ (010) \\
x_{-3} < x(n) \leq x_{-2} \Rightarrow \hat{x}_{-3} \ (001) \\
-\infty < x(n) \leq x_{-3} \Rightarrow \hat{x}_{-4} \ (000)
\end{align*}
\]

**range**  **level**  **codeword**

- codewords are arbitrary!! => there are good choices that can be made (and bad choices)
Uniform Quantization

- choice of quantization ranges and levels so that signal can easily be processed digitally

mid-riser

\[ x_i - x_{i-1} = \Delta \]

\[ \hat{x}_i - \hat{x}_{i-1} = \Delta \]

\[ \Delta = \text{quantization step size} \]
Mid-Riser and Mid-Tread Quantizers

- **mid-riser**
  - origin \((x=0)\) in middle of rising part of the staircase
  - same number of positive and negative levels
  - symmetrical around origin

- **mid-tread**
  - origin \((x=0)\) in middle of quantization level
  - one more negative level than positive
  - one quantization level of 0 (where a lot of activity occurs)

- code words have direct numerical significance (sign-magnitude representation for mid-riser, two's complement for mid-tread)

  - for **mid-riser** quantizer:
    \[
    \hat{x}(n) = \frac{\Delta}{2} \text{sign}[c(n)] + \Delta c(n)
    \]
    
    where \(\text{sign}[c(n)] = +1\) if first bit of \(c(n) = 0\)
    \[
    = -1\quad \text{if first bit of } c(n) = 1
    \]

  - for **mid-tread** quantizer code words are 3-bit two's complement representation, giving
    \[
    \hat{x}(n) = \Delta c(n)
    \]
A-to-D and D-to-A Conversion

\[ \hat{x}[n] = x[n] - e[n] \]

Quantization error

Original signal

\( e[n] = \hat{x}[n] - x[n] \)
Quantization of a Sine Wave

Illustration of Quantization of a Sinewave

Unquantized sinewave

3-bit quantization waveform

3-bit quantization error

8-bit quantization error
Quantization of Complex Signal

\[ x[n] = \sin(0.1n) + 0.3 \cos(0.3n) \]
Uniform Quantizers

- Uniform Quantizers characterized by:
  - number of levels—$2^B$ (B bits)
  - quantization step size-$\Delta$
- if $|x(n)| \leq X_{max}$ and $x(n)$ is a symmetric density, then
  $$\Delta 2^B = 2 X_{max}$$
  $$\Delta = 2 X_{max}/ 2^B$$
- if we let
  $$\hat{x}(n) = x(n) + e(n)$$
- with $x(n)$ the unquantized speech sample, and $e(n)$ the quantization error (noise), then
  $$-\frac{\Delta}{2} \leq e(n) \leq \frac{\Delta}{2}$$
  (except for last quantization level which can exceed $X_{max}$ and thus the error can exceed $\Delta/2$)
Quantization Noise Model

1. Quantization noise is a zero-mean, stationary white noise process
   \[ E[e(n)e(n + m)] = \sigma_e^2, \quad m = 0 \]
   
   \[ = 0 \quad \text{otherwise} \]

2. Quantization noise is uncorrelated with the input signal
   \[ E[x(n)e(n + m)] = 0 \quad \forall m \]

3. Distribution of quantization errors is uniform over each quantization interval
   \[ p_\delta(e) = \frac{1}{\Delta} \quad -\Delta/2 \leq e \leq \Delta/2 \Rightarrow \bar{e} = 0, \quad \sigma_e^2 = \frac{\Delta^2}{12} \]

   \[ = 0 \quad \text{otherwise} \]
Quantization Examples

Speech Signal

3-bit Quantization Error

8-bit Quantization Error (scaled)

how good is the model of the previous slide?
Typical Amplitude Distributions

LaPlacian Amplitude Distribution

A Laplacian distribution is often used as a model for speech signals.

Amplitude Distribution of 3-bit Quantized Signal

A 3-bit quantized signal has only 8 different values.
3-Bit Speech Quantization

Input to quantizer

3-bit quantization waveform
3-Bit Speech Quantization Error

Input to Quantizer

Quantization Error for 3-bits

3-bit quantization error
5-Bit Speech Quantization

Input to quantizer

5-bit quantization waveform
5-Bit Speech Quantization Error

Input to Quantizer

Quantization Error for 5-bits

5-bit quantization error
Histgrams of Quantization Noise

(a) 3-bit quantization histogram

(b) 8-bit quantization histogram
Spectra of Quantization Noise

Power Spectra of Speech and Quantization Noise

- Speech spectrum
- 2-bit quantizer
- 4-bit quantizer
- 6-bit quantizer
- 8-bit quantizer

\[ \approx 12 \text{ dB} \]
Correlation of Quantization Error

assumptions look good at 8-bit quantization, not as good at 3-bit levels (however only 6 db variation in power spectrum level)
Sound Demo

- “Original” speech sampled at 16kHz, 16 bits/sample
- Quantized to 10 bits/sample
- Quantization error (x32) for 10 bits/sample
- Quantized to 5 bits/sample
- Quantization error for 5 bits/sample
SNR for Quantization

• can determine SNR for quantized speech as

\[
SNR = \frac{\sigma_x^2}{\sigma_e^2} = \frac{E(x^2(n))}{E(e^2(n))} = \frac{\sum x^2(n)}{\sum e^2(n)}
\]

\[
\Delta = \frac{2X_{\text{max}}}{2^B}
\]

(uniform quantizer step size)

• assume \( p(e) = \frac{1}{\Delta} \quad -\frac{\Delta}{2} \leq e \leq \frac{\Delta}{2} \) (uniform distribution)

\[
= 0 \quad \text{otherwise}
\]

\[
\sigma_e^2 = \frac{\Delta^2}{12} = \frac{X_{\text{max}}^2}{(3)2^{2B}}
\]
SNR for Quantization

• can determine SNR for quantized speech as

\[
SNR = \frac{\sigma_x^2}{\sigma_e^2} = \frac{E(x^2(n))}{E(e^2(n))} = \frac{\sum x^2(n)}{\sum e^2(n)}
\]

\[
\sigma_e^2 = \frac{\Delta^2}{12} = (3)2^{2B}
\]

\[
SNR = \left(\frac{3)2^{2B}}{\sum x_{max}^2}ight)^2; \quad SNR(dB) = 10\log_{10}\left[\frac{\sigma_x^2}{\sigma_e^2}\right] = 6B + 4.77 - 20\log_{10}\left[\frac{X_{max}}{\sigma_x}\right]
\]

• if we choose \(X_{max} = 4\sigma_x\), then \(SNR = 6B - 7.2\)

\(B = 16, \quad SNR = 88.8 \ dB\)

\(B = 8, \quad SNR = 40.8 \ dB\)

\(B = 3, \quad SNR = 10.8 \ dB\)

The term \(X_{max}/\sigma_x\) tells how big a signal can be accurately represented.
Variation of SNR with Signal Level

Overload from clipping

SNR improves 6 dB/bit, but it decreases 6 dB for each halving of the input signal amplitude
Clipping Statistics

Percentage of Clipped Samples for 8-Bit Quantizer

Percent Clipped Samples

$X_{max}/\sigma_x$
Review--Linear Noise Model

\[ E\{ (x[n])^2 \} = \sigma_x^2 \]

- assume speech is stationary random signal.
- error is uncorrelated with the input.
  \[ E\{x[n]e[n]\} = E\{x[n]\}E\{e[n]\} = 0 \]
- error is uniformly distributed over the interval
  \[ -\left(\frac{\Delta}{2}\right) < e[n] \leq \left(\frac{\Delta}{2}\right). \]
- error is stationary white noise, (i.e. flat spectrum)
  \[ P_e(\omega) = \sigma_e^2 = \frac{\Delta^2}{12}, \quad |\omega| \leq \pi \]
Review of Quantization Assumptions

1. input signal fluctuates in a complicated manner so a statistical model is valid
2. quantization step size is small enough to remove any signal correlated patterns in quantization error
3. range of quantizer matches peak-to-peak range of signal, utilizing full quantizer range with essentially no clipping
4. for a uniform quantizer with a peak-to-peak range of \( \pm 4 \sigma_x \), the resulting \( SNR(dB) \) is \( 6B-7.2 \)
Uniform Quantizer SNR Issues

\[ SNR = 6B - 7.2 \]

- to get an SNR of at least 30 dB, need at least \( B \geq 6 \) bits (assuming \( X_{\text{max}} = 4 \sigma_x \))
  - this assumption is weak across speakers and different transmission environments since \( \sigma_x \) varies so much (order of 40 dB) across sounds, speakers, and input conditions
  - \( SNR(dB) \) predictions can be off by significant amounts if full quantizer range is not used; e.g., for unvoiced segments => need more than 6 bits for real communication systems, more like 11-12 bits
  - need a quantizing system where the SNR is independent of the signal level => constant percentage error rather than constant variance error => need non-uniform quantization
Instantaneous Companding

- to order to get constant percentage error (rather than constant variance error), need logarithmically spaced quantization levels
  - quantize logarithm of input signal rather than input signal itself
\( y(n) = \ln | x(n) | \)

\( x(n) = \exp[y(n)] \cdot \text{sign}[x(n)] \)

- where \( \text{sign}[x(n)] = +1 \) if \( x(n) \geq 0 \)
  \( = -1 \) if \( x(n) < 0 \)

- the quantized log magnitude is

\( \hat{y}(n) = Q[\log | x(n) |] \)

\( = \log | x(n) | + \varepsilon(n) \) new error signal
μ-Law Companding

• assume that \( \varepsilon(n) \) is independent of \( \log| x(n) | \). The inverse is
\[
\hat{x}(n) = \exp[\hat{y}(n)] \cdot \text{sign}[x(n)] \\
= |x(n)| \cdot \text{sign}[x(n)] \cdot \exp[\varepsilon(n)] \\
= x(n) \cdot \exp[\varepsilon(n)]
\]

• assume \( \varepsilon(n) \) is small, then \( \exp[\varepsilon(n)] \approx 1 + \varepsilon(n) + ... \)
\[
\hat{x}(n) = x(n)[1 + \varepsilon(n)] = x(n) + \varepsilon(n)x(n) = x(n) + f(n)
\]

• since we assume \( x(n) \) and \( \varepsilon(n) \) are independent, then
\[
\sigma_f^2 = \sigma_x^2 \cdot \sigma_{\varepsilon}^2 \\
\text{SNR} = \frac{\sigma_x^2}{\sigma_f^2} = \frac{1}{\sigma_{\varepsilon}^2}
\]

• \( \text{SNR} \) is independent of \( \sigma_x^2 \)--it depends only on step size
Pseudo-Logarithmic Compression

- Unfortunately true logarithmic compression is not practical, since the dynamic range (ratio between the largest and smallest values) is infinite => need an infinite number of quantization levels
- Need an approximation to logarithmic compression => \( \mu \)-law/A-law compression
\( y(n) = F[x(n)] \)

\[
\begin{align*}
\log \left[ 1 + \mu \left| \frac{x(n)}{X_{\text{max}}} \right| \right] \\
= X_{\text{max}} \frac{\log(1 + \mu)}{\log(1 + \mu)} \cdot \text{sign}[x(n)]
\end{align*}
\]

- when \( x(n) = 0 \) \( \Rightarrow \) \( y(n) = 0 \)
- when \( \mu = 0 \), \( y(n) = x(n) \) \( \Rightarrow \) linear compression
- when \( \mu \) is large, and for large \( |x(n)| \)

\[
|y(n)| \approx \frac{X_{\text{max}}}{\log \mu} \cdot \log \left[ \frac{\mu |x(n)|}{X_{\text{max}}} \right]
\]
Histogram for $\mu$-Law Companding

Histogram of Speech

Histogram of $\mu=255$ Quantizer

Speech waveform

Output of $\mu$-Law compander
μ-law approximation to log

- μ-law encoding gives a good approximation to constant percentage error \(|y(n)| \approx \log|x(n)|\)
SNR for $\mu$-law Quantizer

$$SNR(dB) = 6B + 4.77 - 20\log_{10} \left[ \ln(1 + \mu) \right] - 10\log_{10} \left[ 1 + \left( \frac{X_{\text{max}}}{\mu \sigma_x} \right)^2 + \sqrt{2} \left( \frac{X_{\text{max}}}{\mu \sigma_x} \right) \right]$$

- $6B$ dependence on $B \Rightarrow \text{good}$
- much less dependence on $\frac{X_{\text{max}}}{\sigma_x} \Rightarrow \text{good}$
- for large $\mu$, $SNR$ is less sensitive to changes in $\frac{X_{\text{max}}}{\sigma_x} \Rightarrow \text{good}$
  - $\mu$-law quantizer used in wireline telephony for more than 40 years
μ-Law Companding

(a) Unquantized Speech Signal

(b) Output of 255-Law Compressor

(c) Quantization Error for Compander

Mu-law compressed signal utilizes almost the full dynamic range (±1) much more effectively than the original speech signal.
μ-Law Quantization Error

(a) Histogram of Input Speech

(b) Histogram of Output of μ = 255 Quantizer

(c) Histogram of 8-Bit Companded Quantization Error
Comparison of Linear and μ-law Quantizers

Dashed line – linear (uniform) quantizers with 6, 7, 8 and 12 bit quantizers

Solid line – μ-law quantizers with 6, 7 and 8 bit quantizers (μ=100)

- can see in these plots that $X_{max}$ characterizes the quantizer (it specifies the ‘overload’ amplitude), and $\sigma_x$ characterizes the signal (it specifies the signal amplitude), with the ratio ($X_{max}/\sigma_x$) showing how the signal is matched to the quantizer
Comparison of Linear and $\mu$-law Quantizers

Dashed line – linear (uniform) quantizers with 6, 7, 8 and 13 bit quantizers

Solid line – $\mu$-law quantizers with 6, 7 and 8 bit quantizers ($\mu=255$)
Analysis of μ-Law Performance

- curves show that μ-law quantization can maintain roughly the same SNR over a wide range of $X_{\text{max}}/\sigma_x$, for reasonably large values of $\mu$
  - for $\mu=100$, SNR stays within 2 dB for $8 < X_{\text{max}}/\sigma_x < 30$
  - for $\mu=500$, SNR stays within 2 dB for $8 < X_{\text{max}}/\sigma_x < 150$
  - loss in SNR in going from $\mu=100$ to $\mu=500$ is about 2.6 dB => rather small sacrifice for much greater dynamic range
  - $B=7$ gives $\text{SNR}=34$ dB for $\mu=100$ => this is 7-bit μ-law PCM-the standard for toll quality speech coding in the PSTN => would need about 11 bits to achieve this dynamic range and SNR using a linear quantizer
CCITT G.711 Standard

- **Mu-law** characteristic is approximated by 15 linear segments with uniform quantization within a segment.
  - uses a mid-tread quantizer. +0 and –0 are defined.
  - decision and reconstruction levels defined to be integers

- **A-law** characteristic is approximated by 13 linear segments
  - uses a mid-riser quantizer
Summary of Uniform and $\mu$-Law PCM

- Quantization of sample values is unavoidable in DSP applications and in digital transmission and storage of speech.
- We can analyze quantization error using a random noise model.
- The more bits in the number representation, the lower the noise. The ‘fundamental theorem’ of uniform quantization is that “the signal-to-noise ratio increases 6 dB with each added bit in the quantizer”; however, if the signal level decreases while keeping the quantizer step-size the same, it is equivalent to throwing away one bit for each halving of the input signal amplitude.
- $\mu$-law compression can maintain constant SNR over a wide dynamic range, thereby reducing the dependency on signal level remaining constant.
Quantization for Optimum SNR (MMSE)

• goal is to match quantizer to actual signal density to achieve optimum SNR
  – $\mu$-law tries to achieve constant SNR over wide range of signal variances => some sacrifice over SNR performance when quantizer step size is matched to signal variance
  – if $\sigma_x$ is known, you can choose quantizer levels to minimize quantization error variance and maximize SNR
Quantization for MMSE

- if we know the size of the signal (i.e., we know the signal variance, $\sigma_x^2$) then we can design a quantizer to minimize the mean-squared quantization error.
  - the basic idea is to quantize the most probable samples with low error and least probable with higher error.
  - this would maximize the SNR
  - general quantizer is defined by defining $M$ reconstruction levels and a set of $M$ decision levels defining the quantization “slots”.
Quantizer Levels for Maximum SNR

- variance of quantization noise is:
  \[ \sigma_e^2 = E\left[ e^2(n) \right] = E\left[ (\hat{x}(n) - x(n))^2 \right] \]

- with \( \hat{x}(n) = Q[x(n)] \). Assume \( M \) quantization levels
  \[ \left[ \hat{x}_{-(M/2)}, \hat{x}_{-(M/2)+1}, \ldots, \hat{x}_{-1}, \hat{x}_1, \ldots, \hat{x}_{(M/2)} \right] \]

- associating quantization level with signal intervals as:
  \( \hat{x}_j = \) quantization level for interval \( [x_{j-1}, x_j] \)

- for symmetric, zero-mean distributions, with large amplitudes (\( \infty \))
  it makes sense to define the boundary points:
  \( x_0 = 0 \) (central boundary point), \( x_{\pm M/2} = \pm \infty \)

- the error variance is thus
  \[ \sigma_e^2 = \int_{-\infty}^{\infty} e^2 p_e(e) de \]
Optimum Quantization Levels

- by definition, \( e = \hat{x} - x \); thus we can make a change of variables to give:
  \[ p_e(e) = p_e(\hat{x} - x) = p_{x/\hat{x}}(x / \hat{x}) = p_x(x) \]
- giving
  \[ \sigma_e^2 = \sum_{i=-M/2+1}^{M/2} \int_{x_{i-1}}^{x_i} (\hat{x}_i - x)^2 p_x(x)dx \]
- assuming \( p(x) = p(-x) \) so that the optimum quantizer is antisymmetric, then
  \( \hat{x}_i = -\hat{x}_{-i} \) and \( x_i = -x_{-i} \)
- thus we can write the error variance as
  \[ \sigma_e^2 = 2 \sum_{i=1}^{M/2} \int_{x_{i-1}}^{x_i} (\hat{x}_i - x)^2 p_x(x)dx \]
- goal is to minimize \( \sigma_e^2 \) through choices of \( \{x_i\} \) and \( \{\hat{x}_i\} \)
Solution for Optimum Levels

- to solve for optimum values for \( \{\hat{x}_i\} \) and \( \{x_i\} \), we differentiate \( \sigma_e^2 \) wrt the parameters, set the derivative to 0, and solve numerically:

\[
\int_{x_{i-1}}^{x_i} (\hat{x}_i - x)^2 p_x(x)dx = 0 \quad i = 1, 2, ..., M/2 \quad (1)
\]

\[
x_i = \frac{1}{2} (\hat{x}_i + \hat{x}_{i+1}) \quad i = 1, 2, ..., M/2 - 1 \quad (2)
\]

- with boundary conditions of \( x_0 = 0, \ x_{\pm M/2} = \pm \infty \) \( (3) \)

- can also constrain quantizer to be uniform and solve for value of \( \Delta \) that maximizes SNR
  - optimum boundary points lie halfway between \( M/2 \) quantizer levels
  - optimum location of quantization level \( \hat{x}_i \) is at the centroid of the probability density over the interval \( x_{i-1} \) to \( x_i \)

  - solve the above set of equations \( (1,2,3) \) iteratively for \( \{\hat{x}_i\}, \{x_i\} \)
## Optimum Quantizers for Laplace Density

Assumes $\sigma_x = 1$

<table>
<thead>
<tr>
<th>$N$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
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</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$x_i$</td>
<td>$\hat{x}_i$</td>
<td>$x_i$</td>
<td>$\hat{x}_i$</td>
<td>$x_i$</td>
</tr>
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<td>1</td>
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<td>1.102</td>
<td>0.395</td>
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<td>1.810</td>
<td>1.181</td>
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<td>2.285</td>
<td>1.576</td>
<td>0.910</td>
<td>0.726</td>
</tr>
<tr>
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<td>1.095</td>
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<tr>
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<td>1.540</td>
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<tr>
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<td>2.103</td>
<td>1.031</td>
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<tr>
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<td>2.895</td>
<td>1.250</td>
<td>1.136</td>
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<tr>
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<td></td>
<td>1.490</td>
<td>1.365</td>
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<tr>
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<td>4.371</td>
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</tr>
<tr>
<td>16</td>
<td>$\infty$</td>
<td>5.768</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$MSE$ | 0.5 | 0.1765 | 0.0548 | 0.0154 | 0.00414 |

$SNR$ dB | 3.01 | 7.53 | 12.61 | 18.12 | 23.83 |

Optimum Quantizer for 3-bit Laplace Density; Uniform Case

- Quantization levels get further apart as the probability density decreases.
- Step size decreases roughly exponentially with increasing number of bits.
Performance of Optimum Quantizers

Table 5.3 Signal-to-Noise Ratios for 3-bit Quantizers. (After Noll [12]).

<table>
<thead>
<tr>
<th>Nonuniform Quantizers</th>
<th>SNR (dB)</th>
<th>Smallest Level ((\sigma_x=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu)-law ((x_{\text{max}} = 8\sigma_x, \mu=100))</td>
<td>9.5</td>
<td>0.062</td>
</tr>
<tr>
<td>Gaussian</td>
<td>14.6</td>
<td>0.245</td>
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<tr>
<td>Laplace</td>
<td>12.6</td>
<td>0.222</td>
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<tr>
<td>Gamma</td>
<td>11.5</td>
<td>0.149</td>
</tr>
<tr>
<td>Speech</td>
<td>12.1</td>
<td>0.124</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Uniform Quantizers</th>
<th>SNR</th>
<th>Smallest Level ((\sigma_x=1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>14.3</td>
<td>0.293</td>
</tr>
<tr>
<td>Laplace</td>
<td>11.4</td>
<td>0.366</td>
</tr>
<tr>
<td>Gamma</td>
<td>11.5</td>
<td>0.398</td>
</tr>
<tr>
<td>Speech</td>
<td>8.4</td>
<td>0.398</td>
</tr>
</tbody>
</table>
Summary

• examined a statistical model of speech showing probability densities, autocorrelations, and power spectra
• studied instantaneous uniform quantization model and derived SNR as a function of the number of bits in the quantizer, and the ratio of signal peak to signal variance
• studied a companded model of speech that approximated logarithmic compression and showed that the resulting SNR was a weaker function of the ratio of signal peak to signal variance
• examined a model for deriving quantization levels that were optimum in the sense of matching the quantizer to the actual signal density, thereby achieving optimum SNR for a given number of bits in the quantizer