Iterative Joint Decoding and Sparse Channel Estimation for Single-Carrier Modulation

Ronald A. Iltis

Department of Electrical and Computer Engineering

University of California, Santa Barbara, CA 93106

iltis@ece.ucsb.edu

http://stnlabs.ece.ucsb.edu

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OFDM vs. SC-FDE Systems

OFDM

Single-Carrier Frequency Domain Equalization

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OFDM vs SC-FDE

• OFDM
  • Disadvantage – High peak-to-average power ratio (PAR) on the order of number of carriers.

• SC-FDE
  • Does not achieve Shannon capacity.
  • Much lower PAR than OFDM, can use constant-envelope waveform if necessary.
  • Performance of coded SC-FDE is similar to that of OFDM on mobile wireless channel. (Benvenuto and Tomasin, IEEE Trans. Comm. 02, Falconer, et. al. IEEE Comm. Mag. 02.)
Uncoded OFDM vs. SC-FDE-ZF
Revisited

Received SC-FDE and OFDM waveforms after Nyquist sampling and vectorization. Channel is a convolution matrix.

\[
\begin{align*}
\mathbf{r} &= \mathbf{F} \mathbf{c} + \mathbf{n} \text{ SC-FDE} \\
\mathbf{r} &= \mathbf{F} \mathbf{W} \mathbf{c} + \mathbf{n} \text{ OFDM}
\end{align*}
\]

\[
\mathbf{F} = \begin{bmatrix}
f_0 & f_1 & \cdots & f_{N_f-1} & \cdots & 0 \\
0 & f_0 & \cdots & 0 & f_{N_f-1} & 0 \\
\vdots \\
0 & 0 & \cdots & 0 & 0 & f_0
\end{bmatrix}
\]
ZF OFDM vs. SC-FDE Detection

\[ W^H F W = H = \]
\[ \text{diag}\{H_{1,1}, H_{2,2}, \ldots, H_{N_s,N_s}\} \]

\[ \hat{c} = H^{-1} W^H r \]
\[ = c + H^{-1} W^H n \quad \text{OFDM} \]

\[ \hat{c} = WH^{-1} W^H r \]
\[ = c + WH^{-1} W^H n \quad \text{SC-FDE} \]
Uncoded Error Rates OFDM-ZF vs. SC-FDE-ZF

OFDM

\[ P_b = \frac{1}{N_d} \sum_{i=0}^{N_d-1} \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}} |H'_{ii}|^2 \right) \]

SC-FDE

\[ P_b = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0 \frac{1}{N_d} \sum_{i=0}^{N_d-1} \frac{1}{|H'_{ii}|^2}}} \right) \]
Uncoded Error Rates

**Proposition 1** Consider case (a) where $|H'_{ii}|^2 < 3/(2\gamma_b)$ for $i = 0, \ldots, N_d-1$, where $\gamma_b = E_b/N_0$. In case (a), the zero-forcing SC-FDE BER is lower-bounded by the OFDM BER. Now for case (b), let $|H'_{ii}|^2 > 3/(2\gamma_b)$ for all $i$. Then the OFDM BER is lower bounded by the ZF SC-FDE result. The OFDM and SC-FDE system performance is equivalent when the channel is allpass.
Uncoded Error Rates

Proof: Define $\alpha_i = \frac{1}{|H_{ii}^\prime|^2}$. For case (a), we can show that $\text{erfc}(\sqrt{\gamma_b/\alpha_i})$ is concave in $\alpha_i$ in the region $\{\alpha_i\} \in [2\gamma_b/3, \infty)^{N_d}$. Then using Jensen’s inequality

\[
\frac{1}{N_d} \sum_{i=0}^{N_d-1} \frac{1}{2} \text{erfc} \left( \sqrt{\frac{\gamma_b}{\alpha_i}} \right) < \frac{1}{2} \text{erfc} \left( \sqrt{\frac{\gamma_b}{\frac{1}{N_d} \sum_{i=0}^{N_d-1} \alpha_i}} \right).
\]

The erfc is convex in case (b), which reverses the inequality, so that the SC-FDE BER is less than OFDM.
Iterative SC-FDE Receiver

Incorporate Effect of Doppler Shift

\[ r = V(\beta)F \mathbf{c} + n \]
\[ V(\beta) = \text{diag}\{e^{i2\pi \beta f_c(N_d-1)T_s}, e^{i2\pi \beta f_c(N_d-2)T_s}, \ldots, 1\} \]

![Diagram of the Iterative SC-FDE Receiver system](image)
Turbo Equalization via Gaussian Approximation for Variable Messages

$p(f)$ → $f$ → $\mathcal{CN}(r; Fc, \sigma_n^2 I)$

$\mu_p \rightarrow c_k$  $\mu c_k \rightarrow p$

$\mu_p \rightarrow c_k$

$H_1$  $H_2$  $C_{n-1}$  $C_n$  $H_m$
Turbo Equalization

• Turbo equalization has recently been justified via statistical physics methods (Nissila and Pasupathy IEEE TC July 07.)

• Measurement likelihood to code variable message. Marginalization under Gaussian approximation for the variable priors yields more direct justification for Turbo Equalization.

\* \mu_{p \rightarrow c_k} = 
\int \mathcal{CN}(r; F_{(k)} c_{(k)} + f_k c_k, \sigma^2_n I) \mathcal{CN}(c_{(k)}; \bar{c}_{(k)}, Q) dc_{(k)}.

Means computed by decoder using total log-APPs

\bar{c}_{l,j} = \tanh(\lambda^e(c_{l,j})/2)
Proposition 1 The density function to code variable message for SC modulation, when the decoder extrinsics are approximated as independent Gaussian, is given by

\[
\mu_{p\rightarrow c_k} \propto \exp\left(-|c_k - \hat{c}_k|^2/p_k\right)
\]

\[
\hat{c}_k = \frac{1}{f_k^H \Sigma^{-1} f_k} f_k^H \Sigma^{-1} \left(r - F(k) \bar{c}(k)\right)
\]

\[
\Sigma = f_k Q f_k^H + \sigma_n^2 I, \quad p_k = (f_k^H \Sigma^{-1} f_k)^{-1}.
\]

That is, the message is a Gaussian density with mean and covariance given by a MMSE Turbo equalizer.
Outline of Proof – Turbo Equalization

The integral (*) has a closed form solution identical to that for the Kalman filter innovations likelihood. The result is

$$\mu_{p \rightarrow c_k} = \mathcal{CN}(r - F_k \bar{c}_k; f_k c_k, F_k Q F_k^H + \sigma^2 I).$$

The density (**) is then readily manipulated to yield the density (***) in terms of $c_k$. 
MMSE Frequency-Domain Turbo Equalization

• Problems with exact belief propagation (marginalization)

  • Closed-form marginalization w.r.t. both channel coefficients and code variables is not tractable.

  • MMSE Turbo Equalizer for exact code variable covariance $Q$ cannot be computed in frequency domain.

• Solutions:

  • Design MMSE turbo equalizer using frequency-domain transformation of $r$.

  • Replace marginalization over channel coefficients by a “good” estimate of channel matrix $F$.

    • Use Matching Pursuits to estimate channel for target application (underwater acoustic communications.)
Frequency Domain Turbo Equalizer

Use offset-corrected received vector

\[ r' = W^H V (\hat{\beta})^H r \approx HW^H c + n', \]

Resulting MMSE equalizer is in the frequency domain and diagonal

\[ D_{ii} = \frac{H_{ii}^*}{|H_{ii}|^2 + \frac{N_0}{T_s}} \]

\[ \hat{c} = WDW^H V (\hat{\beta})^H r \]
Turbo Equalizer and Likelihood

MMSE detector output including effect of ISI

\[ y_k = (W\hat{D}W^H V (\hat{\beta})^H r)_k \]
\[ \approx \frac{1}{N_d} \sum_{i=0}^{N_d-1} \frac{|\hat{H}_{ii}|^2}{|\hat{H}_{ii}|^2 + \frac{N_0}{T_s}} c_k + \sum_{j \neq k} (WD'W'^H)_{kj} (\tilde{c}_j + \tilde{c}_j) + n'_k. \]
Turbo Equalizer and Likelihood

ISI cancelled signal

\[ y'_k = y_k - \sum_{j \neq k} (W \hat{D}' W^H)_{kj} \hat{c}_j \]

\[ \hat{c}_j = \tanh(\lambda^e(c_j,1)/2) + j \tanh(\lambda^e(c_j,2)/2) \]

Likelihood sent to decoder

\[ p(y'_k|c_k) = C\mathcal{N} \left( y'_k, \frac{1}{N_d} \sum_{i=0}^{N_d-1} \frac{|\hat{H}_{ii}|^2}{|\hat{H}_{ii}|^2 + \frac{N_0}{T_s} c_k, \sigma^2_k} \right), \]
Turbo Channel Estimation

Given soft symbol decisions and pilots, the following convolution matrix representations are equivalent.

\[ \mathbf{r} = \mathbf{V}(\beta)\mathbf{F}\hat{\mathbf{c}} + \mathbf{n} = \mathbf{V}(\beta)\hat{\mathbf{C}}\mathbf{f} + \mathbf{n} \]

Matching Pursuits – find sparse fit to \( \mathbf{r} \) using columns of \( \mathbf{C} \).
Matching Pursuits – Sufficient Statistics Representation

Sufficient statistics

\[ v^1 = \hat{C}^H V (\hat{\beta})^H r \quad A = \hat{C}^H \hat{C} \]

Successive least-squares channel estimates and cancellation

\[ v^k = v^{k-1} - A p_{k-1} \hat{f} p_{k-1} \]

\[ p_k = \arg \max_{l \neq \{p_1, \ldots, p_{k-1}\}} |v^k_l|^2 / A_{p_l, p_l} \]

\[ \hat{f} p_k = v^1_{p_k} / A_{p_k, p_k} \]
Turbo Doppler Shift Estimation

Use Doppler shift estimator incorporating unconstrained LS channel estimates and decoder soft decisions plus pilots.

\[ \hat{\beta} = \arg \min_\beta \| \mathbf{r} - \mathbf{V}(\beta) \mathbf{C} \hat{\mathbf{f}}_{LS} \|^2 \]

\[ \hat{\mathbf{f}}_{LS} = \left( \mathbf{C}^H \mathbf{C} \right)^{-1} \mathbf{C}^H \mathbf{V}(\beta)^H \mathbf{r} \]

\[ \hat{\beta} = \arg \max_\beta \mathbf{r}^H \mathbf{V}(\beta) \hat{\mathbf{C}} \left( \mathbf{C}^H \mathbf{C} \right)^{-1} \mathbf{C}^H \mathbf{V}(\beta)^H \mathbf{r} \]
Simulation Parameters

Multipath spread < 25 msec.
Doppler spread < 1 Hz.
Packet length $T_d = 105$ msec.
Cyclic Prefix $T_p = 25$ msec.
Symbol rate 9600 sps.
Gallager LDPC code (1008,504)
3-ray Rayleigh fading channel, delays [0 17.2 22.2] msec.
AR-1 models for channel coefficients
Doppler spreads .05 Hz and .1 Hz.
Doppler velocity .05 m/s and .25 m/s.
Pilot percentage – 50%.
BER Results – Low Doppler Scenario

- **f_D = 0.05 Hz**
- **v_r = 0.05 m/s**
- **N_α = 3**

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BER Results – Moderate Doppler Scenario

- $f_D = 0.1 \text{ Hz}$
- $v_r = 0.25 \text{ m/s}$
- $N_x = 3$

Eb/N0 (dB)

BER

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Channel Estimation Error – Low Doppler Scenario

- $f_D = 0.05$ Hz
- $v = 0.05$ m/s
- $N_\alpha = 3$

Normalized Channel MSE vs. $E_b/N_0$ (dB) for different iterations.
Conclusions

- Uncoded SC-FDE performance can exceed OFDM in strongly Ricean channels. (Always exceeds OFDM for non-fading channels with effective $E_b/N_0 > 3/2$.)
- Direct justification for Turbo Equalization using Gaussian priors for code variables.
- Overall channel and Doppler estimator appear robust for underwater acoustic channel scenario.
- Adequate BER performance for UAC requires large percentage of pilots for SC-FDE. Need to use more accurate UAC channel models and implement in hardware/field test.