ECE202A 11/22/89

Noise!

Why build an amplifier? Because we want to turn a small signal into a larger one.

Relevant questions are then:

How big can the output signal get?
(Power, distortion)

and

How small can the input signal get?

(and still see it) → Noise

A device with neither low noise nor significant output power probably isn't much good to anyone.

Noise is Many topics, not one:

Mathematics: Probability theory, distributions, ...

Physics: The origins of random processes

Device theory: Noise Models of devices
How to design low-noise devices.

Circuit Theory: How device noise acts in circuit
How to best take advantage of low device noise.

or "Black Box" circuit theory: Noise figure circles, noise match.

System theory: Ok, you've got noise... How do you process your signals to minimize errors (data errors, radar recognition, ...

Information theory: optimum methods of modulating or coding and decoding messages so as to maximize information transmission rate.

ECE 202A- Lectures on Noise

The first step-random Variables

A single random variable X:

the random variable X can take on any particular value x. The probability that x lies between limits A and B is given by the probability density function $f_X(x)$.

$$P\{A < x < B\} = \int_A^B f_X(x)dx$$

A common probability density function is the Gaussian distribution:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\frac{-(x-\bar{x})^2}{2\sigma_x^2}\right)$$

(sketch on the board)

Mean values and expectations

The expectation of a function g(x) is as follows:

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x) f_X(x) dx$$

Important definitions using this are the mean value or average value of x

$$\langle X \rangle = \overline{X} = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

and the expected value of x^2:

$$\langle X^2 \rangle = E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx$$

the variance of x is the R.M.S. deviation of x from its mean value:

$$\sigma_X^2 = \langle (X - \bar{x})^2 \rangle = E[(X - \bar{x})^2] = \int_{-\infty}^{+\infty} (x - \bar{x})^2 f_X(x) dx$$

The standard deviation σ_X is the square root of the variance.

The notation describing the Gaussian distribution should now be clear

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(\frac{-(x-\bar{x})^2}{2\sigma_x^2} \right)$$

An important identity is as follows:

$$\begin{aligned}
\sigma_X^2 &= \langle (X - \bar{x})^2 \rangle = \langle (X - \bar{x})(X - \bar{x}) \rangle \\
&= \langle X^2 - 2X\bar{x} + (\bar{x})^2 \rangle \\
&= \langle X^2 \rangle - 2\bar{x}\langle X \rangle + \langle (\bar{x})^2 \rangle \\
&= \langle X^2 \rangle - 2\bar{x} \bullet \bar{x} + (\bar{x})^2 \\
&= \langle X^2 \rangle - (\bar{x})^2
\end{aligned}$$

In other words, the variance is equal to the expectation of X^2 minus the square of the expectation of x.

Pairs of random variables

In order to study random processes we must first understand pairs of RVs and correllations

A pair of random variables X and Y can take on values x and y. Again we must describe this with a (joint) probability density function $f_{XY}(x,y)$

$$P\{A < x < B \text{ and } C < y < D\} = \int_C^D \int_A^B f_{XY}(x,y)dxdy$$

Marginal distributions must also be defined:

$$P\{A < x < B\} = \int_{-\infty}^{+\infty} \int_A^B f_{XY}(x,y)dxdy$$

$$= \int_A^B f_X(x)dx$$

and similarly for Y:

$$P\{C < y < D\} = \int_C^D \int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy$$

$$= \int_C^D f_Y(y)dy$$

In the case where

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

the variable are said to be *independent*. This is not generally expected.

Expectations of a pair of random variables

Again we must define expectations. The expectation of a function g(X,Y) of the random variables Y and Y is.

$$E[g(x,y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) f_{XY}(x,y)dxdy$$

The expectation of X is

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{+\infty} x f_X(x)dxdy \end{aligned}$$

and the expectation of X^2 is

$$\begin{aligned} E[X^2] = \langle X^2 \rangle &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{+\infty} x^2 f_X(x)dxdy \end{aligned}$$

and similarly for Y and Y^2.

The important expectations now follow. The correlation of X and Y is defined as

$$R_{XY} = E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \bullet f_{XY}(x,y)dxdy$$

and the covariance of X and Y is defined as

$$\begin{aligned} C_{XY} &= E[(X-\overline{x})(Y-\overline{y})] \\ &= E[XY - \overline{x}Y - X\overline{y} + \overline{xy}] \\ &= R_{XY} - \overline{xy} \end{aligned}$$

Note that the correllation and the covariance are the same if the random variables X and Y have zero mean.

<u>We are working with noise in circuits. Noise will be a perturbation of the circuit voltages (etc) about the DC bias conditions. Since we are dealing with DC bias separately, all random variables will therefore have zero mean. This will make our lives much easier. We will therefore get sloppy and talk about correllation and covariance as if they were the same.</u>

Jointly Gaussian Random Variables

To make my life easier, I will assume zero means throughout. Define a random vector $\vec{X}$ (arrows will be used for vectors to avoid confusion with expected values, while matrices will be denoted by an underbar)

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}$$

To write these neatly, lets define some intermediate terms. The correllation (or covariance-- since zero mean) matrix is as below, the matrix of the individual correllations

$$R_{\vec{X}} = E\left[\vec{X}\vec{X}^T\right] = E\left[\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix}\begin{bmatrix} X_1 & \cdots & X_N \end{bmatrix}\right]$$

$$= \begin{bmatrix} R_{X_1X_1} & \cdots & R_{X_1X_N} \\ \vdots & \ddots & \vdots \\ R_{X_NX_1} & \cdots & R_{X_NX_N} \end{bmatrix}$$

$\vec{X}$ is jointly Gaussian if it has the following distribution

$$f_{\vec{X}}(x) = \frac{1}{(2\pi)^{N/2}\det(R_{\vec{X}})}\exp\left[-\frac{1}{2}\vec{X}^T R_{\vec{X}}^{-1}\vec{X}\right]$$

1/2.

Sum of TWO Random Variables

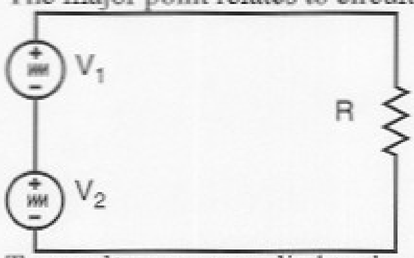
One important case we need to look at is the sum of two random variables X and Y:
Z=Y+Y both X and Y being zero-mean.

$$
\begin{aligned}
E[Z^2] &= \langle Z^2 \rangle = \langle (X+Y)^2 \rangle \\
&= \langle X^2 + 2XY + Y^2 \rangle \\
&= \langle X^2 \rangle + 2C_{XY} + \langle Y^2 \rangle \\
&= \langle X^2 \rangle + 2\sigma_{XY} + \langle Y^2 \rangle \\
&= \langle X^2 \rangle + \langle Y^2 \rangle + 2\rho_{XY}\sigma_X\sigma_Y
\end{aligned}
$$

We've assumed zero means. I have implicitly defined a number of variables here. I have given the covariance (correllation) another name $\sigma_{XY} = C_{XY} = \rho_{XY}\sigma_X\sigma_Y$ and defined a correllation coefficient which is the covariance normalized to the individual standard deviations.

Two random variables are said to be uncorrellated if their covariance is zero. "uncorrellated" is a somewhat weaker statement than "independent", but the two become the same for Jointly Gaussian random variables, which will be our major concern .

The major point relates to circuits:



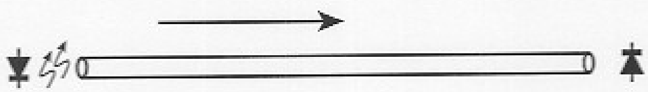
Two voltages are applied to the resistor R
The power dissipated in the resistor is a random variable P

$$
\begin{aligned}
E[P] &= \langle P \rangle = \frac{1}{R}\langle (V_1 + V_2)^2 \rangle \\
&= \frac{1}{R}\langle V_1^{\,2} + 2V_1V_2 + V_2^{\,2} \rangle \\
&= \frac{1}{R}\langle V_1^{\,2} \rangle + \frac{1}{R}2C_{V_1V_2} + \frac{1}{R}\langle V_2^{\,2} \rangle \\
&= \frac{1}{R}\langle V_1^{\,2} \rangle + \frac{1}{R}2\sigma_{V_1V_2} + \frac{1}{R}\langle V_2^{\,2} \rangle \\
&= \frac{1}{R}\langle V_1^{\,2} \rangle + \frac{1}{R}\langle Y^2 \rangle + \frac{1}{R}2\rho_{V_1V_2}\sigma_{V_1}\sigma_{V_2} \\
&= \frac{1}{R}\langle V_1^{\,2} \rangle + \frac{1}{R}2\langle V_1V_2 \rangle + \frac{1}{R}\langle V_2^{\,2} \rangle
\end{aligned}
$$

This is an essential point. The noise powers of two random generators do not add--a correllation term must be included. Their random noise voltages do add.

Some random variable examples
These are not "pretty" examples, but things which we need to know later

shot noise.



Above is an optical fiber illuminated by an optical source which has produced an input pulse with N photons. The attenuation of the fiber is p: how many photons M reach the detector?

It is tempting to say that the number m of photons reaching the detector is m=Np, but think: if we send one fiber, and the attenuation is 1/2, do we get 1/2 a photon at the receiving end?

Instead, each photon passes through the fiber with probability p. The probability of receiving m photons is given by basic combinational arguments:

$$P\{M=m\}=\binom{N}{m}p^m(1-p)^{N-m}$$

The funny term is the binomial coefficient

$$\binom{N}{m}=\frac{N!}{m!(N-m)!}$$

The mean value of M is Np.

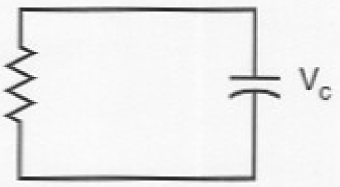
How does this behave under certain limiting conditions? Depending on N and p, the binomial distribution can converge on several different limiting cases. If N becomes very large and p very small, such that Np remains finite but is large compared to one, then the binomial distribution converges on a Gaussian distribution with mean value and variance

$$\langle M\rangle = Np$$

$$\sigma_M^2=\langle M\rangle$$

The variance is equal to the expected (mean) value. This is shot noise, which arises from a large number of independent events.

Thermal Noise



A resistor R is connected to a capacitor C. The resistor is in thermal equilibrium with a "reservoir" (a warm room) at temperature T--in other words R can exchange energy with the room in the form of heat. C can dissipate no power and hence can establish thermal equilibrium with the room only through the resistor.

From statistical thermodynamics, any independent degree of freedom of a system at temperature T must have mean energy kT/2, hence

$$\langle E \rangle = kT / 2$$
$$\langle CV^2 / 2 \rangle = kT / 2$$
$$\langle V^2 \rangle = kT / C$$

the noise voltage has a variance of kT/C! More on this later.

Random Processes

We now at last have enough background to tackle Random processes

Noise is a random process, a voltage or current whose magnitude varies in a random fashion with time. To describe it, we must be able to describe both the probability of the voltage taking on a particular variable, and of how the value of the voltage at one time is related to the voltage at another time. In general, this is an unmanageable problem

A random process is a random variable X(t) varying with time. If we pick times t_1, t_2, etc, then $X_1=X(t_1)$, etc, then X_1,X_2, etc. are a random variables.

First, remember not to confuse how the process varies with voltage (probability distribution) with how it varies with time (its power spectrum or autocorrellation function).

We will make following restrictions on the random variable in order to make the problem concise and tractable. Without these we have no hope:

The process will be **Ergodic**. This is hard to explain. An Ergodic process has the same statistics over time as it has over its statistical ensemble. Ask if you want me to explain this.

The process will be **stationary**. This means that all statistical properties are independent of time. A surprising number of processes are non stationary, including the "drunken sailor's walk", and 1/f noise.

The process will be **Jointly Gaussian**. This means that if the values of a random process X(t) are sampled at times t_1, t_2, etc, to form random variables $X_1=X(t_1)$, etc, then X_1,X_2, etc. are a jointly Gaussian random variable.

$$f_{\vec{X}}(x) = \frac{1}{(2\pi)^{N/2}\det(R_{\vec{X}})}\exp\left[-\frac{1}{2}\vec{X}^T R_{\vec{X}}^{-1}\vec{X}\right]$$

The central limit theorem saves us here. If a random variable arises from the result of a sum of a very large number of random stimuli, then it converges on a Gaussian.

Variation of a random process with time

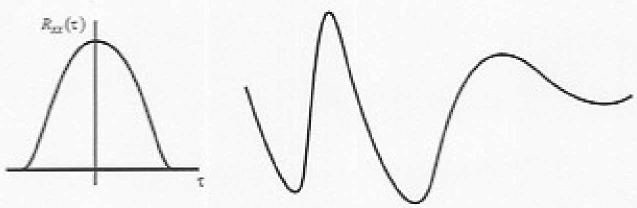
For the random process X(t), look at $X_1=X(t_1)$ and $X_2=X(t_2)$.

$$R_{X_1X_2} = E[X_1X_2] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1x_2 \bullet f_{X_1X_2}(x_1,x_2)dx_1dx_2$$

To compute this we need to know the joint probability distribution. We have assumed a Gaussian process. The above is called the Autocorrellation function. IF the process is stationary, it is a function only of $(t_1\text{-}t_2)$=tau, and hence

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

this is the autocorrellation function. It describes how rapidly a random voltage varies with time



Note that $R_{XX}(0) = E[X(t)X(t)] = \sigma_X^2$ gives the variance of the random process.

The autocorrelation function gives us variance of the random process and the correlation between its values for two moments in time. If the process is Gaussian, this is enough to completely describe the process.

Estimation

I haven't described estimation, so the value of a correllation is not yet clear to you.

If we know that two random variable X and Y are **Jointly Gaussian**, and have **zero mean**, then knowledge of one leads to a best estimate of the other as follows:

$$E[X|Y=y] = \langle X|Y=y \rangle = \frac{R_{XY}}{\sigma_Y^2} y$$

"The expected value of the random variable X, given that the random variable Y has value y is ..."

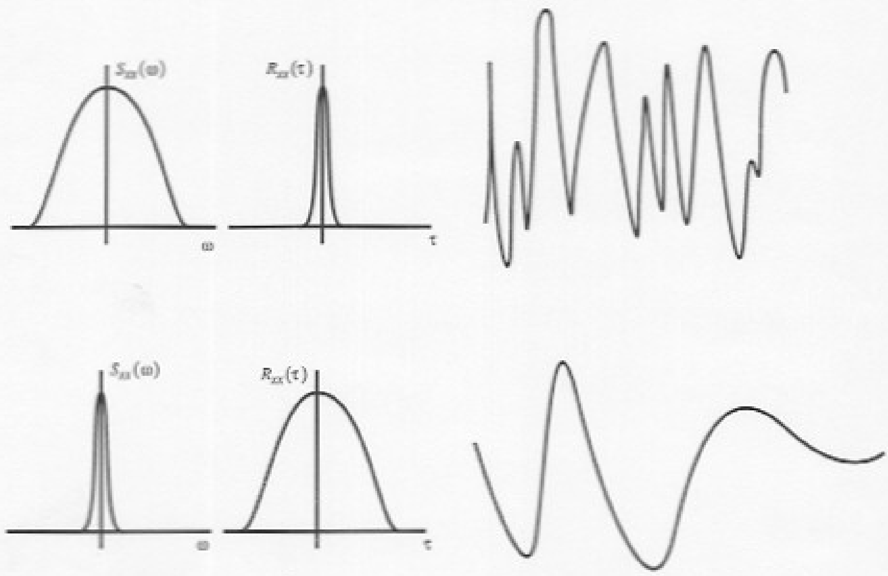
$$\left\langle \left(X - \langle X|Y=y \rangle \right)^2 \right\rangle$$

<u>Power spectral densities.</u>

The autocorrellation function describes how a random process evolves with time. Find its Fourier transform:

$$S_{XX}(\omega)=\int_{-\infty}^{+\infty} R_{XX}(\tau)\exp(-j\omega\tau)d\tau$$

This is called (somewhat confusingly) the power spectrum of the signal.



Remembering the usual Fourier transform relationships, if the power spectrum is broad, the autocorrellation function is narrow, and the signal varies rapidly--it has content at high frequencies, and the voltages of any two points are strongly related only if the two points are close together in time.

If the power spectrum is narrow, the autocorrellation function is broad, and the signal varies slowly--it has content only at low frequencies and the voltages of any two points are strongly related unless if the two points are broadly separated in time.

The inverse transform holds, so that

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)\exp(-j\omega\tau)d\omega$$

specifically

$$R_{XX}(0) = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega$$

So if σ_X^2 is called the power in the process, then integrating the power spectral density versus frequency will give us the power. The name makes sense

In working signal processing problems, the above definitions are popular. In device or circuit problems, we use a bit different notation:

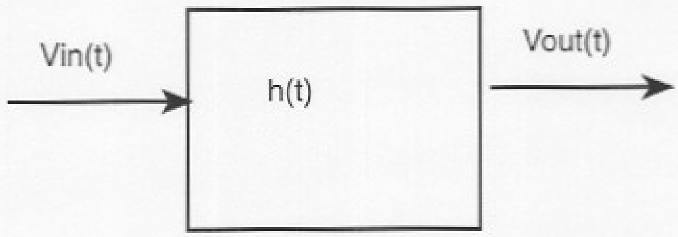
$$\langle V_N^2(t)\rangle = \int_0^{+\infty}\left[\frac{d\langle V_N^2(t)\rangle}{df}\right]df$$

The spectral density of a random (noise) voltage V_N is written as

$$\frac{d\langle V_N^2(t)\rangle}{df}.$$

Note that in addition to the change in symbology, that the units are now Hertz, and we are integrating from DC, not f=-∞. The former type of spectral densities are "double-sided, radian-based", while the latter are "single-sided, Hz-based". We integrate the spectral density to get the variance of the process, which (if the voltage is applied across a 1Ω resistor) gives us the mean (expected) power dissipation. Hence the name "power spectral density"

Noise passing through "filters" (linear electrical networks)



If Vin(t) has power spectral density

$$\frac{d\langle V_{in}^2(t)\rangle}{df}$$

and the filter's transfer function is H(f) (Fourier transform of h(t)), then

$$\frac{d\langle V_{out}^2(t)\rangle}{df} = \|H(f)\|^2 \times \frac{d\langle V_{in}^2(t)\rangle}{df}$$

Correlated Random Processes

Yes, two processes can **and often are** related. Suppose we have two random processes X(t) and Y(t). We can define the cross-correllation function of the process as follows

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

They will have a **cross-spectral density** as follows:

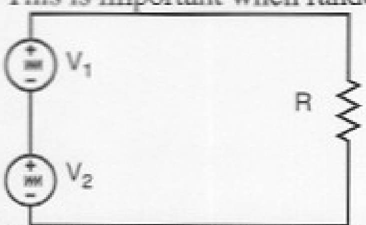
$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau)\exp(-j\omega\tau)d\tau$$

again, in circuits we use a bit different formalism:

$$\langle X(t)Y(t)\rangle = \int_0^{+\infty}\left[\frac{d\langle X(t)Y(t)\rangle}{df}\right]df$$

e.g. the cross spectral density is written as $\dfrac{d\langle X(t)Y(t)\rangle}{df}$.

This is important when random processes add:



Two voltages are applied to the resistor R

$$V = V_1 + V_2$$

$$\frac{d}{df}\langle V^2\rangle = \frac{d}{df}\langle (V_1+V_2)^2\rangle$$
$$= \frac{d}{df}\langle V_1^2\rangle + \frac{d}{df}\langle V_2^2\rangle + 2\frac{d}{df}\langle V_1V_2\rangle$$

So the power in the resistor is given by

$$P = V^2 / R$$

$$\frac{d}{df}\langle P\rangle = \frac{1}{R}\frac{d}{df}\langle V^2\rangle$$
$$= \frac{1}{R}\frac{d}{df}\langle V_1^2\rangle + \frac{1}{R}\frac{d}{df}\langle V_2^2\rangle + \frac{2}{R}\frac{d}{df}\langle V_1V_2\rangle$$

Integrating with respect to frequency (over whatever bandwidth is relevant) gives the total (expected) power dissipated in R. Note that the cross-spectral density is relevant.

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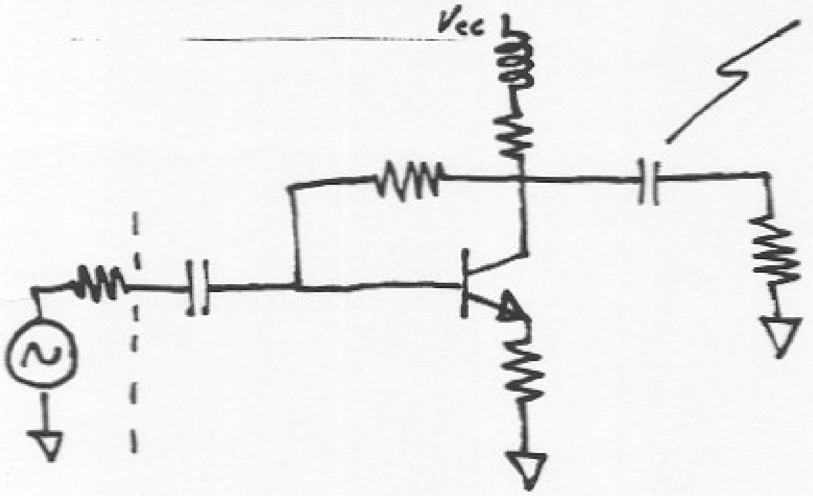
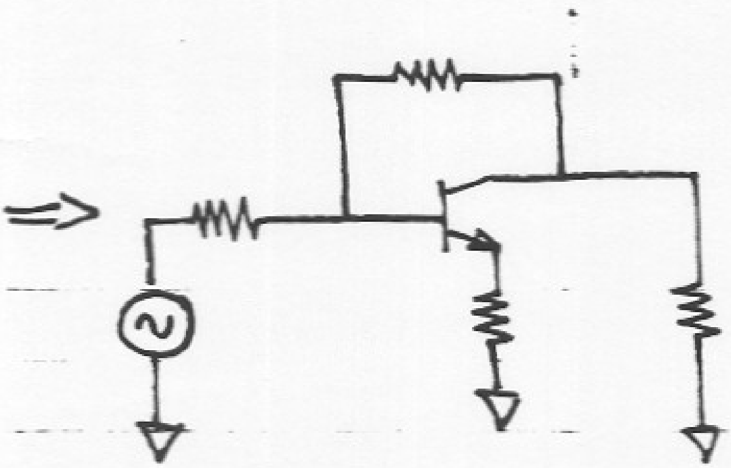
Noise In Circuits:



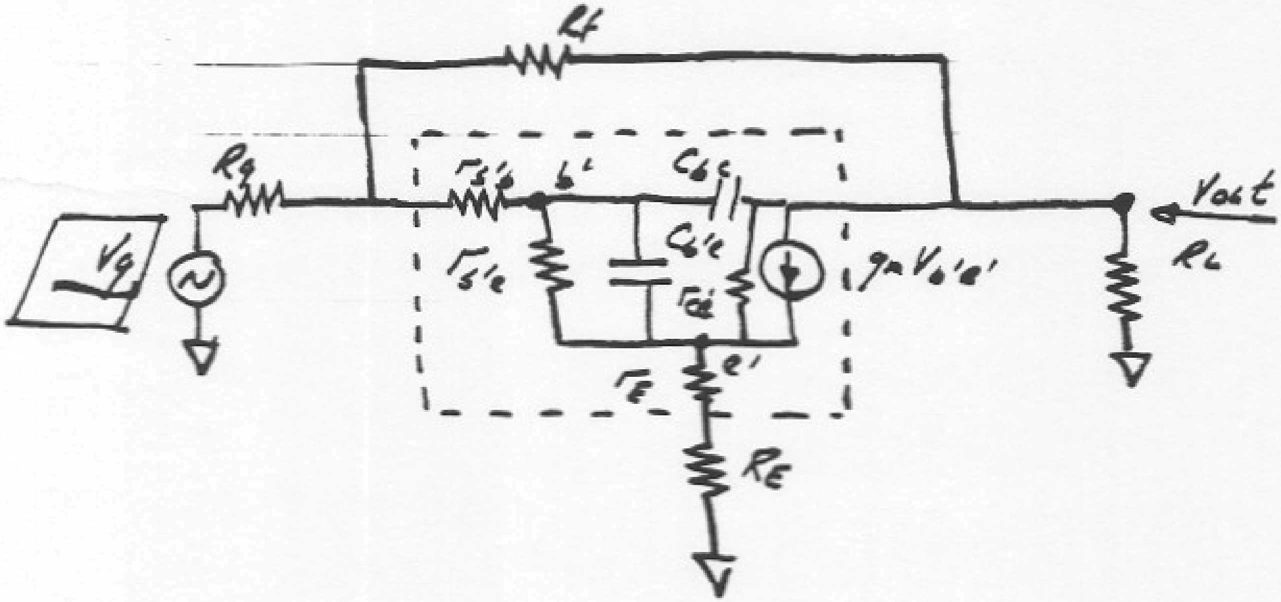
example circuit.

first start with circuit



... then replace with small-signal equivalent circuit model...

and replace the device with its equivalent circuit model...



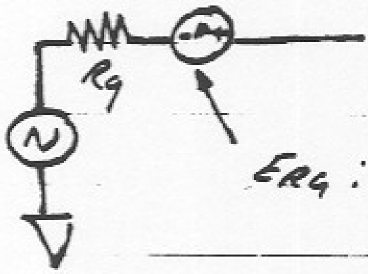
At this point we have just followed the steps necessary to find the gain & frequency response of any circuit. We would analyze the circuit above by direct nodal analysis, through computer, through Mason's gain rules, or through "tricks" (Thevenin/Norton, series/parallel, ...). Pretty much the same course can/will be followed for noise analysis.

First: Identify which resistances are real — these all have thermal (Johnson) noise associated with them. Note that $r_{b'e}$ is not a real physical resistance, but $r_{b'b}$ is.

To each element with noise, add its noise model to the small-signal circuit model: 

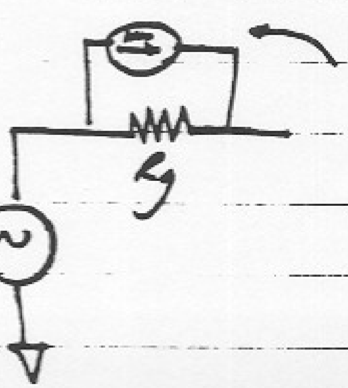
for example:

add thermal noise of generator resistance

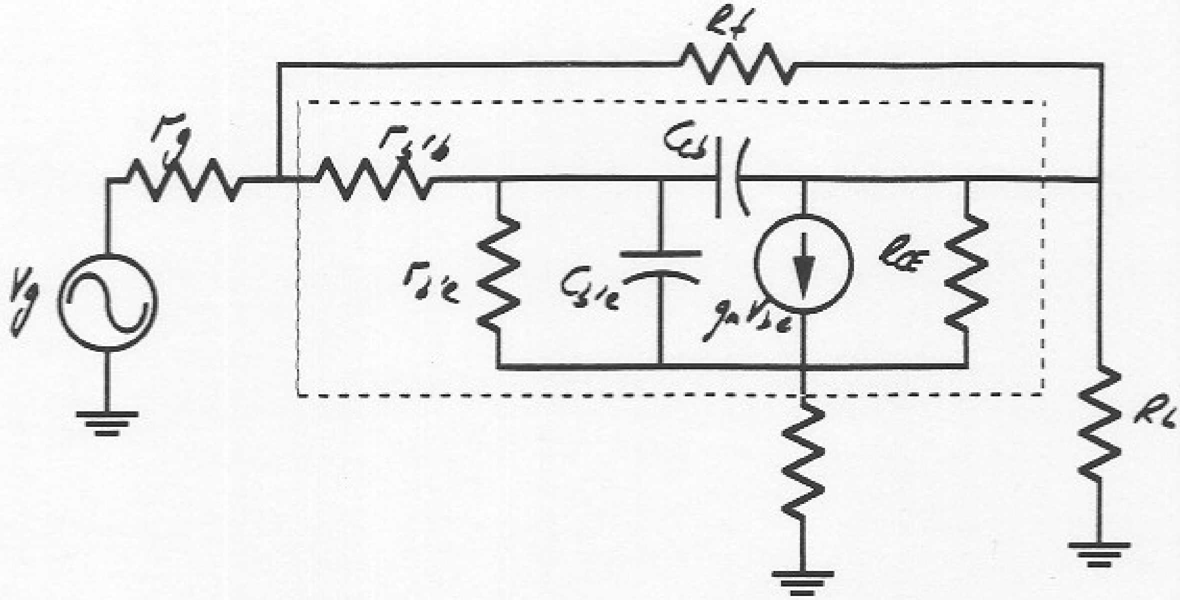


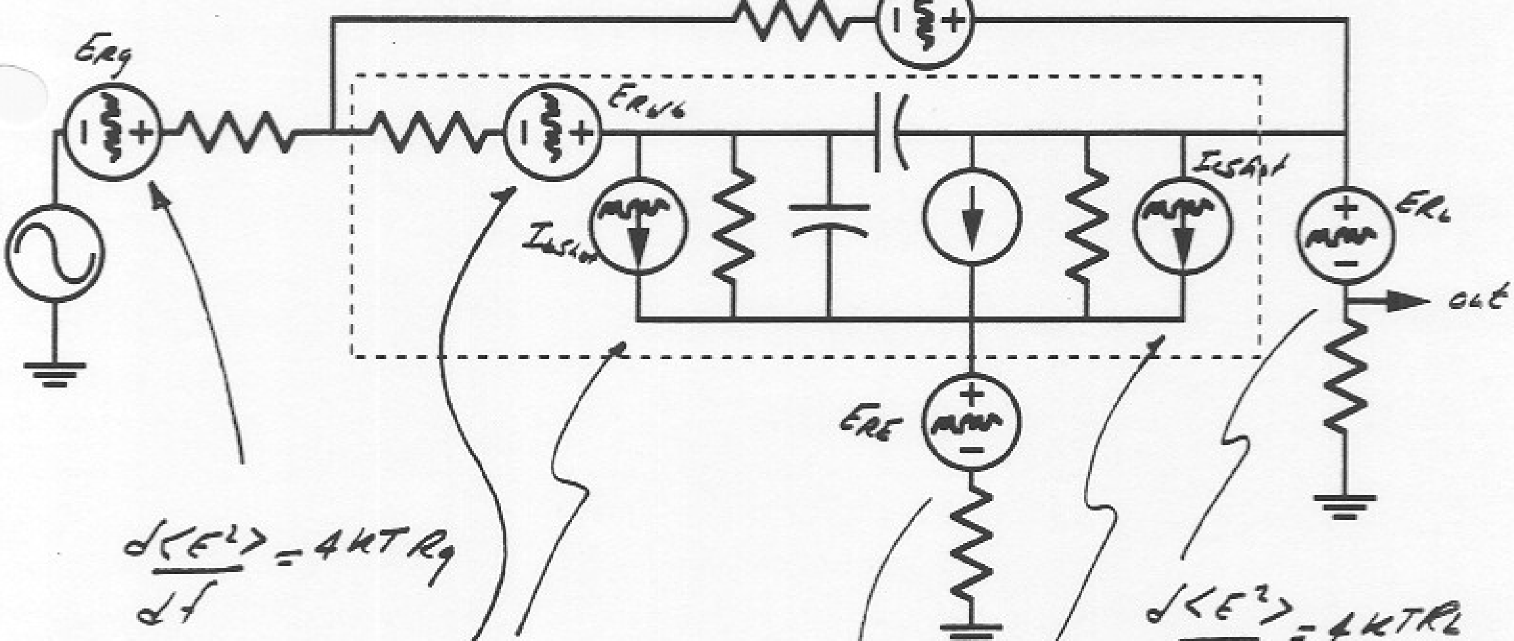
E_{Rg}: $\dfrac{d\langle E_{Rg}^2\rangle}{df} = 4kTR_g$

or:



I_{Rg}: $\dfrac{d\langle i_{Rg}^2\rangle}{df} = 4kT/R_g$





$$\frac{d\langle E^2\rangle}{df} = 4kTR_f$$

$$\frac{d\langle E^2\rangle}{df} = 4kTR_g$$

$$\frac{d\langle E^2\rangle}{df} = 4kTR_L$$

$$\frac{d\langle I^2\rangle}{df} = 2q I_B$$

$$\frac{d\langle E^2\rangle}{df} = 4kTR_{b'b}$$

$$\frac{d\langle I^2\rangle}{df} = 2q I_C$$

$$\frac{d\langle E^2\rangle}{df} = 4kTR_E$$

I_{bshot} = base shot noise

I_{cshot} = collector shot noise

Without the noise, we calculated the output voltage as a function of the input voltage:

$$\frac{V_{out}}{V_g} = A_v \quad \Rightarrow \quad V_{out} = A_v V_g$$

Now we have to calculate the output **noise** voltage, too

We have in the circuit E_{Rg}, E_{RE}, $E_{R_b'b}$, E_{RL}, I_{bshot}, I_{cshot}.

We (painfully) analyze the circuit by nodal analysis to find:

$$V_{out} = \underbrace{A_v V_g}_{\text{signal}} + \alpha_1 E_{Rg} + \alpha_2 E_{RE} + \alpha_3 E_{R_b'b} + \alpha_4 I_{bshot} + \alpha_5 I_{cshot}.$$

$$= \underbrace{A_v V_g}_{\text{amplified signal}} + E_{Nout} \leftarrow \text{output referred noise voltage.}$$

te that:

$$E_{N\,out} = \alpha_1 E_{Ng} + \alpha_2 E_{RE} + \alpha_3 E_{Rb'b} + \alpha_4 I_{b\,shot} + \alpha_5 I_{c\,shot}$$

But the random noise voltages don't add in amplitude:

Huh!

Remember, if $V_T = V_1 + V_2$, then $E\langle V_T^2\rangle = E\langle V_1^2\rangle + E\langle V_2^2\rangle + \underbrace{2E\langle V_1 V_2\rangle}_{\text{correlation}}$

$$\frac{d\langle E_{N\,out}^2\rangle}{df} = \alpha_1^2 \underbrace{\frac{d\langle E_{Ng}^2\rangle}{df}}_{4kTR_g} + \alpha_2^2 \underbrace{\frac{d\langle E_{NRE}^2\rangle}{df}}_{4kTR_E} + \alpha_1\alpha_2 \underbrace{\frac{d\langle E_{Rg} E_{RE}\rangle}{df}}_{\text{cross spectral density } = 0}$$

$$+ \ldots \qquad (\text{15 total terms})$$

As the problem has been set up, all noise terms are uncorrellated, and so all cross-spectral densities = 0

$$\frac{d\langle E_{NOUT}^2\rangle}{df} = \alpha_1^2 \frac{d\langle E_{Ng}^2\rangle}{df} + \alpha_2^2 \frac{d\langle E_{RE}^2\rangle}{df} + \alpha_3^2 \frac{d\langle E_{Rs6}^2\rangle}{df}$$

$$+ \alpha_4^2 \frac{d\langle I_{bshot}^2\rangle}{df} + \alpha_5^2 \frac{d\langle I_{cshot}^2\rangle}{df}$$

$$= \alpha_1^2 \cdot 4kTR_{gs} + \alpha_2^2 4kTR_E + \alpha_3^2 4kT r_{b's}$$

$$+ \alpha_4^2 \cdot 2q I_B + \alpha_5^2 2q I_c$$

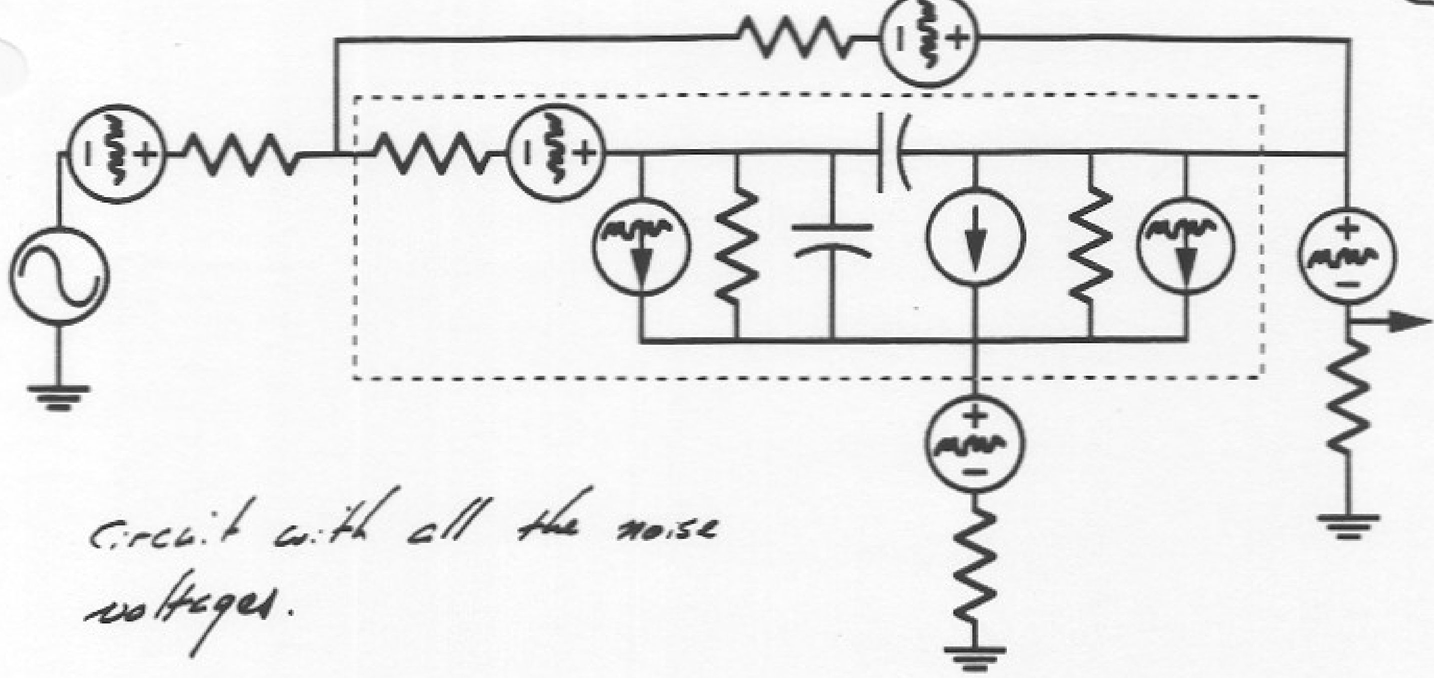
again, the $\alpha_1, \alpha_2 \ldots$ have to be found by nodal analysis.

$V_{out} = A_v V_g + E_{N\,out}$, $\frac{d\langle E_{n\,out}^2\rangle}{df} = \ldots$

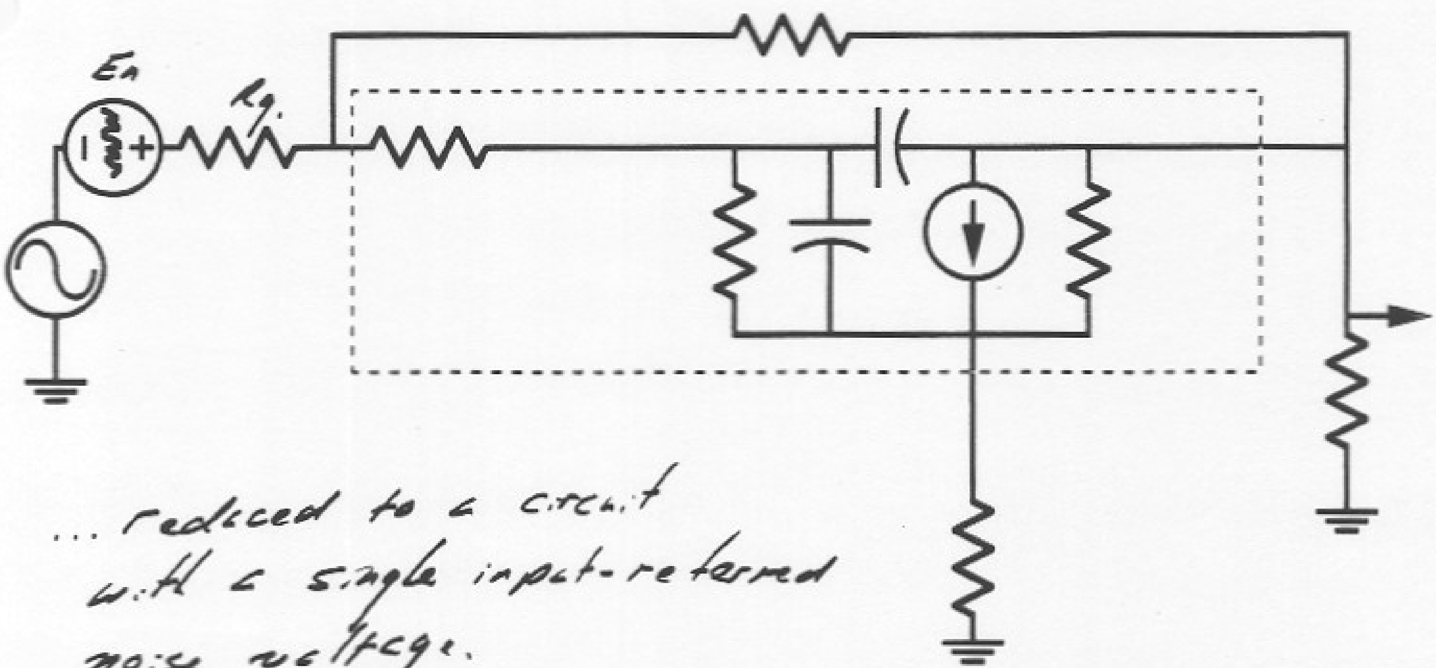
Concept: Input-Referred Noise voltage (for a particular R_g)

$$V_{out} = A_v\left[V_g + \frac{E_{N\,out}}{A_v}\right] = A_v\left[V_g + E_{N\,in}\right]$$

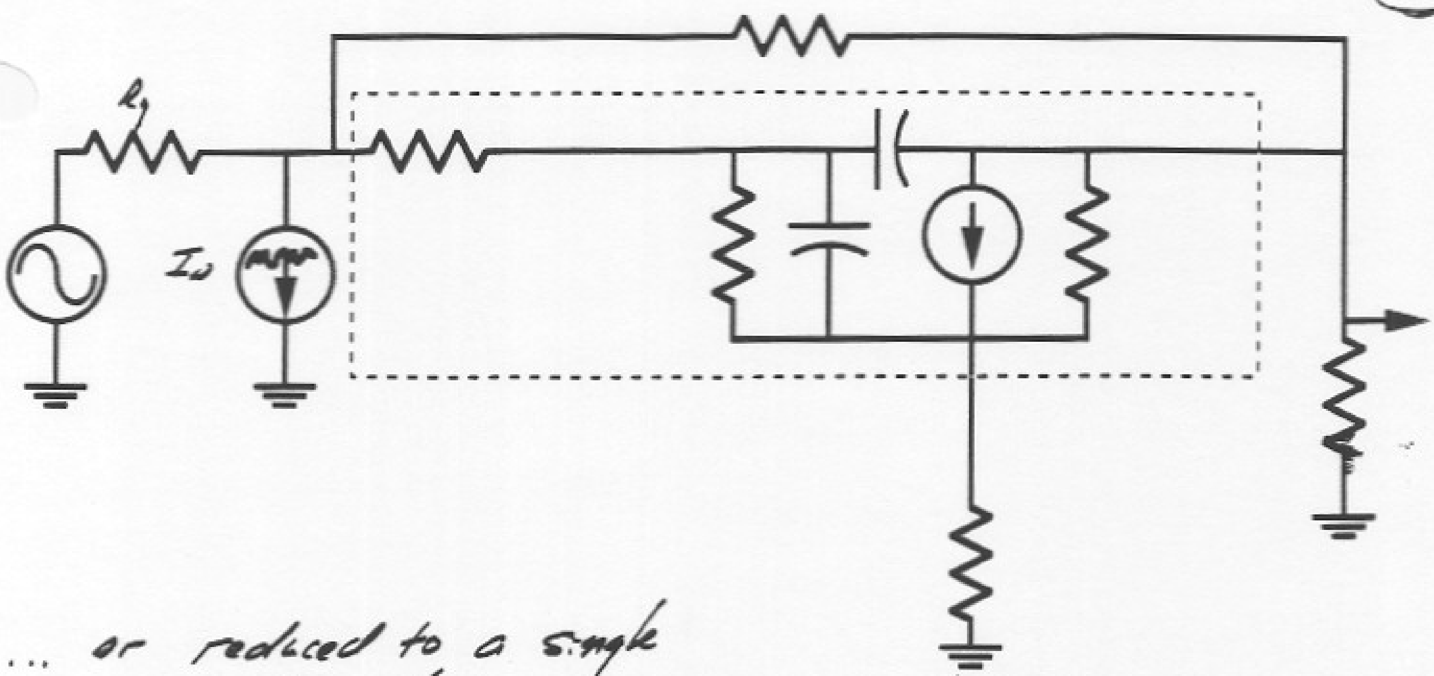
$E_{N\,in}$ represents the whole amplifiers' noise as a fictitious single random (noise) voltage at the input.



circuit with all the noise voltages.



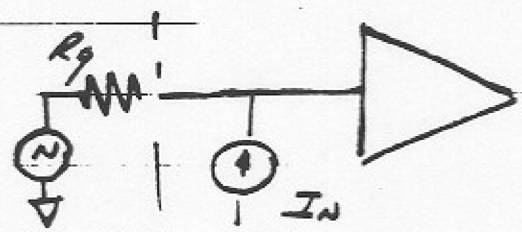
... reduced to a circuit with a single input-referred noise voltage.



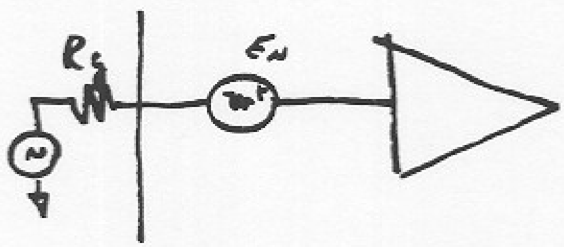
... or reduced to a single
input-referred noise current

These reductions in general dependend on having a known and specific generator resistance R_g. The total input noise voltage (or noise current) will depend on R_g. To represent the amplifier noise independent of a particular R_g, need a slightly more complex representation.

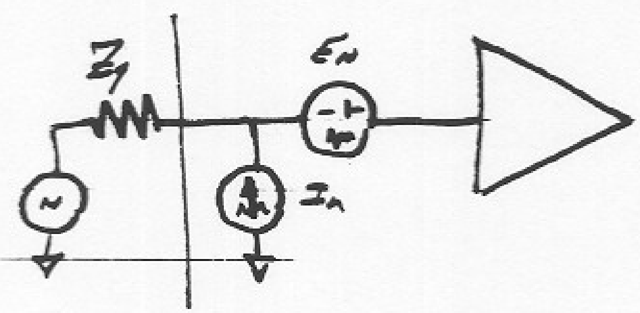
First analyze the circuit for noise with a very large R_g & find the input-referred noise current



Then analyze the circuit for noise with a zero R_g & find the input-referred noise voltage:



The total amplifier noise can then be represented by the pair of noise generators



Total input-referred noise voltage =

$$\frac{d\langle E_{NT}^2\rangle}{df} = \frac{d\langle E_N^2\rangle}{df} + \|Z_g^2\| \frac{d\langle I_N^2\rangle}{df} + 2R_E\left\{\frac{d\langle E_N I_N\rangle}{df} Z_g^*\right\}$$

This is how circuits are usually described at lower frequencies. Again note that we find $\langle E_N \rangle$ & $\langle I_N^2 \rangle$ & $\langle E_N I_N \rangle$ by nodal analysis of the basic circuit with noise voltage/current generators.

Signal/Noise Ratio:

RMS

$$\frac{S}{N} = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{\langle V_{signal, out}^2 \rangle / R_L}{\frac{d\langle E_N^2 \rangle}{df} \cdot \frac{1}{R_L} \cdot \Delta f}$$

what Δf ??

depends upon what we are talking about...

If we are talking about a system with a specific bandwidth => signal / (noise integrated over that b.w.)

otherwise normally stated as

$$\frac{\text{Signal Power}}{\text{Noise power in 1 Hz bandwidth}}$$