Controllability (and Observability)

Today: Look at meaning of "controllability" in more depth.

* Note, observability is a very analogous concept.

Review: From last time, recall that for a linear dynamic system, there are two definitions we can (equivalently) use to test for controllability.

1. If we can pick \( n \) gain values in \( K \) to set locations of \( n \) poles (in an \( n \)-th order system) arbitrarily.

2. If we can get to an arbitrarily new state vector \((n \text{ states})\) given an appropriately chosen \( u(t) \) (control action).

Let's look at each definition more closely.

\[
X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix},
\begin{align*}
X &= AX + Bu \\
Y &= CX + DU
\end{align*}
\]

- \( X \) \((n \times 1)\) states
- \( A \) \((n \times n)\)
- \( B \) \((n \times m)\)
- \( U \) \((m \times 1)\)
- \( Y \) \((k \times 1)\)
- \( C \) \((k \times n)\)
- \( D \) \((k \times m)\)

* Usually, \( D = 0 \)

\( n \) states \( m \) inputs \( K \) outputs

Assume \( m = 1 \) for now (single input).

a) Open-loop poles are at \( \text{eig}(A) \).

b) From poles, we can create the characteristic eqn:

\[
s^n + a_1 s^{n-1} + \ldots + a_{n-1} = 0
\]
For example:

\[ m_1 \ddot{x}_1 + (b_1 + b_2) \dot{x}_1 + (k_1 + k_2) x_1 = u + b_2 \dot{x}_2 + k x_2 \]
\[ m_2 \ddot{x}_2 + b_2 \dot{x}_2 + k_2 x_2 = b_2 \dot{x}_1 + k_2 x_1 \]

Let

\[ X = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1 + k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-b_2}{m_1} & \frac{k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{-b_2}{m_2} & \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{bmatrix} \]

For \( m_1 = 1, \quad m_2 = 2 \)
\( b_1 = 1, \quad b_2 = 2 \)
\( k_1 = 5, \quad k_2 = 10 \)

\[ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 10 & -3 & 2 \\ 5 & -5 & 1 & -1 \end{bmatrix} \]

and \( B = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \)

\[ S = \text{eig}(A) \]

\[ p = \text{conv}(p, [1, -s(n)]) \]

\[ p = \left[ 1, 4, 21, 10, 25 \right] \]

Characteristic Equation given A matrix.

\[ s^4 + 4s^3 + 21s^2 + 10s + 25 = 0 \]
Now, given open-loop poles, which determine a (unique) characteristic equation (if highest order in $s$ has coeff. 1),

...we can generally define a new dynamic system

with such that eigenvalues are the same.

For example, the **canonical** form, where top row of $A_c$ is built directly from the char. poly.:

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$$

$$A_c = \begin{bmatrix}
-a_1 & -a_2 & -a_3 & \cdots & -a_n \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}$$

$$B_c = \begin{bmatrix}
1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}$$

Here, $X = \begin{bmatrix}
x \\
x \\
x \\
\vdots \\
x
\end{bmatrix}$  \textit{Not easy to interpret physically.}
Controllability matrix

\[ C = \begin{bmatrix} B, AB, A^2B, \ldots, A^{n-1}B \end{bmatrix} \]

\((n \times n)\)

* If \( C \) is **full rank**, then it is invertible, and then you can set pole arbitrarily by picking \( K \).

Why? Let

\[ T = CC_c^{-1} \]

For the canonical system, control gains are:

\[ K_c = \begin{bmatrix} K_{1c}, K_{2c}, \ldots, K_{nc} \end{bmatrix} \]

Control law is:

\[ u = -Kx \]

\[ \dot{x} = A_c x + B_c u \]

\[ = A_c x - B_c K_c x \]

\[ \dot{x} = (A_c - B_c K_c)x \]

Since \( B_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), \( B_c K_c = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), \( A_c - B_c K_c \) are at:

\[ S = \text{eig}(A_c - B_c K_c) \]

Our example

\[ \begin{bmatrix} \begin{array}{cccc} -a_3 & K_4 & 0 & 0 \\ K_1 & K_2 & K_3 & K_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \end{bmatrix} \]
Say we want poles $s = \{ \begin{array}{c} -1+j \\ -1-j \\ -10 \\ -10 \end{array} \}$

Desired char. eq. is: $s^4 + 22s^3 + 142s^2 + 240s + 200 = 0$

So, since G.L. was: $s^4 + 4s^3 + 21s^2 + 10s + 25 = 0$

We need: $k_1 + a_1 = 22 \Rightarrow k_1 = 22 - 4 = 18$
$k_2 + a_2 = 142 \Rightarrow k_2 = 142 - 21 = 121$
$k_3 + a_3 = 240 \Rightarrow k_3 = 240 - 10 = 230$
$k_4 + a_4 = 200 \Rightarrow k_4 = 200 - 25 = 175$

Finally, to get from $k_c$ (for the control canonical form) to $k$ (for original A & B matrices):

$A_c - B_c k_c$
$T(A_c)T^{-1} - T(B_c k_c)T^{-1}$

$A - B(k_c T^{-1})$

$A - Bk$

Where $K = k_c T^{-1}$
2. Arbitrary states in finite time?

With 4 states, we expect to need 4 tunable variables to set each state independently.

So, say we do the following: (Discrete-time control)

\[ x_{n+1} = A_d x_n + B_d u_n \]

Can we find values for each \( u_i \) to change all 4 states independently?

Note - It's a lot easier to visualize this for the DT case, but same concept of "getting to a desired state" applies in CT (cont-time) case, too.

So, for 4 time steps,

- after \( u_0 \), \( x_1 = A x_0 + B u_0 \)
- after \( u_1 \), \( x_2 = A x_1 + B u_1 = A (A x_0 + B u_0) + B u_1 = A^2 x_0 + A B u_0 + B u_1 \)
- after \( u_2 \), \( x_3 = A (A^2 x_0 + A B u_0 + B u_1) + B u_2 = A^3 x_0 + A^2 B u_0 + A B u_1 + B u_2 \)
- after \( u_3 \), \( x_4 = A x_3 + B u_3 \)

\[ x_4 = A^4 x_0 + A^3 B u_0 + A^2 B u_1 + A B u_2 + B u_3 \]

We want \( x_4 = x_{des} \)
\((X_{\text{des}} - A^4 X_0) = \begin{bmatrix} B & AB & A^2 B & A^3 B \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \end{bmatrix} \)

This is \(C\)!
(Controllability)

To solve for vector \(u\), use matrix algebra:

\((X_{\text{des}} - A^4 X_0) = C u\)

\(C^{-1} (X_{\text{des}} - A^4 X_0) = C^{-1} C u = u\)

\(C\) must be INVERTIBLE for a solution to be guaranteed!

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**Observability**: Given some initial vector of states, \(x_i\), can we identify all values in \(x_i\) by observing the available outputs, \(y\), over some finite time?

\[y = C x + D u\]

\[\downarrow\]

\[y_o = C x_o\]

\[x_1 = A x_0 + B u_0\]

\[y_1 = C (A x_0 + B u_0)\]

\[x_2 = A x_1 + B u_1\]

\[y_2 = C [A (A x_0 + B u_0) + B u_1]\]

\[x_3 = A x_2 + B u_2\]

\[y_3 = C [A [A (A x_0 + B u_0) + B u_1] + B u_2]\]
Rearranging

\[ y_0 = CX_0 \]
\[ y_1 = CAX_0 + CBU_0 \]
\[ y_2 = CA^2X_0 + CABU_0 + CBU_1 \]
\[ y_3 = CA^3X_0 + CA^2BU_0 + CABU_1 + CBU_2 \]

All other terms \( y_0, y_1, y_2, y_3, C, A, B \) are measured or known at the start.

\[
\begin{bmatrix}
C \\
CA \\
CA^2 \\
CA^3 \\
\end{bmatrix}
\begin{bmatrix}
X_0 \\
y_0 \\
y_1 - CBU_0 \\
y_2 - CABU_0 - CBU_1 \\
y_3 - CA^2BU_0 - CABU_1 - CBU_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
Z_0 \\
Z_1 \\
Z_2 \\
Z_3 \\
\end{bmatrix}
\]

\[ X_0 = (A^T)^{-1} \cdot \begin{bmatrix}
Z_0 \\
Z_1 \\
Z_2 \\
Z_3 \\
\end{bmatrix} \]

Can be found if \( A \) is full rank, i.e., invertible.

Matlab

\( \text{eig}(A-BK) \)
\( \text{eig}(A-LC) \)

Control gain

Observer gain

Matlab to find \( K = \text{place}(A, B, pcon) \)

\( L = \text{place}(A', C', pobs) \)

Note 3 transpose char