Additive Successive Refinement

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Scalable coding rate-distortion bounds, and conditions for their coincidence with non-scalable rate-distortion bounds, were derived in [1, 3]. However, there has always been an implicit assumption of tree structured vector quantization (TSVQ) scheme, which requires extensive memory usage. In most of the practical applications, on the other hand, additive structures such as multi-stage vector quantization (MSVQ) are preferred. In this paper, we analyze the bounds for MSVQ.

Let \{X_i\}_{i=1}^n be a discrete memoryless source taking values from an alphabet \(\mathcal{X}\). We encode this source in two MSVQ stages, i.e., we have two stage-coding functions \(f_i: \mathcal{X}^n \rightarrow \mathcal{F}_i\), and stage-decoding functions \(g_i: \mathcal{F}_i \rightarrow \mathcal{X}^n\), where \(\mathcal{F}_i\) is the reproduction alphabet.

The quadruple \((R_1, R_2, D_1, D_2)\) is called \((\alpha, \beta)\)-achievable if there exist stage-coding functions \(f_i\) and stage-decoding functions \(g_i\) with
\[
\begin{align*}
  n(R_i + \alpha) &\geq \log |\mathcal{F}_i|, \quad i = 1, 2, \\
  n(D_1 + \beta) &\geq E \sum_{i=1}^n d(X_i, \tilde{X}_i), \\
  n(D_2 + \beta) &\geq E \sum_{i=1}^n d(X_i, \tilde{X}_i + \tilde{X}_2).
\end{align*}
\]

**Theorem 1 (Sufficient Conditions for Achievability)**

If there exist random variables \(\tilde{X}_1\) and \(\tilde{X}_2\), jointly distributed with source variable \(X\), such that
\[
\begin{align*}
  I(X; \tilde{X}_1) &\leq R_1 + \alpha, \\
  I(X; \tilde{X}_2) &\leq R_2 + \alpha, \\
  I(X; \tilde{X}_1, \tilde{X}_2) &+ I(\tilde{X}_1; \tilde{X}_2) \leq R_1 + R_2 + 2\alpha, \\
  Ed(X, \tilde{X}_1) &\leq D_1 + \beta, \\
  Ed(X, \tilde{X}_1 + \tilde{X}_2) &\leq D_2 + \beta,
\end{align*}
\]

then the quadruple \((R_1, R_2, D_1, D_2)\) is \((\alpha, \beta)\)-achievable.

The proof is very similar to the one used by El Gamal and Cover for the multiple descriptions problem [2].

**Theorem 2 (Necessary Conditions for Achievability)**

If the quadruple \((R_1, R_2, D_1, D_2)\) is \((\alpha, \beta)\)-achievable, then there exist random variables \(\tilde{X}_1\) and \(\tilde{X}_2\), jointly distributed with source variable \(X\), such that
\[
\begin{align*}
  I(X; \tilde{X}_1) &\leq R_1 + \alpha, \\
  I(X; \tilde{X}_2) &\leq R_2 + \alpha, \\
  I(X; \tilde{X}_1, \tilde{X}_2) &\leq R_1 + R_2 + 2\alpha, \\
  Ed(X, \tilde{X}_1) &\leq D_1 + \beta, \\
  Ed(X, \tilde{X}_1 + \tilde{X}_2) &\leq D_2 + \beta.
\end{align*}
\]

We call \((R_1, R_2, D_1, D_2)\) achievable, if it is \((\alpha, \beta)\)-achievable for all \(\alpha, \beta > 0\).

**Definition 1 (Scalable Coding Bound)**

The rate-distortion bound for scalable coding is given by
\[
R(D_1, D_2, R_1) = \inf_{p(\tilde{X}_1, \tilde{X}_2) \mid \text{s.t. } I(X; \tilde{X}_1) \leq R_1, Ed(X, \tilde{X}_1) \leq D_1, Ed(X, \tilde{X}_1 + \tilde{X}_2) \leq D_2} I(X; \tilde{X}_1, \tilde{X}_2).
\]

**Theorem 3 (The Special Case of No-excesse rate)**

When \(R_1 + R_2 = R(D_1, D_2, R_1)\), a quadruple \((R_1, R_2, D_1, D_2)\) is achievable if and only if there exist random variables \(X_1\) and \(X_2\), jointly distributed with source variable \(X\), such that
\[
\begin{align*}
  I(X; X_1) &\leq R_1, \\
  I(X; X_2) &\leq R_2, \\
  I(X_1; X_2) &\leq R_1 + R_2, \\
  Ed(X, X_1) &\leq D_1, \\
  Ed(X, X_1 + X_2) &\leq D_2.
\end{align*}
\]

The proof follows virtually the same lines as the multiple descriptions proof in [1, Section IV].

**Example**

Let \(X = \tilde{X} = R = \mathcal{R}\), and let \(X \sim p(x)\). If the Shannon Lower Bound (SLB) is tight at distortion \(D_1\), we observe that \(p(x, \tilde{x}_1, \tilde{x}_2)\) satisfies all conditions in (7) for \(D_2 < D_1\) and \(R_1 = R(D_1)\). Then, \(p(x, \tilde{x}_1, \tilde{x}_2)\) is achievable in the usual TSVQ sense [3], i.e., \(R(D_1, D_2, R_1) = R(D_2)\).

**Counterexample**

Let \(X = \tilde{X} = \{0, 1, 2\}\) with Hamming distortion measure, and let \(\alpha, \beta\) be defined as modulo 3 summation. Let \(p_0 \geq p_1 \geq p_2\). This source is successively refinable in the usual (TSVQ) sense [3]. However, the conditions (7) are not satisfied for any \(D_1 > 2p_2\). Hence this is an example of a source that is everywhere successively refinable but is not everywhere additively refinable.

**REFERENCES**


