Lecture 1a: Review of The Wave Equation in Dielectric Media
Notation

• MKS units
• Lower case for time varying quantities
• Capitals for the amplitudes of time varying quantities
• Complex quantities used to represent amplitude and phase:

\[ a(t) = \text{Re}[A e^{i\omega t}] \]

• (at least in Chapter 1. In Chapter 2, \[ E(x,y,z,t) = \text{Re} [E(x,y,z) e^{i\omega t}] \]
Maxwell’s Equations

\[ \nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t} \]

\[ \nabla \times \vec{e} = -\frac{\partial \vec{b}}{\partial t} \]

\[ \nabla \cdot \vec{d} = 0 \]

\[ \nabla \cdot \vec{b} = 0 \]

where \( \vec{e} \) and \( \vec{h} \) are the electric and magnetic field vectors
\( \vec{d} \) and \( \vec{b} \) are the electric and magnetic displacement vectors
No free charge.
Constitutive Relations

\[ \vec{d} = \varepsilon_0 \vec{e} + \vec{p} \]
\[ \vec{b} = \mu_0 (\vec{h} + \vec{m}) \]

\( p \) and \( m \) are the electric and magnetic polarizations of the medium
\( \varepsilon_0 \) and \( \mu_0 \) are the electric and magnetic permeabilities of vacuum
\( e \) and \( h \) are the electric and magnetic field vectors
\( d \) and \( b \) are the electric and magnetic displacement vectors
Electric Susceptibility $\chi$ (Isotropic)

Isotropic Media: $\chi$ is a complex number

$$P = \varepsilon_0 \chi E$$

The real part determines the index (velocity) and the imaginary part determines the gain or absorption.

Isotropic media: Vacuum, gasses, glasses (optical fibers)
Anisotropic media: Semiconductors, crystalline materials.
Electric Susceptibility $\chi$ (Anisotropic media)

Anisotropic Media: $\chi$ is a complex second rank tensor

$$\vec{P} = \varepsilon_0 \vec{\chi} \vec{E}$$

$$P_i = \varepsilon_0 \sum \chi_{ij} E_j$$

$$P_x = \varepsilon_0 (\chi_{xx} E_x + \chi_{xy} E_y + \chi_{xz} E_z)$$

One can always choose a coordinate system such that off axis elements are zero. These are the principal dielectric axes of the crystal. We will only use the principal coordinate system.

$$P_x = \varepsilon_0 \chi_{11} E_x$$

$$P_y = \varepsilon_0 \chi_{22} E_y$$

$$P_z = \varepsilon_0 \chi_{33} E_z$$
Principal Axes

D, E and P are not parallel in general. D and E are related by the electric permeability tensor $\varepsilon$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{D} = \varepsilon \vec{E}$$

Principal axes can always be chosen such that D and E are parallel and the off diagonal elements of $\varepsilon$ are zero.

$$\varepsilon_{11} = \varepsilon_0 (1 + \chi_{11})$$

$$\varepsilon_{22} = \varepsilon_0 (1 + \chi_{22})$$

$$\varepsilon_{33} = \varepsilon_0 (1 + \chi_{33})$$
Wave Propagation in Lossless, Isotropic Media

• Lossless: $\sigma=0$, $\chi$ is real, $\varepsilon$ is real.
• Isotropic: $\chi$, $\varepsilon$ are scalars (not tensors).

\[
\nabla \times \vec{e} = i + \frac{\partial \vec{b}}{\partial t} = 0 + \mu \frac{\partial \vec{h}}{\partial t}
\]

\[
\nabla \times \vec{h} = i + \frac{\partial \vec{d}}{\partial t}
\]

\[
\nabla \times (\nabla \times \vec{e}) = \mu \frac{\partial (\nabla \times \vec{h})}{\partial t} = \mu \frac{\partial^2 \vec{d}}{\partial^2 t} = \mu \varepsilon \frac{\partial^2 \vec{e}}{\partial^2 t}
\]

\[
\nabla \times (\nabla \times \vec{e}) = \nabla^2 \vec{e} - \nabla (\nabla \cdot \vec{e})
\]

\[
\nabla^2 \vec{e} = \mu \varepsilon \frac{\partial^2 \vec{e}}{\partial^2 t} \quad \text{Wave Equation}
\]
Wave Equation

\[ e(x, y, z, t) = \text{Re}[E(x, y, z)e^{i\omega t}] \]
\[ \nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0 \]
\[ \nabla^2 \vec{E} + k^2 \vec{E} = 0 \]

where

\[ k = \omega \sqrt{\mu \varepsilon} = \omega n / c \]
\[ c = 1 / \sqrt{\mu_0 \varepsilon_0} \]
\[ n = \sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} \]
Step Index Circular Waveguide (lossless, isotropic)

• Simplest type of fiber
• (Most fiber these days is far more complex)
• Cylindrical symmetry

Figure 3-1 Structure and index profile of a step-index circular waveguide.
Step Index Circular Waveguide (lossless, isotropic)

- Simplest type of fiber
- (Most fiber these days is far more complex)
- Cylindrical symmetry
- Express Laplacian operator in cylindrical coordinates

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \]

Separate variables

\[ E_r = \psi (r) \Phi (\phi) e^{i(\omega t - \beta z)} \]
Separable Solutions

\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + (k^2 - \beta^2) \right] E_z = 0 \]

\( E_r = \psi(r) \Phi(\phi) e^{i(\omega t - \beta z)} \)

\( \Phi(\phi) = e^{\pm il\phi} \quad \text{where} \quad l = 0, 1, 2, \ldots \)
Separable Solutions

\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + (k^2 - \beta^2) \right] E_z = 0 \]

\[ E_r = \psi (r) \Phi (\phi) e^{i(\omega t - \beta z)} \]

\[ \Phi (\phi) = e^{\pm il \phi} \quad \text{where} \quad l = 0, 1, 2, \ldots \]

\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + (k^2 - \beta^2 - \frac{l^2}{r^2}) \right] \psi = 0 \]

Bessel differential equation

\[ \psi = c_1 J_l (hr) + c_2 Y_l (hr) \quad k^2 - \beta^2 = h^2 > 0 \]

\[ \psi = c_1 I_l (qr) + c_2 K_l (qr) \quad k^2 - \beta^2 = -q^2 > 0 \]
Boundary Conditions

Decaying fields for $r > a$
$q > 0$

For fields in the core $r < a$, we need finite fields
(which eliminates $Y$ and $K$ which go to infinity as $r$ approaches 0.)
TE $l=0$ Modes

Figure 3-2 Graphical determination of the propagation constants of TE modes ($l = 0$) for a step-index waveguide.
l=1 (not TE or TM, but EH)
Figure 3-4 Graphical determination of the propagation constants of the $l = 1$ HE modes for a step-index dielectric waveguide.
Figure 3-5 Normalized propagation constant as a function of $V$ parameter for a few of the lowest-order modes of a step-index waveguide [4].
For $n_1-n_2 \ll n_1$, LP approximation is valid.

Figure 3-6: Normalized propagation constant $b$ as function of normalized frequency $V$ for the guided modes of the optical fiber, $b = (\beta/k_c - n_2)/(n_1 - n_2)$. (After Reference [5].)

Single mode cut off: $V = 2.405$
Degenerate Modes $\text{LP}_{11}$

Figure 3-8 Sketch of the fiber cross section and the four possible distributions of $\text{LP}_{11}$. 
Modes as a function of V parameter

At cutoff, all the power is in the cladding.

Figure 3-9 Fractional power contained in the cladding as a function of the frequency parameter V. (After Reference [5].)
Dispersion

\[ \tau = D L \sigma \]

\( D \) is dispersion parameter  
\( L \) is the propagation length  
\( \sigma \) is the spectral width
Dispersion (sum of material and waveguide dispersion)

Figure 3-10 Group velocity dispersion of (a) dispersion-unshifted 1.3 \( \mu \text{m} \) fiber and (b) dispersion-flattened and dispersion-shifted fibers. (After Reference [1].)
Loss in early optical fibers  
(now the O-H peaks around 1.4 µm are small)

![Graph showing loss in single-mode fiber]

**Figure 3-19** Observed loss spectrum of a germanosilicate single-mode fiber. Estimated loss spectra for various intrinsic materials effects and waveguide imperfections are also shown. (From Reference [20].)
Summary

- Single mode condition required for high performance
- Multimode fiber used for low cost
- Dispersion is designable.
- 1.3 micron: zero of dispersion
- 1.55 micron: minimum loss
- Zero dispersion is not good because of nonlinearity