The Nyquist Feedback Stability Criterion

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Feedback loop stability

\[ A_{CL}(s) = \frac{A_{OL}(s)}{1 + A_{OL}(s)\beta(s)} = \frac{A_{OL}(s)}{1 + T(s)} \]
\[ = \frac{N(s)}{D(s)} \]

We want to know whether \( A_{CL}(s) \) has any poles in the right half of the S-plane.

Key point 1: poles of \( A_{CL}(s) \) are zeros of \( D(s) = 1 + A_{OL}(s)\beta(s) \)
Key point 2: zeros of \( A_{CL}(s) \) are poles of \( D(s) = 1 + A_{OL}(s)\beta(s) \)
We have a variable $s$. We have a function $F(s)$.

First: the trivial function $F(s) = s$

If we move the point $s$ around the $s$ – plane,
The point $F(s)$ moves in an identical trajectory (of course).
Walking around the S-plane (2): a zero

Now consider a zero \( F(s) = s - s_z \)

\( s \to F(s) \)

If we move the point \( s \) once in a clockwise circle around the zero, then the point \( F(s) \) moves in one clockwise circle around the origin.
Given that $F(s) = s - s_z$, the angle of the point $s$ with respect to the zero has to equal to the angle of the point $F(s)$ with respect to the origin.

So, when $s$ circles the zero, $F(s)$ must circle the origin, and clockwise circling leads to clockwise circling.
Walking around the S-plane (3): missing the zero

\[ F(s) = s - s_z \]

If our path in the s-plane does not circle the zero, then the path in the F(s) plane will not circle the origin.
Walking around the S-plane (3): multiple zeros

\[ F(s) = (s - s_{z1})(s - s_{z2}) \cdots (s - s_{zM}) \]

We can now see that, if our path in the s-plane wraps around \( N \) zeros, going clockwise, then the path in the F(s) plane will circle the origin \( N \) times, going clockwise.
Walking around the S-plane (2): a pole

Now consider a pole \( F(s) = \frac{1}{(s - s_p)} \)

\[ s \rightarrow F(s) \]

Note that because \( \angle \left( \frac{1}{(s - s_z)} \right) = -1 \cdot \angle(s - s_z) \),
the angle has **changed sign**.

(Also, the radius has inverted, but that is not important here.)

If we move the point \( s \) once in a *clockwise* circle around the pole, then
the point \( F(s) \) moves in one *counter-clockwise* circle around the origin.
Our prize: Cauchy's principle

Let us travel clockwise around a closed loop in the s-plane which wraps around $Z$ zeros and $P$ poles.

Then $F(s)$ will wrap $N$ times clockwise around the origin, where

$$N = Z - P$$
Towards Nyquist's criterion

If \( s \) follows the marked trajectory, then the number of times \(*N*\) that \((1+T(s))\) circles the origin, in a clockwise direction, equals the number of zeros, \(Z\), in \((1+T(s))\), minus the number of poles, \(P\), in \((1+T(s))\),

\[
N = Z - P, \quad \text{or} \quad Z = P + N
\]
Towards Nyquist's criterion

But: $Z = \# \text{ unstable poles in } A_{CL}(s)$, the closed loop gain
and: $P = \# \text{ unstable poles in } A_{OL}(s)\beta(s)$, the loop transmission.

So: $Z = P + N$, where
$Z = \# \text{ unstable poles in } A_{CL}(s)$, the closed loop gain
$P = \# \text{ unstable poles in } A_{OL}(s)\beta(s)$, the loop transmission.
$N = \# \text{ times (1+T(s)) wraps clockwise around the origin}$
Nyquist's criterion (finally)

Let's plot $T(s)$ instead of $(1+T(s))$.

So: $Z = P + N$, where

$Z = \# \text{ unstable poles in } A_{CL}(s), \text{ the closed loop gain}$

$P = \# \text{ unstable poles in } A_{OL}(s)\beta(s), \text{ the loop transmission}$

$N = \# \text{ times } T(s) \text{ wraps clockwise around the point } (-1+j0)$
Nyquist's criterion: simplified case: stable before feedback

Nyquist criterion applies even for systems which are unstable before feedback is applied! Example: pitch (nose up/down) control on some fighter planes.

NOW: let's consider cases where the system is stable before feedback is applied. In that case: \( P = \# \text{unstable poles in } A_{OL}(s)\beta(s),\text{ the loop transmission} = *\text{zero}* \)

In that case: \( Z = N, \text{ where } \)

\( Z = \# \text{unstable poles in } A_{CL}(s), \text{ the closed loop gain} \)

\( N = \# \text{times } T(s) \text{ wraps clockwise around the point } (-1+j0) \)
Nyquist stability test: feedback with one pole

Here the loop transmission has one pole. 

T(s), in the Nyquist test, does not wrap around the point (-1+j0)
Nyquist stability test: feedback with two poles

Here the loop transmission has two poles.

$T(s)$, in the Nyquist test, still does not wrap around the point $(-1+j0)$.
Nyquist stability test: feedback with three poles

Here the loop transmission has three poles. Depending on the numerical parameters, \( T(s) \), in the Nyquist test, might wrap twice clockwise around the point \((-1+j0)\).

\( \rightarrow \) Two unstable poles in \( A_{CL}(s) \)
Here the loop transmission has three poles and two zeros
Depending on the numerical parameters, as shown
T(s), in the Nyquist test, might wrap *zero times* clockwise around the point (-1+j0).
→ No unstable poles in $A_{CL}(s)$