NOTES
on
SELF-ADJUSTING CONTROL SYSTEMS

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Contents

Chapter 1

A FRAMEWORK FOR SELF-ADJUSTING CONTROL

The aim of this section is to explain in general terms, what an adaptive control system is. We start this reviewing what is meant by a conventional *non-adaptive* control system.

1.1 A Conventional Control System

By a conventional *control system* is usually meant the interconnection of a controlled plant or *process* $\mathbb{P}$ and a *controller* $\mathbb{C}$ as shown in Figure 1.1. Here w, u, z and y are signal vectors of *exogeneous inputs*, *control inputs*, *controlled outputs*, and *measured outputs* respectively.

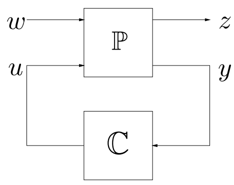

Figure 1.1: Control System

The purpose of $\mathbb{C}$ is to cause $\mathbb{P}$'s controlled outputs to respond to w in a prescribed fashion. The components of w typically include process disturbances, sensor noise, command or reference signals, etc. The components of y include all sensor outputs as well as all exogeneous inputs which can be measured. Thus with this labeling, some of $\mathbb{P}$'s inputs and outputs may appear as components in more than one of $\mathbb{P}$'s signal vectors. For example, a sensor output which is to be controlled would appear as a component of both z and y.

Most control design problems begin with a *nominal process model* $\mathbb{N}$ which explains, to first approximation, how $\mathbb{P}$'s outputs respond to its inputs. Precise models of $\mathbb{P}$ usually are not known exactly - rather they are taken to be unknown but fixed members of some suitably defined *family of admissible process models* which we denote by $\mathcal{M}$. The models in $\mathcal{M}$ are typically the nominal $\mathbb{N}$ plus suitably defined perturbed versions of $\mathbb{N}$. The *leap-of-faith assumption* is that one of $\mathcal{M}$'s members exactly models $\mathbb{P}$.

Given $\mathcal{M}$, the conventional *controller design problem* is then to design or select a controller $\mathbb{C}$ from a specific class of admissible control algorithms which, for each admissible model $\mathbb{M}$ in $\mathcal{M}$, causes the controlled system shown in Figure 1.2 to meet prescribed specifications.

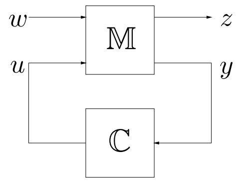

Figure 1.2: Model-Based Control System

These specifications might include a relative stability requirement, a steady-state tracking requirement, a transient response requirement, an disturbance to controlled output attenuation requirement, etc. The design problem is approached by devising a controller for $\mathbb{N}$ which meets specifications and continues to meet specifications under those purturbations which deform $\mathbb{N}$ into the other models in $\mathcal{M}$. Such perturbations are typically norm-bounded and one appeals to arguments based on small gains to determine whether or not a proposed controller meets specifications.

1.2 A Parameter-Adaptive Control System

The need for *adaptive* control arises in those situations where $\mathcal{M}$ is so large that it is not possible to design a *single* controller to meet specifications for every possible model in $\mathcal{M}$. Such large modeling uncertainties typically occur when one seeks to account for the behavior of a process which is likely to undergo large, possibly abrupt, unpredictable changes in its dynamics. Such uncertainties might arise in real-time, because of changes in operating environment, component aging or failure, or perhaps because of an external influence. To cope with these types of uncertainties obviously requires a controller which is a good deal more flexible that that which conventional control theory can provide. What's clearly needed is a controller which can *change* or be changed in response to perceived changes in plant dynamics. If plant changes can be accurately predicted in advance or directly sensed when they occur, then controller gain scheduling will often suffice. But if plant changes cannot be predicted or directly sensed, on-line controller selection or tuning must be carried out using whatever real-time operating data is available. This is where adaptive control comes in.

The first step in the development of an adaptive control system is to define a *class of candidate controllers* $\mathcal{C}$ in such a way so that for each admissible process model $\mathbb{M}$ in $\mathcal{M}$ there is at least one controller $\mathbb{C}$ in $\mathcal{C}$ which causes the controlled system shown in Figure 1.2 to meet prescribed specifications. The *adaptive control design problem* is then to devise a procedure for automatically picking a controller {or sequence of controllers} from $\mathcal{C}$, based on real-time operating data, which when applied {in sequence} to the process, causes the process to perform in a prescribed manner[1]. More often than not, $\mathcal{C}$ is of the form $\mathcal{C} = \{\mathbb{C}_q : q \in \mathcal{Q}\}$ where $\mathcal{Q}$ is a set of indices or parameter values and for each $q \in \mathcal{Q}$, $\mathbb{C}_q$ is the *qth candidate controller.* In these notes we shall consider $\mathcal{Q}$'s which are either finite sets or subsets of real finite dimensional, normed linear spaces. From this point-of-view, a *parameter-adaptive control system* is then the interconnection shown in Figure 1.3, consisting of a process to be controlled $\mathbb{P}$, a switchable or tunable controller $\mathbb{C}_{\widehat{q}}$, and a high-level algorithm called a "supervisor" which carries out the on-line controller selection process.

Here $\widehat{q}$ is a suitably defined *tuning or switching signal* which takes its values in $\mathcal{Q}$. In the sequel we refer to $\mathbb{C}_{\widehat{q}}$ as a *multi-controller.* Note that $\mathbb{C}_{\widehat{q}}$'s inputs are both y *and* $\widehat{q}$. The reason for using this notation is to emphasize that whenever $\widehat{q}$ is held fixed at some $q \in \mathcal{Q}$, $\mathbb{C}_{\widehat{q}}$ becomes the *qth* -candidate controller in $\mathcal{C}$.

The upper *control loop* shown in Figure 1.3 can be thought of as a conventional control system with $\mathbb{C}_{\widehat{q}}$

[1]As we shall see, there is generally no guarantee that such a controller sequence will converge to a "good" controller or, for that matter, to any controller at all. In fact, because of noise and disturbances, controller convergence proves to be more the exception than the rule.

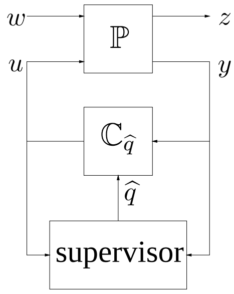

Figure 1.3: Parameter-Adaptive Control System

playing the role of a controller. The lower *adaptive loop* is where controller adjustment or adaptation takes place. This is done by a supervisor who generates $\widehat{q}$ on the basis of real-time observations of u and y.

Throughout these notes, $\mathbb{C}_{\widehat{q}}$ will be taken to be a dynamical system with a real, finite dimensional state space $\mathcal{X}_{\mathbb{C}}$ and defining equations of the form

$$\dot{x}_{\mathbb{C}} = F_{\widehat{q}}(x_{\mathbb{C}}, y), \qquad u = G_{\widehat{q}}(x_{\mathbb{C}}, y) \tag{1.1}$$

where for each fixed $q \in \mathcal{Q}$, the equations

$$\dot{\bar{x}}_{\mathbb{C}} = F_q(\bar{x}_{\mathbb{C}}, y), \qquad u_q = G_q(\bar{x}_{\mathbb{C}}, y)$$

model $\mathbb{C}_q$ and the F_q and G_q are locally Lipschitz continuous functions on $\mathcal{X}_{\mathbb{C}} \oplus \mathcal{Y}$. We say that $\mathbb{C}_{\widehat{q}}$ is implemented using *state-sharing* in that all of the $\mathbb{C}_q$, $q \in \mathcal{Q}$, share the same state $x_{\mathbb{C}}$. State sharing requires all $\mathbb{C}_q$ to have the same finite dimension. As we shall see later, this is in fact a very mild requirement.

The characterization of a parameter adaptive control system just given proves to be remarkably general. With proper interpretation it captures the essential features of most of the parameter-adaptive control systems proposed to date. Of course as is typical with such attempts at generalization, there are exceptions. Most require one to modify slightly the diagram in Figure 1.3. For example, the state $x_{\mathbb{C}}$ is sometimes fed to the supervisor and signals generated by the supervisor are often fed back to $\mathbb{C}_{\widehat{p}}$. We will discuss many of these modifications and generalizations in later chapters.

1.2.1 Supervisors

By a supervisor is meant a "hybrid" dynamical system with inputs u and y {and sometimes $x_{\mathbb{C}}$} and output $\widehat{q}$. Supervisors which adjust $\widehat{q}$ continuously are typically called *continuous tuning supervisors* while those which generate $\widehat{q}$'s which are piecewise constant are referred to as *switching supervisors*. Note that switching supervisors can be devised for $\mathcal{Q}$'s which are either continuums or finite sets whereas continuous tuning supervisors require $\mathcal{Q}$ to be a continuum. Of course, it is also possible to devise supervisors which perform both switching and tuning; in this case $\widehat{q}$ would be piecewise-continuous and $\mathcal{Q}$ might be a finite or countable union of continuums.

Although there are a great many different types of supervisors, all have certain features in common. All possess either a *switching logic* $\mathbb{S}$, a *tuner* $\mathbb{T}$ or a combination of the two, whose output is $\widehat{q}$. The vast majority of supervisors considered to date employ some form of "tuning error" and are accordingly classified as *tuning error supervisors*. The simplest example of a tuning error supervisor is a hybrid system of the form shown in Figure 1.4 where e is a suitably defined *tuning error* which drives $\mathbb{T}$, and $\mathbb{D}$ is a smooth dynamical system. Such supervisors are said to be *non-estimator based*.

The most common type of tuning error supervisor encountered in adaptive control carries out tuning not in $\mathcal{Q}$, but rather in a suitably defined *nominal process model parameter space* $\mathcal{P}$. The introduction of such

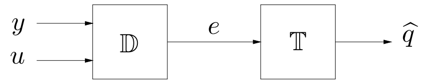

Figure 1.4: Non-Estimator Based Tuning Error Supervisor

a parameter space usually arises in connection with the definition of the class of admissible process models $\mathcal{M}$. In most applications $\mathcal{M}$ is taken to be of the form

$$\mathcal{M} = \bigcup_{p \in \mathcal{P}} \mathcal{M}_p$$

where $\mathcal{M}_p$ is a subset consisting of a *nominal process model* $\mathbb{N}_p$ plus suitably defined purturbed versions of $\mathbb{N}_p$. In these notes we shall consider $\mathcal{P}$'s which like $\mathcal{Q}$, are either finite sets or subsets of real finite dimensional, normed linear spaces. Tuning error supervisors associated with such $\mathcal{M}$'s are said to be *estimator-based* and are of the general form

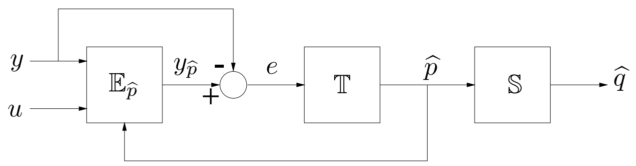

Figure 1.5: Estimator-Based, Tuning - Error Supervisor

Here $\mathbb{E}_{\widehat{p}}$ is a parameter-varing dynamical system, called a *multi-estimator*, which is defined in such a way so that if $\widehat{p}$ were frozen at any given value $p \in \mathcal{P}$, then $y_{\widehat{p}}$ would be an asymptotically correct estimate of y if $\mathbb{N}_p$ were the exact model of $\mathbb{P}$ and were there were no process disturbances or sensor noise. For this type of supervisor, the so-called *output estimation error* $y_{\widehat{p}} - y$ serves as the tuning error input to $\mathbb{T}$. Throughout these notes, $\mathbb{E}_{\widehat{p}}$ will usually be taken to be a dynamical system with a real, finite dimensional state space $\mathcal{X}_{\mathbb{E}}$ and defining equations of the form

$$\dot{x}_{\mathbb{E}} = H_{\widehat{p}}(x_{\mathbb{E}}, y, u), \qquad y_{\widehat{p}} = J_{\widehat{p}}(x_{\mathbb{E}}, y) \tag{1.2}$$

where for each fixed $p \in \mathcal{P}$, the H_p and J_p are locally Lipschitz continuous functions on $\mathcal{X}_{\mathbb{E}} \oplus \mathcal{Y} \oplus \mathcal{U}$. For each $p \in \mathcal{P}$, we often make reference to the dynamical system

$$\dot{\bar{x}}_{\mathbb{E}} = H_p(\bar{x}_{\mathbb{E}}, y, u), \qquad \bar{y}_p = J_p(\bar{x}_{\mathbb{E}}, y)$$

which we call the *pth estimator* and denote by the symbol $\mathbb{E}_p$.

By far the most common situation is when $\mathcal{P} = \mathcal{Q}$ and $\widehat{q} \stackrel{\Delta}{=} \widehat{p}$ in which case $\mathbb{S}$ is the identity system. Note that in this case $\widehat{p}$ is actually playing two roles. On the one hand it is the index of multi-estimator's current output signal while on the other it is the index of the controller currently being applied to the process. It is worth pointing out here that there is no compelling reason to equate $\widehat{q}$ with $\widehat{p}$, and in fact there are situation when it is advantageous not to do so. We will discuss these situations in subsequent chapters.

The flow diagrams shown in Figures 1.4 and 1.5 characterize the architectures of most of the tuning error supervisors proposed to date. There are however, exceptions. The most common of these can be dealt with by minor modification of the diagrams in Figures 1.4 and 1.5. In addition, there are several especially important examples of estimator-based supervisors which are not of the tuning error type. These will be discussed in later chapters.

1.2.2 The Self-Tuning Paradigm

Both estimator-based and non-estimator based supervisors tune either by continuous adjustment, by switching or by a combination of the two. The main function of a tuner is to adjust its output so as to cause e to become "small." The *self-tuning paradigm* is to define $\mathbb{D}$ and $\mathbb{C}_{\widehat{q}}$ in the non-estimator based case, or $\mathbb{E}_{\widehat{p}}$, $\mathbb{C}_{\widehat{q}}$ and $\mathbb{S}$ in the estimator-based case, so that

1. with proper tuning, e can in fact be made small, and
2. if this occurs and tuning ceases, the entire adaptive control system subsequently performs in a satisfactory manner.

Each of these factors imposes specific requirements on $\mathcal{C}$ and on $\mathbb{D}$ in the non-estimator based case, or on $\mathcal{C}$, $\mathbb{S}$ and the $\mathbb{E}_p$, $p \in \mathcal{P}$ in the estimator-based case, which are not dependent on the way in which $\mathbb{T}$ is defined. The first implies in both cases, that $\mathcal{C}$ must be large enough so that for each admissible process model $\mathbb{M} \in \mathcal{M}$, there is a controller $\mathbb{C}_q \in \mathcal{C}$, for which the closed-loop system shown in Figure 1.6 meets prescribed specifications.

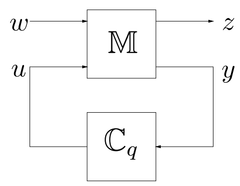

Figure 1.6: Supervised Control Loop with $\widehat{q} = q$

The second implies, for non-estimator based case, that $\mathbb{D}$ be defined so that it is impossible for any signal in the adaptive control system to grow without bound along a system trajectory on which $\widehat{q}$ is constant and e is identically zero. Similarly, for the estimator-based case, condition 2 implies that $\mathbb{S}$, the $\mathbb{C}_q$ and the $\mathbb{E}_p$ be defined so that it is impossible for any signal in the adaptive control system to grow without bound along a system trajectory on which $\widehat{p}$ is constant and e is identically zero. These are *detectability requirements* about which we will say more in the sequel.

The diagram is shown in Figure 1.7 provides a useful overall description of a tuning error adaptive control system, be it estimator-based or non-estimator based. For the non-estimator based case $\mathbb{S}$ would be the identity map {i.e., $\widehat{q} \stackrel{\Delta}{=} \widehat{p}$} , and there would be no parameter feedback from the output of $\mathbb{T}$ to $\mathbb{D}_{\widehat{p}}$ {i.e., $\mathbb{D}_{\widehat{p}} \stackrel{\Delta}{=} \mathbb{D}$}. For the estimator-based case $\mathbb{D}_{\widehat{p}}$ would denote the subsystem in Figure 1.5 consisting of $\mathbb{E}_{\widehat{p}}$ together with the comparator who's inputs are y an $y_{\widehat{p}}$ and who's output is e. To avoid unnecessary repetitions of definitions and concepts applicable to both estimator-based and non-estimator based systems, we shall often refer to this diagram when the topic under consideration is applicable to both types of systems.

1.2.3 Tuners

Tuners may be nonlinear dynamical systems, switching logics or a combination of the two. Tuners can therefore be thought of as hybrid dynamical systems possessing continouus states, discrete states, or a combination of the two. Consider first the estimator-based case. Although the specifics of tuners used by estimator-based supervisors vary greatly from algorithm to algorithm, in most instances tuning is carried out in one of two fundamentally different ways depending on whether the 'path' $\widehat{p}$ takes in $\mathcal{P}$ is "pre-routed" or not. We say that $\mathbb{T}$ is *pre-routed* if its strategy is configured to move or adjust $\widehat{p}$ through $\mathcal{P}$ along a pre-specified path or *tuning route*, using real-time data to decide only if and when or how fast to change $\widehat{p}$

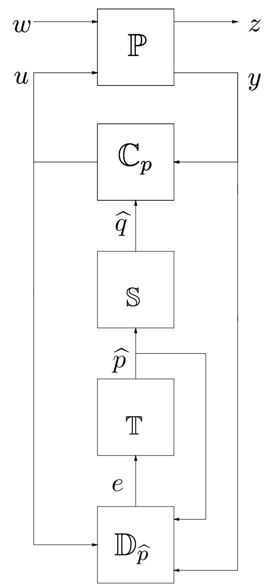

Figure 1.7: Tuning Error Based Adaptive Control System

from one value along the path to the next. In sharp contrast, for the non-prerouted case, the path in $\mathcal{P}$ along which $\widehat{p}$ is adjusted is not prespecified off-line but instead is determined in real time from the measured values of u and y. The same classifications and definitions apply to tuners used by non-estimator based supervisors. However in the non-estimator based case, tuning is typically pre-routed.

Because a tuner is a hybrid dynamical system, it will usually possess certain "rest" or "equlibrium" values in $\mathcal{P}$ at which $\widehat{p}$ remains fixed if $e = 0$. To be more precise, let us agree to say that a tuner $\mathbb{T}$ has an an *equilibrium value* $p_0 \in \mathcal{P}$ at time $t_0 \geq 0$, if there is a value for $\mathbb{T}$'s state at $t = t_0$ such that for every pair of input signals y and u consistent with the constraints $e = 0, \;\; t \geq t_0$, $\mathbb{T}$'s output $\widehat{p}$ remains fixed at p_0 for $t \geq t_0$. We assume that $\mathbb{T}$ is stationary to the extent that its possible equilibrium values at t_0 are independent of t_0 and we define the $\mathbb{T}$'s *equilibrium set* $\mathcal{E}_{\mathbb{T}}$ to be the set of all such values of p in $\mathcal{P}$. The same definitions apply to tuners used by non-estimator based supervisors.

To grasp what is meant by these ideas, consider for example a tuner of the form

$$\dot{\widehat{p}} = f_1(\widehat{p}) + f_2(\widehat{p}, y, u, x_{\mathbb{E}})e \tag{1.3}$$

where the $f_i(\cdot)$ are nonlinear functions. A great many continuous time tuning algorithms can be so modelled. For this example take $\mathcal{P}$ to be the whole linear space $\mathbb{R}^{n_p}$, n_p being the size of $\widehat{p}$. Then $\mathcal{E}_{\mathbb{T}} = \{p : f_1(p) = 0, \; p \in \mathcal{P}\}$. Note that the one-parameter, pre-routed tuner $\dot{\widehat{p}} = e^2, \;\; e \stackrel{\Delta}{=} y$ fits this description; in this case $\mathcal{E}_{\mathbb{T}} = \mathcal{P} = \mathbb{R}$. The two-parameter non-prerouted tuner

$$\begin{aligned} e &= y \\ \widehat{p} &= \begin{bmatrix} x_{\mathbb{C}} \\ y \end{bmatrix} e + x_{\mathbb{T}} \\ \dot{x}_{\mathbb{T}} &= \left(2\lambda \begin{bmatrix} x_{\mathbb{C}} \\ y \end{bmatrix} - \begin{bmatrix} u \\ 0 \end{bmatrix} \right) e \end{aligned}$$

is also of this form. In this case $\mathcal{E}_{\mathbb{T}} = \mathcal{P} = \mathbb{R}^2$. Note that for the diagram in Figure 1.3 to accurately describe this particular adaptive control system, it would be necessary to add an arrow from the multi-controller to the supervisor to account $x_{\mathbb{C}}$.

As another example, take for $\mathbb{T}$ a pre-routed switching algorithm of the form

$$\widehat{p}(t) = \rho(i), \quad t \in [t_i, t_{i+1}), \quad i \geq 0$$

where ρ is a function from the non-negative integers to $\mathbb{R}^{n_p}$, $t_0 = 0$, and for $i > 0$

$$t_i = \begin{cases} \min_{t > t_{i-1}}\{t : \int_0^t \|e_{\mathbb{T}}\| d\tau = i!\}, & \text{if this set is nonempty} \\ \infty, & \text{otherwise} \end{cases}$$

In this case $\mathcal{P}$ is the image of ρ and $\mathcal{E}_{\mathbb{T}} = \mathcal{P}$. Note also that the path through $\mathcal{P}$, namely $\rho(0), \rho(1), \rho(2), \rho(3), \ldots$ is prespecified and fixed.

1.3 Linear Adaptive Control System

By a *linear adaptive control system* is meant an adaptive control system in which the admissible process models in $\mathcal{M}$, the candidate controllers in $\mathcal{C}$, and the $\mathbb{D}_p$ shown in Figure 1.7 are finite dimensional time-invariant, linear systems. Thus in the linear case, the sub-system shown in Figure 1.8 is a parameter-varying

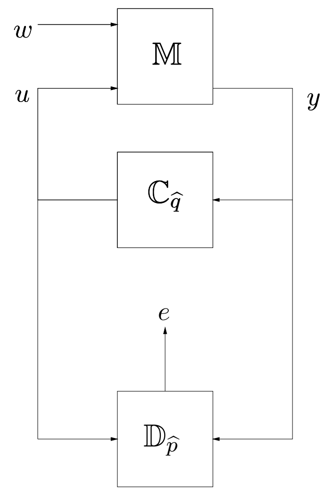

Figure 1.8: Closed-Loop Parameterized System

linear dynamical system depending on two parameter vectors, namely $\widehat{p}$ and $\widehat{q}$. We call this model the adaptive control system's *closed-loop parameterized sub-system* {at $\mathbb{M}$}.

As we've already stated, the most common situation is when $\mathcal{Q} = \mathcal{P}$ and

$$\widehat{q} \triangleq \widehat{p} \tag{1.4}$$

For the remainder of this chapter we will restrict our attention to this case, leaving generalizations for discussions in late chapters.

Under condition (1.4), the subsystem shown in Figure 1.8 admits a parameter-dependent linear model of the form

$$\dot{x} = A_{\widehat{p}}x + B_{\widehat{p}}w \tag{1.5}$$
$$e = C_{\widehat{p}}x + D_{\widehat{p}}w \tag{1.6}$$

where x is is a composite state vector whose sub-vectors are the state of $\mathbb{M}$, the state of $\mathbb{C}_{\widehat{p}}$ and the state of $\mathbb{D}_{\widehat{p}}$ respectively. We denote this system by $\Sigma_{\widehat{p}}$. In the sequel we uncover some basic properties $\Sigma_{\widehat{p}}$ must possess in order for the overall adaptive control system under consideration to meet the minimal requirement of stability.

1.3.1 The Detectability Requirement

As we've already said, the purpose of a tuner $\mathbb{T}$ within an adaptive control system is to adjust $\widehat{p}$ to make e small - and when e is small enough, tuning should subsequently stop. The issue we want to address here is this: What happens to $\Sigma_{\widehat{p}}$'s state if e is identically zero and $\widehat{p}$ is constant? Consideration of this question leads naturally to the following concept.

With $\mathcal{E}$ any fixed, nonempty subset of $\mathbb{T}$'s parameter space $\mathcal{P}$, let us agree to call the parameter-dependent system (1.5)-(1.6) *detectable on* $\mathcal{E}$ if for each $p \in \mathcal{E}$, x goes to zero as $t \to \infty$ along each trajectory on which $\widehat{p} = p$ and both e and w equal zero for $t \in [0, \infty)$. It is easy to see that $\Sigma_{\widehat{p}}$ is detectable on $\mathcal{E}$ just in case the matrix pair (C_p, A_p) is detectable in the usual linear system theory sense, for each fixed value of $p \in \mathcal{E}$.

Suppose $\Sigma_{\widehat{p}}$ is detectable on $\mathcal{E}$. We define $\Sigma_{\widehat{p}}$'s *detectability margin on* $\mathcal{E}$ to be the supremum of the detectability margins[2] of the (C_p, A_p) as p ranges over $\mathcal{E}$. The definition implies that if $\Sigma_{\widehat{p}}$ is detectable on $\mathcal{E}$, then its detectability margin on $\mathcal{E}$ is nonnegative. Moreover if $\mathcal{E}$ is compact, a positive detectability margin in assured.

Our aim here is to briefly explain why it is important for $\Sigma_{\widehat{p}}$ to be detectable. Towards this end, suppose we call a proposed tuner $\mathbb{T}$, a *stabilizer* of $\Sigma_{\widehat{p}}$ if for each tuner initial state, and each initial state $x(0)$ of $\Sigma_{\widehat{p}}$, the state response x of $\Sigma_{\widehat{p}}$ tends to zero as $t \to \infty$ assuming that $w \equiv 0$ and that $\widehat{p}$ tuned by $\mathbb{T}$.

Suppose $\mathbb{T}$ is a candidate tuner of $\widehat{p}$. The definition of detectability implies that if $\Sigma_{\widehat{p}}$ is not detectable on $\mathbb{T}$'s equilibrium set $\mathcal{E}_{\mathbb{T}}$, then for some initial state $x(0)$, and parameter value $p_0 \in \mathcal{E}_{\mathbb{T}}$, the tuned system Σ_{p_0}, with $w = 0$, will admit a state response $x(t)$ which does not go to zero and which generates the tuning error signal $e(t) = 0$. Moreover since p_0 is in $\mathbb{T}$'s equilibrium set, for some initial tuner state, $\mathbb{T}$'s response to the input $e \equiv 0$ will be $\widehat{p} \equiv p_0$. Therefore if $\mathbb{T}$ is so initialized, then the tuning error generated by $\Sigma_{\widehat{p}}$ with $\widehat{p}$ tuned by $\mathbb{T}$, will be identically zero, $\widehat{p}$ will remain fixed at p_0, and x will not go to zero. In other words, if $\Sigma_{\widehat{p}}$ is not detectable on $\mathbb{T}$'s equilibrium set, then $\mathbb{T}$ cannot be a stabilizer of $\Sigma_{\widehat{p}}$. We are led to the following theorem.

Theorem 1 (Necessity of Detectability) *A necessary condition for tuner $\mathbb{T}$ to be a stabilizer of the closed-loop parameterized system $\Sigma_{\widehat{p}}$, is that $\Sigma_{\widehat{p}}$ be detectable on the equilibrium set $\mathcal{E}_{\mathbb{T}}$ of $\mathbb{T}$.*

The preceding theorem makes clear why it is important for $\Sigma_{\widehat{p}}$ to be detectable on the equilibrium set of its tuner. In assessing the significance of this finding, reader should keep in mind that in most cases of interest, a tuner's equlibrium set is all of $\mathcal{P}$. In other words, most of the time one wants to have a closed-loop parameterized system which is detectable on the entire space $\mathcal{P}$ within which $\widehat{p}$ takes values. When this is so, $\Sigma_{\widehat{p}}$ is said to be *globally detectable*.

The definition of a stabilizing tuner of $\Sigma_{\widehat{p}}$ stipulates that $\mathbb{T}$ should be able to tune $\widehat{p}$ so as to ensure $x \to 0$ if $w = 0$. If $w \neq 0$, the most one can usually expect of a tuner is that its tuning of $\widehat{p}$ ensure that x remain bounded. If boundedness of x is all thats required, then detectability of $\Sigma_{\widehat{p}}$ on a tuner's equalibrium set is no longer necessary. However without $\Sigma_{\widehat{p}}$ being detectable on $\mathcal{E}_{\widehat{p}}$, w typically must satisfy stringent requirements {commonly known as persistency of excitation conditions} in order boundedness of x to be assured.

[2]The *detectability margin* of a fixed, parameter independent, detectable pair (C, A) is the largest real number μ such that $(C, \lambda I + A)$ is detectable for all $\lambda \in [0, \mu)$

Example 1 Suppose we take for $\mathbb{P}$'s model class $\mathcal{M}$, the set of all one-dimensional system $\dot{y} = ay + gu$. Suppose that $a_{\mathbb{P}} > 0$ and $g_{\mathbb{P}} \neq 0$ are numbers such that

$$\dot{y} = a_{\mathbb{P}} y + g_{\mathbb{P}} u \tag{1.7}$$

is a correct model of $\mathbb{P}$. To stabilize this system, consider using an simple multi-controller of the form

$$u_{\mathbb{C}} = \widehat{f} y \tag{1.8}$$

where, if we knew the values of $a_{\mathbb{P}}$ and $g_{\mathbb{P}}$ we'd choose $\widehat{f}$ so that $a_{\mathbb{P}} + g_{\mathbb{P}}\widehat{f} = -1$ since this would stabilize (1.7); but since $a_{\mathbb{P}}$ and $g_{\mathbb{P}}$ are not known, we might instead try to choose $\widehat{f}$ in accordance with the the idea of "certainty-equivalence" {to be discussed later}, so that

$$\widehat{a} + \widehat{g}\widehat{f} = -1 \tag{1.9}$$

where $\widehat{a}$ and $\widehat{g}$ are estimates of $a_{\mathbb{P}}$ and $g_{\mathbb{P}}$ respectively. Suppose these estimates are generated by the non-prerouted, "pseudo-gradient" algorithm

$$\dot{\widehat{a}} = -\bar{y}e \tag{1.10}$$

$$\dot{\widehat{g}} = -\bar{u}e \tag{1.11}$$

where e is the output estimation error

$$e = \widehat{y} - y \tag{1.12}$$

and $\widehat{y}$ is the output of the multi-estimator defined by the equations

$$\widehat{y} = (\widehat{a}+1)\bar{y} + \widehat{g}\bar{u} - y \tag{1.13}$$

$$\dot{\bar{y}} = -\bar{y} + y \tag{1.14}$$

$$\dot{\bar{u}} = -\bar{u} + u \tag{1.15}$$

Note that there is no provision in (1.11) for preventing $\widehat{g}$ from passing through zero. Thus situations could occur under which (1.9) could not be solved for $\widehat{f}$. Thus to avoid this possibility, rather than using (1.9) to define $\widehat{f}$, let us instead consider tuning $\widehat{f}$ with the "gradient" adjustment law

$$\dot{\widehat{f}} = -\frac{1}{2}\frac{\partial e_D^2}{\partial \widehat{f}} = -\widehat{g}e_D \tag{1.16}$$

where e_D is the "design error"

$$e_D \triangleq \widehat{a} + \widehat{g}\widehat{f} + 1 \tag{1.17}$$

For the adaptive system under consideration, (1.10), (1.11), (1.16) and (1.17) define a tuner $\mathbb{T}$ with parameter vector

$$\widehat{p} \triangleq \begin{bmatrix} \widehat{g} \\ \widehat{a} \\ \widehat{f} \end{bmatrix}$$

and parameter space $\mathcal{P} \triangleq \mathrm{I\!R}^3$. Note that $\mathbb{T}$'s equilibrium set is

$$\mathcal{E}_{\mathbb{T}} = \left\{ \begin{bmatrix} g \\ a \\ f \end{bmatrix} : g(a + gf + 1) = 0,\ \begin{bmatrix} g \\ a \\ f \end{bmatrix} \in \mathcal{P} \right\}$$

For this example, $\mathbb{C}_{\widehat{p}}$ is defined by (1.8), $\mathbb{E}_{\widehat{p}}$ by (1.13)-(1.15), and tuning error e by (1.12). Therfore the closed-loop parameterized system $\Sigma_{\widehat{p}}$ determined by these equations, process model (1.7), and the feedback connection $u \triangleq u_{\mathbb{C}}$ is

$$e = [\,-1 \quad \widehat{a}+1 \quad \widehat{g}\,]\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} \qquad \begin{bmatrix} \dot{y} \\ \dot{\bar{y}} \\ \dot{\bar{u}} \end{bmatrix} = \begin{bmatrix} a_{\mathbb{P}} + g_{\mathbb{P}}\widehat{f} & 0 & 0 \\ 1 & -1 & 0 \\ \widehat{f} & 0 & -1 \end{bmatrix}\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} \tag{1.18}$$

where $\widehat{p} \triangleq [\,\widehat{g} \quad \widehat{a} \quad \widehat{f}\,]'$. Fix f at any value such that $a_{\mathbb{P}} + g_{\mathbb{P}}f > 0$ and observe that the point $\bar{p} \triangleq [\,0 \quad a_{\mathbb{P}} + g_{\mathbb{P}}f \quad f\,]' \in \mathcal{E}_{\mathbb{T}}$. It is easy to verify that for $\widehat{p}$ fixed at $\bar{p}$, $\Sigma_{\bar{p}}$ admits the unbounded solution

$$\begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} e^{(a_{\mathbb{P}}+g_{\mathbb{P}}f)t} \\ \frac{e^{(a_{\mathbb{P}}+g_{\mathbb{P}}f)t}}{1+a_{\mathbb{P}}+g_{\mathbb{P}}f} \\ 0 \end{bmatrix}$$

along which $e = 0$ so $\Sigma_{\widehat{p}}$ is not detectable on $\mathcal{E}_{\mathbb{T}}$. Careful examination of the equations involved reveals that $\Sigma_{\widehat{p}}$ is not detectable on $\mathcal{E}_{\mathbb{T}}$ because $\mathcal{E}_{\mathbb{T}}$ contains points of the form $[\,g \quad a \quad f\,]'$ at which $a + gf > 0$. We will return to this example at the end of this chapter.

Example 2 Take $\mathcal{P} = \mathbb{R}$ and let $\Sigma_{\widehat{p}}$ be the closed-loop parameterized system defined by $e = C_{\widehat{p}}x$ and $\dot{x} = A_{\widehat{p}}x$ where

$$C_p = [\,1 \quad 0\,] \qquad A_p \begin{bmatrix} 1 & 1 \\ 0 & p \end{bmatrix}$$

Since this system is detectable on $\mathcal{P}$, $\Sigma_{\widehat{p}}$ must be detectable on each nonempty subset $\mathcal{E} \subset \mathcal{P}$. In spite of this, observe that no matter how $\widehat{p}$ is tuned, $\Sigma_{\widehat{p}}$ can have an unbounded state response {e.g. with $x(0) = [\,1 \quad 0\,]'$, $x(t) = e^t\,[\,1 \quad 0\,]'$}, so it is impossible to keep x from blowing up no matter how $\widehat{p}$ is tuned. This example therefore shows that detectability of $\Sigma_{\widehat{p}}$ on a known subset $\mathcal{E} \subset \mathcal{P}$, is not sufficient to insure that there is a tuner $\mathbb{T}$ which is a stabilizer of $\Sigma_{\mathbb{T}}$. Sufficient conditions for such a tuner to exist are discussed in the next section.

1.4 Non-Estimator Based Supervision

As Example 2 has just established, merely requiring a closed-loop parameterized system to be detectable on a given subset $\mathcal{E} \subset \mathcal{P}$, is not enough to insure the existence of a stabilizing tuner of $\Sigma_{\widehat{p}}$, even if $\mathcal{E} = \mathcal{P}$. However it turns out that if $\Sigma_{\widehat{p}}$ is detectable on such a subset, and in addition, if $\mathcal{E}$ is countable and contains a parameter value p_0 for which Σ_{p_0} is internally stable, then without knowing p_0 it is possible to construct a tuner $\mathbb{T}$, with equilibrium set $\mathcal{E}_{\mathbb{T}} = \mathcal{E}$, which is a stabilizer of $\Sigma_{\widehat{p}}$. Thus for the case when $\mathcal{P} = \mathbb{R}^{n_P}$, for such a tuner to exist, it's enough to design $\mathbb{C}_{\widehat{p}}$ and $\mathbb{D}$ so that the resulting closed-loop parameterized system $\Sigma_{\widehat{p}}$ is detectable on a countable subset $\mathcal{E} \subset \mathcal{P}$ containing a point p_0 at which Σ_{p_0} is stable. The following example shows that this is very easy to do without assuming very much about $\mathbb{P}$.

Example 3 For a fixed integer $n > 0$, define parameter vector $p = [\,p_1 \quad p_2 \quad \cdots \quad p_{2n+1}\,]'$, and parameter-dependent polynomials

$$\begin{aligned} \beta_p(s) &= s^n + p_n s^{n-1} + \ldots p_2 s + p_1 \\ \alpha_p(s) &= p_{2n+1}s^n + p_{2n}s^{n-1} + \ldots p_{n+2}s + p_{n+1} \end{aligned}$$

Choose $\gamma(s)$ to be any, monic, stable polynomial of degree n and let $\mathbb{A}_p$ and $\mathbb{B}_p$ be n-dimensional realizations of $\frac{\alpha_p(s)}{\gamma(s)}$ and $\frac{\gamma(s)}{\beta_p(s)}$ respectively. Define $\mathbb{C}_{\widehat{p}}$ to be the cascade interconnection of $\mathbb{A}_{\widehat{p}}$ with $\mathbb{B}_{\widehat{p}}$ as shown in Figure

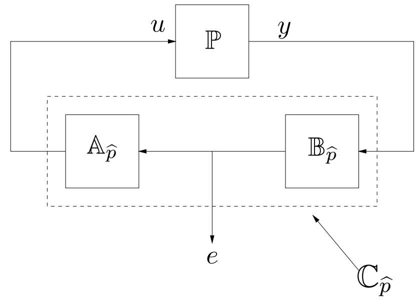

Figure 1.9: Parameterized System $\Sigma_{\widehat{p}}$

1.9. Let e be as shown in the diagram. Note that the diagram in Figure 1.5 would have to be modified slightly to account for this particular definition of e.

Let λ_S be γ's stability margin[3]. Fix $\lambda \in (0, \lambda_S)$ and $\widehat{p}$ at any value $p \in \mathcal{P} \stackrel{\Delta}{=} \mathbb{R}^{2n+1}$. Observe that along any closed loop trajectory on which e is identically zero, both u and y must go to zero as fast as $e^{-\lambda t}$ since $\mathbb{A}_p$ and the inverse of $\mathbb{B}_p$ are proper stable linear systems with the same stability margin λ_S. From this and the standing assumption that $\mathbb{P}$ is an observable system, it follows that for each $p \in \mathcal{P}$, $\Sigma_{\widehat{p}}$ must be detectable through e with detectability margin λ_S; in other words, $\Sigma_{\widehat{p}}$ is globally detectable[4] with detectability margin λ_S. Let $\mathcal{E}$ be a countable, dense subset of $\mathcal{P}$; clearly $\Sigma_{\widehat{p}}$ is also detectable on $\mathcal{E}$. Moreover, if $\mathbb{P}$'s transfer function is sufficiently close {in say the $\mathcal{H}^\infty$ sense} to a transfer function of McMillan degree not exceeding n, then there must exist a vector $p_0 \in \mathcal{E}$ for which Σ_{p_0} is stable. Thus $\mathcal{E}$ and $\Sigma_{\widehat{p}}$ will have what's required so long as $\mathbb{P}$'s transfer function is close enough to a transfer function of McMillan degree no greater than n.

1.4.1 A Pre-Routed Stabilizing Tuner for a Closed-Loop Parameterized System

What we want to do now is to explain how to construct a stabilizing tuner $\mathbb{T}$ for an unforced, closed-loop parameterized system $\Sigma_{\widehat{p}}$

$$\dot{x} = A_{\widehat{p}} x \tag{1.19}$$
$$e = C_{\widehat{p}} x \tag{1.20}$$

whose parameter space $\mathcal{P}$ contains a known countable subset $\mathcal{E}$ with the following properites:

1. $\Sigma_{\widehat{p}}$ is detectable on $\mathcal{E}$.
2. There exists a parameter value $p_0 \in \mathcal{E}$ at which A_{p_0} is a stability matrix.

Apart from this nothing is assumed about $\Sigma_{\widehat{p}}$ other that e can be measured. In particular, neither the parameter-dependent pair (C_p, A_p) nor p_0 are presumed to be known. Of course with so little known, one should not expect to come up with a tuner $\mathbb{T}$ worthy of actual implementation unless perhaps $\mathcal{E}$ is a finite set with a small number of elements.

The type of tuner which we will consider, consists of the cascade connection of two subsystems, one a *scheduling logic* $\mathbb{S}$ and the other a memoryless map $\rho : \{1, 2, \ldots, \infty\} \rightarrow \mathcal{P}$ called a *routing function.*

[3]The *stability margin* of a real stable polynomial $\alpha(s)$ is tha largest value of μ for which $\alpha(\lambda + s)$ is a stable polynomial for $\lambda \in [0, \mu)$. The stability margin of a constant stability matrix A or a time-invariant linear system (A, B, C, D) is the stability margin of the characteristic polynomial of A.

[4]It is interesting to note that this will no longer be true if the positions of $\mathbb{A}_{\widehat{p}}$ and $\mathbb{B}_{\widehat{p}}$ are interchanged, unless $\mathbb{P}$ is restricted to be minimum phase with all zeros possessing real parts more negative than $-\lambda_S$

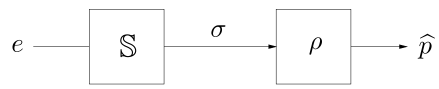

Figure 1.10: Prerouted Tuner

All that is required of ρ is that it have the *revisitation property:* That is, for any $p \in \mathcal{E}$ and any positive integer i there must exist an integer $j \geq i$ at which $\rho(j) = p$. In other words, ρ must have the property that the pre-routed path $\rho(1), \rho(2), \ldots$ *revisits* each point in $\mathcal{E}$ infinitely often. One way to define such a function is as follows.

1. If $\mathcal{E} = \{p_1, p_2, \ldots, p_m\}$ is a finite set, define ρ to be the m periodic function whose first m values are $p_1, p_2, \ldots, p_{m-1}$ and p_m respectively.

2. If $\mathcal{E} = \{p_1, p_2, \ldots, \}$ is not a finite set, define ρ be the function whose sequence of values $\rho(1), \rho(2), \ldots$ are the elements of the sequence $p_1, p_1, p_2, p_1, p_2, p_3, p_1, p_2, p_3, p_4, p_1, \ldots$

There are many possible ways to define $\mathbb{S}$, depending what one is trying to accomplish. For our purposes, we shall take $\mathbb{S}$ to be a hybrid dynamical systems whose input is e and whose output is a piecewise-constant switching signal σ taking values in the set of positive integers. $\mathbb{S}$'s state consists of four variables - σ, a timing signal τ, a piecewise-continuous *monitoring signal* μ and a piecewise constant 'sampled monotoring signal' $\bar{\mu}$. Both μ and $\bar{\mu}$ take values in $[0, \infty)$. Timing signal τ takes values in the closed interval $[0, \tau_D]$, where τ_D is a preselected positive number called a *dwell time.* $\mathbb{S}$'s dynamics are defined by the computer diagram shown in Figure 1.11.

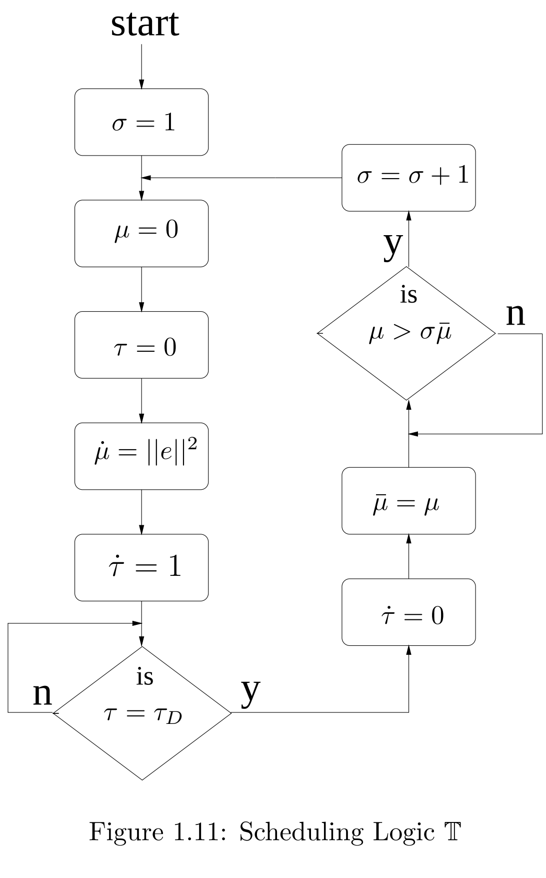

Figure 1.11: Scheduling Logic $\mathbb{T}$

The functioning of $\mathbb{S}$ is as follows. During the first τ_D time units after the algorithm is initiated, τ is increased linearly from 0 to τ_D using a reset integrator and μ is increased from 0 according to the rule

$$\dot{\mu} = ||e||^2 \tag{1.21}$$

where $||e|| \stackrel{\Delta}{=} \sqrt{e'e}$. Just at the end of this period, τ is reset to zero, the reset integrator is turned off, and $\bar{\mu}$ is set equal to the present value of μ. So long as μ remains less than or equal to $\sigma\bar{\mu}$, the updating of μ continues according to (1.21). If and when μ becomes greater than $\sigma\bar{\mu}$, σ is incremented by 1, μ and τ are reset to zero, and the entire process is repeated. Note that the time between any two successive switchings of σ can never be smaller than τ_D. Said differently, σ "dwells" at each of its values for at least τ_D time units. Because of this, infinitely fast switching cannot occur so existence and uniqueness of solutions to the differential equations involved is not an issue.

There is a fairly straight forward way to go about analyzing the type of adaptive control system we've just described. The key step is to prove that switching stops in finite time. In particular, for the problem at hand, the trick is to show that there must be a finite integer $\bar{\sigma}$, depending only on the pair (C_p, A_p) and not on the initial value of x, which σ cannot exceed in value. Before addressing this claim, let us consider its consequences. What the claim implies is that σ can switch at most a finite number of times and therefore that there must be a time $\bar{t}$ beyond which σ is constant and $\mu \leq \bar{\sigma}\bar{\mu}$. The latter assures that

$$\int_{\bar{t}}^{t} ||e(s)||^2 ds \leq \bar{\sigma}\bar{\mu}, \quad t \geq \bar{t}$$

and thus that e is a zeroing signal. Moreover since $\Sigma_{\widehat{p}}$ is detectable on $\mathcal{E}$ and $\widehat{p}$ is fixed at some value $\bar{p} \in \mathcal{E}$, $(C_{\bar{p}}, A_{\bar{p}})$ is detectable[5]. As a consequence, for $t \geq \bar{t}$ it is possible to rewrite (1.19) as

$$\dot{x} = (A_{\bar{p}} + KC_{\bar{p}})x - Ke$$

where K is any matrix which stabilizes $A_{\bar{p}} + KC_{\bar{p}}$. Therefore $x \to 0$ since e is a zeroing signal. In fact, because $\dot{x} = A_{\bar{p}}x$ is a time invariant linear system, x must go to zero as fast at $e^{-\lambda t}$, $-\lambda$ being the largest of the real parts of $A_{\bar{p}}$'s stable eigenvalues. In other words, to prove that this tuner is an {exponential} stabilizer of $\Sigma_{\widehat{p}}$, it is enough to show that there is an integer $\bar{\sigma}$ which σ cannot exceed in value.

Here briefly is the idea. Let q be any point in $\mathcal{P}$ at which A_q is a stability matrix. If $C_q = 0$, let $\bar{\sigma}$ be the first positive integer such that $\rho(\bar{\sigma}) = q$. Then either there is an interval $[t_0, t_1)$ of maximum length on which $\sigma = \bar{\sigma}$ or σ never gets larger than $\bar{\sigma}$. If the latter is true, then we are done. On the other hand, if the former is true, then $\mu = 0$ on $[t_0, t_1)$ so for such t, $\mu < \bar{\sigma}\bar{\mu}$. Because of $\mathbb{S}$'s definition, this means that no more switching can occur, that $t_1 = \infty$ and thus that σ can grow no larger than $\bar{\sigma}$.

Now suppose that $C_q \neq 0$. Reduce (C_q, A_q) to an observable pair $(\bar{C}, \bar{A})$ by picking any full rank matrix R whose kernel is the unobservable space of (C_q, A_q) and solving the linear equations $C_q = \bar{C}R$, $RA_q = \bar{A}R$ for $\bar{C}$ and $\bar{A}$ respectively. Note that $\bar{A}$ must be a stability matrix because A_q is.

Let $G(t)$ denote the observability Gramian

$$G(t) \stackrel{\Delta}{=} \int_0^t e^{\bar{A}'s}\bar{C}'\bar{C}e^{\bar{A}s} ds$$

Note that $G(\infty)$ must exist because of $\bar{A}$'s stability. Moreover, $G(\tau_D)$ must be positive definite because of the observability of $(\bar{C}, \bar{A})$. This implies that

$$\lambda \stackrel{\Delta}{=} \sup_x \frac{x'G(\infty)x}{x'G(\tau_D)x} < \infty$$

[5]There is of course no reason to conclude that $A_{\bar{p}}$ is a stability matrix.

and that

$$G(\infty) \leq \lambda G(\tau_D) \tag{1.22}$$

Now let σ_q be the least integer no smaller than λ for which $\rho(\sigma_q) = q$. Because of the revisitation property, σ_q must necessarily exist. In view of (1.22)

$$G(\infty) \leq \sigma_q G(\tau_D) \tag{1.23}$$

We claim that $\bar{\sigma} \stackrel{\Delta}{=} \sigma_q$ has the desired property. To prove that this is so, we may as well assume that there is an interval on which $\sigma = \bar{\sigma}$. For if this were not the case then $\widehat{p}$ could not exceed $\bar{\sigma}$ and we would be done.

Let $[t_0,\ t_1)$ denote the largest interval on which $\sigma = \bar{\sigma}$. For $t \in [t_0, t_1)$, $\widehat{p} = \rho(\sigma) = q$ and

$$\mu(t) = \int_{t_0}^{t} ||C_q e^{A_q(w-t_0)} x(t_0)||^2 dw \leq \int_0^\infty ||\bar{C} e^{\bar{A}s} R x(t_0)||^2 ds = ||\sqrt{G(\infty)} R x(t_0)||^2$$

From this, (1.23) and the definitions of $\bar{\sigma}$ and $\bar{\mu}$ it follows that for $t \in [t_0, t_1)$,

$$\mu(t) \leq \sigma_q ||\sqrt{G(\tau_D)} R x(t_0)||^2 = \bar{\sigma} \int_0^{\tau_D} ||\bar{C} e^{\bar{A}s} R x(t_0)||^2 ds = \bar{\sigma}\bar{\mu}$$

Thus because of $\mathbb{S}$'s definition, no more switching can occur, $t_1 = \infty$ and thus σ can grow no larger than $\bar{\sigma}$. ■

1.5 Estimator-Based Supervision

In this section we illustrate by means of how a parameterized sub-system's detectability can be used in the analysis of an overall estimator-based adaptive control system. Consider again an unforced closed-loop parameterized system $\Sigma_{\widehat{p}}$ described by

$$\dot{x} = A_{\widehat{p}} x \tag{1.24}$$
$$e = C_{\widehat{p}} x \tag{1.25}$$

where now $\mathcal{P} = \mathbb{R}^{n_p}$ and $x \stackrel{\Delta}{=} [\,x'_{\mathbb{P}} \quad x'_{\mathbb{C}} \quad x'_{\mathbb{E}}\,]'$. Assume that e is a scalar, that $\Sigma_{\widehat{p}}$ is globally detectable and that the matrices A_p and C_p are continuously differentiable on $\mathcal{P}$. Later we will encounter many situations when it is possible to rewrite e as

$$e = x'_E(\widehat{p} - p_0) \tag{1.26}$$

where p_0 is a fixed but unknown vector in $\mathcal{P}$. Assume for the present that (1.26) is a correct formula.

Suppose that $\widehat{p}$ is tuned using the *pseudo-gradient tuner*

$$\dot{\widehat{p}} = -x_{\mathbb{E}}\bar{e} \tag{1.27}$$

where $\bar{e}_T$ is the so-called *normalized tuning error*

$$\bar{e} = \frac{1}{1 + \|x_{\mathbb{E}}\|^2} e \tag{1.28}$$

Note that $\bar{e}$ can also be written as

$$\bar{e} = e + x'_{\mathbb{E}}\dot{\widehat{p}} \tag{1.29}$$

1.5.1 Partial Lyapunov Analysis

Our aim is to anaylze the behavior of the overall system defined by (1.24),(1.25),(1.27), and (1.28). Observe that these equations define a dynamical system with state $\{x, \widehat{p}\}$ and locally Lipschitz vector field. Therefore by standard condition for local existence, we know that for each initial state $\{x_0, \widehat{p}_0\}$ there must by an interval $[0, T)$ of maximal length on which a unique solution $\{x, \widehat{p}\}$ to the equations exists. Evaluation of the partial Lyapunov function

$$V \triangleq ||\widehat{p} - p_0||^2 \tag{1.30}$$

along such a solution gives

$$\frac{d\|\widehat{p} - p_0\|^2}{dt} = 2\dot{\widehat{p}}'(\widehat{p} - p_0)$$

Therefore by (1.27)

$$\frac{d\|\widehat{p} - p_0\|^2}{dt} = -2\bar{e}x'_{\mathbb{E}}(\widehat{p} - p_0)$$

Using (1.26), and then (1.29) we get

$$\begin{aligned} \dot{V} &= -2\bar{e}e \\ &= -2\bar{e}(\bar{e}x'_{\mathbb{E}}\dot{\widehat{p}}) \end{aligned}$$

Therefore by (1.27)

$$\dot{V} = -2\bar{e}^2 - 2||\dot{\widehat{p}}||^2 \tag{1.31}$$

The first thing to note about (1.31) is that $\dot{V}$ is nonpositive which means that V is monotone nonincreasing. Since V is a nonnegative function, it must be that V is bounded between its initial value and 0. In view of (1.30), $\widehat{p}$ must be bounded as well. Therefore $\{x, \widehat{p}\}$ must exist globally, and $\widehat{p}$ must be bounded on $[0, \infty)$.

There are several other properties to be gleaned from this analysis. First observe that $\bar{e}$ must be bounded because of (1.28) and (1.26); similarly, $\dot{\widehat{p}}$ must be bounded because of (1.28) to (1.26). Next note that V must have a finite limit $V(\infty)$ as $t \to \infty$ because V is a bounded monotone function. This and (1.31) enable us to write

$$2\int_0^\infty (\bar{e}^2 + ||\dot{\widehat{p}}||^2)ds = V(0) - V(\infty)$$

Since $V(0) - V(\infty)$ is a finite number, both $\bar{e}$ and $\dot{\widehat{p}}$ have finite $\mathcal{L}^2[0, \infty)$ norms. This is exactly what normalization buys. Without normalization the same anaylsis would provide boundedness of $\widehat{p}$, global existence and $\mathcal{L}^2$ boundedness of the unnormalized tuning error e - but neither boundedness nor $\mathcal{L}^2$ boundedness of $\dot{\widehat{p}}$ would any longer be assured. As we shall see in the sequel, the $\mathcal{L}^2$ boundedness of $\dot{\widehat{p}}$, or more specifically the implied fact that $\dot{\widehat{p}}$ is "nondestabilizing with zero growth rate" , will play a crucial role in establishing the boundedness of x.

1.5.2 Detectability Analysis

So far, the analysis has been reasonably straight-forward. Such arguments have for a long time marked the starting point for the stability analysis of adaptive control systems of all types. To get beyond this point and show that x is bounded can be a challenging matter requiring intricate, lengthy arguments, and a good bit of "signal chasing." It is exactly here where the global detectability of $\Sigma_{\widehat{p}}$ proves its worth. To show that x is bounded, we make use of two technical facts:

1. Because of global detectability, there exists a matrix K_p which stabilizes $A_p + K_pC_p$ for each fixed $p \in \mathcal{P}$; moreover, because C_p and A_p are continuously differentiable, such a K_p exists which is continuously differentiable as well.

2. The time-varying matrix $B(t) \triangleq A_{\widehat{p}} + K_{\widehat{p}}(C_{\widehat{p}} + \begin{bmatrix} 0 & 0 & \dot{\widehat{p}}' \end{bmatrix})$ is exponentially stable along any closed-loop trajectory of the adaptive system under consideration.

Armed with these technical facts, a proof that x is bounded is at hand: All one has to do is to use the expression

$$\bar{e} = (C_{\widehat{p}} + \begin{bmatrix} 0 & 0 & \dot{\widehat{p}}' \end{bmatrix})x$$

which is a consequence of (1.25), (1.29) and the definition of x - i.e., $x \triangleq [\, x'_{\mathbb{P}} \quad x'_{\mathbb{C}} \quad x'_{\mathbb{E}} \,]'$ to rewrite (1.24) as

$$\dot{x} = B(t)x - K_{\widehat{p}}\bar{e}$$

Since B is exponentially stable x can be viewed as the output of an exponentially stable linear system with input $-K_{\widehat{p}}\bar{e}$. Moreover since $K_{\widehat{p}}$ is bounded and $\bar{e}$ has a finite $\mathcal{L}^2[0,\infty)$ norm, $-K_{\widehat{p}}\bar{e}$ must be a zeroing signal. Therefore x must go to zero. In other words, by exploiting the structural property of global detectability together with several technical facts from linear system theory which we will prove in a moment, we have been able to show that $x \to 0$ in a very simple way. As we shall see, this *output injection based analysis* will be useful in studying adaptive control systems of all types.

1.5.3 Technical Tools

In the sequel it will be shown that the two technical claims made in the preceding section are correct. First we will prove the following proposition.

Proposition 1 *Let $\mathcal{P}$ be an open subset of a finite dimensional linear space on which (C_p, A_p) is a continuously differentiable, detectable pair. For each compact subset $\bar{\mathcal{P}} \subset \mathcal{P}$ there exists an open subset $\tilde{\mathcal{P}} \subset \mathcal{P}$ containing $\bar{\mathcal{P}}$ and a continuously differentiable function $p \longmapsto K_p$ on $\tilde{\mathcal{P}}$ which stabilizes $A_p + K_pC_p$ for each fixed $p \in \tilde{\mathcal{P}}$.*

The proof of the preceding proposition depends on the following fact.

Lemma 1 *A constant matrix pair $(C_{m\times n}, A_{n\times n})$ is detectable if and only if there exists a positive definite matrix Q such that*

$$QA + A'Q - C'C < 0 \tag{1.32}$$

Moreover if (1.32) holds for some such Q then the output injection matrix $K \triangleq -\frac{1}{2}Q^{-1}C'$ exponentially stabilizes $A + KC$.

Proof of Proposition 1: In view of Lemma 1, for each $p \in \bar{\mathcal{P}}$ there exists a positive definite matrix H_p such that

$$H_pA_p + A'_pH_p - C'_pC_p < 0 \tag{1.33}$$

Since C_p and A_p are continuously differentiable on $\mathcal{P}$, it follows that for each such $p \in \bar{\mathcal{P}}$ there is a positive number r_p which is sufficiently small so that at all points q in the open ball

$$\mathcal{B}_p \triangleq \{q : ||q - p|| < r_p,\ q \in \mathcal{P}\}$$

the inequality

$$H_pA_q + A_q'H_p - C_q'C_q < 0, \quad q \in \mathcal{B}_p \tag{1.34}$$

hold. The family of balls $\{\mathcal{B}_p : p \in \bar{\mathcal{P}}\}$ is an open cover of $\bar{\mathcal{P}}$. Since $\bar{\mathcal{P}}$ is compact, this family must contain a finite subfamily $\{\mathcal{B}_{p_i} : p_i \in \bar{\mathcal{P}}, i \in \{1, 2, \dots m\}\}$ which also covers $\bar{\mathcal{P}}$. Therefore the set

$$\tilde{\mathcal{P}} \triangleq \bigcup_{i=1}^{m} \mathcal{B}_{p_i}$$

is an open subset of $\mathcal{P}$ containing $\bar{\mathcal{P}}$.

To proceed, for each $i \in \{1, 2, \dots, m\}$ define $w_i : \mathcal{P} \to \mathbb{R}$ so that

$$w_i(p) = \begin{cases} (||p - p_i||^2 - r_{p_i})^2 & \text{if } p \in \mathcal{B}_{p_i} \\ 0 & \text{otherwise} \end{cases}$$

Note that each such w_i is continuously differentiably on $\mathcal{P}$ and positive on $\mathcal{B}_{p_i}$. Thus the function $w : \mathcal{P} \to \mathbb{R}$

$$w \triangleq \sum_{i=1}^{m} w_i$$

is continuously differentiably on $\mathcal{P}$ and positive on $\tilde{\mathcal{P}}$. It follows that for each $i \in \{1, 2, \dots, m\}$, the normalized function $\bar{w}_i : \tilde{\mathcal{P}} \to \mathbb{R}$, defined by

$$\bar{w}_i \triangleq \frac{w_i}{w}$$

is continuously differentiable on $\tilde{\mathcal{P}}$. Moreover

$$\sum_{i=1}^{m} \bar{w}_i(p) = 1, \quad p \in \tilde{\mathcal{P}} \tag{1.35}$$

We claim that the function

$$Q_p \triangleq \sum_{i=1}^{m} \bar{w}_i(p) H_{p_i}$$

is continuously differentiable and positive definite on $\tilde{\mathcal{P}}$ and satisfies

$$Q_pA_p + A_p'Q_p - C_p'C_p < 0, \quad p \in \tilde{\mathcal{P}} \tag{1.36}$$

Continuous differentiability is a consequence of the continuous differentiability of the $\bar{w}_i$. Positive definiteness is a consequence of the positive definiteness of the H_{p_i} and the positivness on the $\bar{w}_i$. To establish (1.36), fix $p \in \tilde{\mathcal{P}}$. If $p \in \mathcal{B}(p_i)$ then

$$H_{p_i}A_p + A_p'H_{p_i} - C_p'C_p < 0, \tag{1.37}$$

because of (1.34). On the other hand, if $p \notin \mathcal{B}(p_i)$ then $\bar{w}_i(p) = 0$ by definition. In either case it follows that

$$\bar{w}_i(p) \left\{ H(p_i)A_p + A_p'H(p_i) - C_p'C_p \right\} < 0$$

Since this must be true for all $i \in \{1, 2, \dots, m\}$

$$\sum_{i=1}^{m} \bar{w}_i(p) \left\{ H(p_i)A_p + A_p'H(p_i) - C_p'C_p \right\} < 0$$

From this, (1.35) and the definition of Q it follows that (1.36) holds. ■

Nondestabilizing Signals

Let us agree to call a bounded, piecewise-continuous matrix $B : [0, \infty) \rightarrow \mathbb{R}^{n \times m}$ is *nondestabilizing with growth rate* $\rho_B \geq 0$ if 1) for each real number $\rho > \rho_B$ there is a finite nonnegative constant b_ρ such that

$$\int_{\tau}^{t} |B(s)| ds \leq b_\rho + \rho(t - \tau) \qquad t \geq \tau \geq 0$$

and 2) ρ_B is the least nonnegative number with this property. Note that any piecewise-continuous bounded matrix $B : [0,\ \infty)$ is nondestabilizing with growth rate

$$\rho_B \leq \sup_{t \in [0, \infty)} |B(t)|$$

Examples of nondestabilizing signals with zero growth rates include all bounded, piecewise-continuous function $B : [0, \infty) \rightarrow \mathbb{R}^{n \times m}$ which tend to zero or which satisfy

$$\int_{0}^{\infty} |B(\mu)|^i d\mu < \infty \tag{1.38}$$

for either $i = 1$ or $i = 2$. We will encounter other examples in the sequel.

The term "nondestabilizing" is prompted by the following basic result.

Theorem 2 (Nondestabilization) *Let $A : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ and $B : [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ be bounded, piecewise-continuous matrices. Suppose that A is exponentially stable and let a and λ be positive constants for which A's state transition matrix $\phi_A(t, \tau)$ satisfies*

$$|\phi_A(t, \tau)| \leq a e^{-\lambda(t - \tau)}, \ \ \forall t \geq \tau \geq 0$$

Suppose B is nondestabilizing with growth rate ρ_B and for each number $\rho > \rho_B$, let b_ρ be a constant for which

$$\int_{\tau}^{t} |B(s)| ds \leq b_\rho + \rho(t - \tau), \quad t \geq \tau \geq 0 \tag{1.39}$$

Then for $\rho > \rho_B$, the state transition matrix of $A + B$ satisfies

$$|\Phi_{A+B}(t, \tau)| \leq a e^{(a b_\rho - (\lambda - a\rho)(t - \tau))}, \ \ t \geq \tau \geq 0$$

As an immediate consequence of this theorem we see that if B's growth rate is sufficiently small, namely

$$\rho_B < \frac{\lambda}{a}$$

then $A + B$ will be exponentially stable. This of course will always be true if B's growth rate is zero.

The preceding theorem is a simple consequence of the following lemma which is of importance in its own right.

Lemma 2 (Bellman-Gronwall) *If for some constant $c \geq 0$ and nonnegative, piecewise-continuous function $\alpha : [t_1, t_2) \rightarrow \mathbb{R}$, $w : [t_1, t_2) \rightarrow \mathbb{R}$ is a continuous function satisfying*

$$w(t) \leq c + \int_{t_1}^{t} \alpha(\mu) w(\mu) d\mu, \quad t \in [t_1, t_2) \tag{1.40}$$

then

$$w(t) \leq c e^{\int_{t_1}^{t} \alpha(\mu) d\mu}, \quad t \in [t_1, t_2) \tag{1.41}$$

The Bellman-Gronwall Lemma becomes quite plausable as soon as one recognizes that the solution to the scalar differential equation,

$$\begin{aligned} \dot{w} &= \alpha w \\ w(t_1) &= c \end{aligned}$$

or equivalant integral equation

$$w(t) = c + \int_{t_1}^{t} \alpha(\mu)w(\mu)du$$

is

$$w(t) = ce^{\int_{t_1}^{t} \alpha(\mu)d\mu}$$

Proof: Set

$$\begin{aligned} v(t) &= w(t)e^{-\int_{t_1}^{t} \alpha(\mu)d\mu} \\ u(t) &= \left(c + \int_{t_1}^{t} \alpha(\mu)w(\mu)d\mu\right) e^{-\int_{t_1}^{t} \alpha(\mu)d\mu} \end{aligned}$$

and note from (1.40) that

$$v(t) \leq u(t), \quad t \in [t_1, t_2) \tag{1.42}$$

Differentiating the expression for u and then replacing w with

$$v(t)e^{\int_{t_1}^{t} \alpha(\mu)d\mu}$$

one gets

$$\begin{aligned} \dot{u} &= -\alpha u + \alpha w e^{-\int_{t_1}^{t} \alpha(\mu)d\mu} \\ &= \alpha(v - u) \end{aligned}$$

Hence by (1.42) $\dot{u} \leq 0$ so $u(t) \leq u(t_1) = c,\ t \in [t_1, t_2)$. Therefore by (1.42),

$$v(t) \leq c, \quad t \in [t_1, t_2)$$

Multiplying through by $e^{\int_{t_1}^{t} \alpha(\mu)d\mu}$ and then replacing $v(t)e^{\int_{t_1}^{t} \alpha(\mu)d\mu}$ by $w(t)$ yields (1.41) which is the desired result. ■

Proof of Theorem: Fix τ and define $\theta(t) \triangleq \phi_{A+B}(t, \tau)$. Then

$$\begin{aligned} \dot{\theta} &= A\theta + B\theta \\ \theta(\tau) &= I \end{aligned}$$

By viewing $B\theta$ as a forcing function in the preceding, one may write the variation of constants formula

$$\theta(t) = \phi_A(t, \tau) + \int_{\tau}^{t} \phi_A(t, s)B(s)\theta(s)ds$$

Therefore

$$\begin{aligned} |\theta(t)| &= |\phi_A(t, \tau) + \int_{\tau}^{t} \phi_A(t, s)B(s)\theta(s)ds| \\ &\leq |\phi_A(t, \tau)| + \int_{\tau}^{t} |\phi_A(t, s)B(s)\theta(s)|ds \\ &\leq ae^{-\lambda(t-\tau)} + \int_{\tau}^{t} ae^{-\lambda(t-\tau)}|B(s)||\theta(s)|ds \end{aligned}$$

Hence

$$e^{\lambda(t-\tau)}|\theta(t)| \leq a + \int_{\tau}^{t} a|B(s)|\{e^{\lambda(s-\tau)}|\theta(s)|\}ds$$

By the Bellman-Gronwall Lemma,

$$e^{\lambda(t-\tau)}|\theta(t)| \leq ae^{\int_{\tau}^{t} a|B(s)|ds}$$

Therefore

$$|\theta(t)| \leq ae^{(-\lambda(t-\tau)+\int_{\tau}^{t} a|B(s)|ds)}$$

Using (1.39) there follows

$$|\theta(t)| \leq ae^{(ab_{\mu}-(\lambda-a\mu)(t-\tau))}$$

which is the desired result. ∎

1.5.4 Completing the Detectability Analysis

We now turn to the problem of showing that the time-varying matrix

$$A_{\widehat{p}} + K_{\widehat{p}}(C_{\widehat{p}} + \begin{bmatrix} 0 & 0 & \dot{\widehat{p}}' \end{bmatrix})$$

is exponentially stable. Note first that $K_{\widehat{p}}\begin{bmatrix} 0 & 0 & \dot{\widehat{p}}(t)' \end{bmatrix}$ is nondestabilizing with zero growth rate, because $\dot{\widehat{p}}$ is. Therefore by the Nondestabilization Theorem, $A_{\widehat{p}} + K_{\widehat{p}}(C_{\widehat{p}} + \begin{bmatrix} 0 & \dot{\widehat{p}}' \end{bmatrix})$ will be exponentially stable provided $A_{\widehat{p}} + K_{\widehat{p}}C_{\widehat{p}}$ is. Exponential stability of the latter is a direct consequence of the following lemma.

Lemma 3 *Let $A : \mathcal{P} \rightarrow \mathbb{R}^{n\times n}$ be a continuously differentiable matrix with the property that A_p is a stability matrix for each fixed $p \in \mathcal{P}$. If $\widehat{p} : [0, \infty) \rightarrow \mathcal{P}$ is a bounded, differentiable function whose derivitive is nondestabilizing with zero growth rate, then $A_{\widehat{p}}$ is an exponentially stable, time-varying matrix.*

Proof: Since A_p is a stability matrix for each fixed $p \in \mathcal{P}$, for each such p the Lyapunov equation

$$A_p' Q_p + Q_p A_p + I = 0 \tag{1.43}$$

has a unique, positive definite solution Q_p. Moreover, because 1.43 is a linear equation, the ijth element of Q_p must be of the form $q_{ij}(A_p)$ where $q_{ij} : \mathbb{R}^{n\times n} \rightarrow \mathbb{R}$ is a rational function. Because of this and the fact that Q_p exists for each $p \in \mathcal{P}$, Q_p must be continuously differentiable on $\mathcal{P}$.

Let $\bar{\mathcal{P}} \stackrel{\Delta}{=} \{p : ||p|| \leq \bar{p}\}$ where $\bar{p} \stackrel{\Delta}{=} \sup_{t\geq 0} ||\widehat{p}||$. Thus $\bar{\mathcal{P}}$ is a closed, bounded subset of $\mathcal{P}$ in which $\widehat{p}$ takes its values. Define the time-varying matrix

$$B \stackrel{\Delta}{=} A_{\widehat{p}} - \frac{1}{2}Q_{\widehat{p}}^{-1}\frac{d}{dt}Q_{\widehat{p}}$$

Note that $Q_{\widehat{p}}^{-1}$ exists and is bounded because $\widehat{p} \in \bar{\mathcal{P}}$ and Q_p is continuous on and positive definite on $\bar{\mathcal{P}}$. Note in addition that $\dot{Q}_{\widehat{p}}$ is nondestabilizing with zero growth rate because $\dot{\widehat{p}}$ is, because Q_p is continuously differentiable on $\bar{\mathcal{P}}$ and because $\widehat{p} \in \bar{\mathcal{P}}$. Thus $A_{\widehat{p}}$ and B differ by a matrix which is nondestabilizing with zero growth rate. In view of the Nondestabilization Theorem, $A_{\widehat{p}}$ will be exponentially stable just in case B is.

To establish the latter, let x be any solution to the differential equation

$$\dot{x} = Bx$$

It follows from (1.43) and the definition of B, that the positive definite function $V \triangleq x'Q_{\widehat{p}}x$ satisfies the differential equation

$$\dot{V} = -||x||^2$$

Clearly

$$\dot{V} \leq -\lambda V$$

where

$$\lambda = \sup_{p \in \bar{p}} \lambda_p$$

λ_p being the largest eigenvalue of Q_p. It follows from the Bellman-Gronwall Lemma that

$$V \leq V(0)e^{-\lambda t}$$

which proves that $V \to 0$ exponentially fast. But

$$||x||^2 \leq \frac{1}{\mu}V$$

where

$$\mu = \inf_{p \in \bar{p}} \mu_p$$

$\mu_p > 0$ being the smallest eigenvalue Q_p, $p \in \bar{\mathcal{P}}$. Therefore $x \to 0$ exponentially fast. Since this is true for all x, B is exponentially stable. ■

Example 1 continued Let us note that with $\bar{u}$ and $\bar{y}$ defined by (1.14) and (1.15) respectively, the signal $z \triangleq (a_{\mathbb{P}} + 1)\bar{y} + g_{\mathbb{P}}\bar{u}$ satisfies $\dot{z} = -z + (a_{\mathbb{P}} + 1)y + g_{\mathbb{P}}u$. Moreover process model $\dot{y} = a_{\mathbb{P}}y + g_{\mathbb{P}}u$ can also be written as $\dot{y} = -y + (a_{\mathbb{P}} + 1)y + g_{\mathbb{P}}u$. Hence $\dot{z} - \dot{y} = -(z - y)$. Therefore $y = (a_{\mathbb{P}} + 1)\bar{y} + g_{\mathbb{P}}\bar{u} - \epsilon$ where $\epsilon = e^{-t}(z(0) - x(0))$. From this, the definition of e in (1.12) and the formula for $\widehat{y}$ in (1.13), we see that e also satisifes the error equation

$$e = \bar{y}(\widehat{a} - a_{\mathbb{P}}) + \bar{u}(\widehat{g} - g_{\mathbb{P}}) + \epsilon \tag{1.44}$$

Let us also note that the design error $e_D \triangleq \widehat{a} + \widehat{g}\widehat{f} + 1$ can also be written as

$$e_D = (\widehat{a} - a_{\mathbb{P}}) + \widehat{f}(\widehat{g} - g_{\mathbb{P}}) + g_{\mathbb{P}}(\widehat{f} - f_{\mathbb{P}}) \tag{1.45}$$

where $f_{\mathbb{P}}$ is the solution to $a_{\mathbb{P}} + g_{\mathbb{P}}f_{\mathbb{P}} + 1 = 0$ These equations suggest that tuning be carried out using the equations

$$\begin{aligned} \dot{\widehat{a}} &= -\bar{y}\bar{e} - e_D && (1.46)\\ \dot{\widehat{g}} &= -\bar{u}\bar{e} - \widehat{f}e_D && (1.47)\\ \dot{\widehat{f}} &= -e_D && (1.48) \end{aligned}$$

where $\bar{e}$ is the normalized tuning error

$$\bar{e} = \frac{1}{1 + \bar{u}^2 + \bar{y}^2}e \tag{1.49}$$

Indeed if we define

$$V \triangleq \frac{1}{2}((\widehat{a} - a_{\mathbb{P}})^2 + (\widehat{g} - g_{\mathbb{P}})^2 + \frac{1}{g_{\mathbb{P}}}(\widehat{f} - f_{\mathbb{P}})^2 + \int_t^\infty \epsilon^2(s)ds$$

then

$$\dot{V} = -\bar{e}^2 - e_D^2 - \dot{\widehat{a}}^2 - \dot{\widehat{g}}^2 + \bar{e}\epsilon - \epsilon^2$$

Using the algebraic inequality $2xz \leq (x^2 + y^2)$ to deal with the term $\bar{e}\epsilon$ we thus obtain

$$\dot{V} \leq -\frac{1}{2}\bar{e}^2 - e_D^2 - \dot{\widehat{a}}^2 - \dot{\widehat{g}}^2 - \frac{1}{2}\epsilon^2$$

From this we can conclude, in much the same way as before, that solutions exist globally, that $\widehat{p}$ and $\bar{e}$ are bounded, and that $\dot{\widehat{p}}, \bar{e}$ and e_D have finite $\mathcal{L}^2[0,\infty)$ norms.

To proceed, let us note that (1.46)-(1.49) define a tuner with parameter $\widehat{p}$ and equilibrium set $\mathcal{E}_{\mathbb{T}} \triangleq \{[\,a \quad g \quad f\,]' : a + bf + 1 = 0\}$. Moreover the closed-loop parameterized system (1.18) is detectable on $\mathcal{E}_{\mathbb{T}}$. From this and the definition of e_D it follows that the "extended" closed-loop parameterized system

$$e^* \triangleq \begin{bmatrix} e \\ e_D\bar{y} \\ e_D\bar{u} \end{bmatrix} = \begin{bmatrix} -1 & \widehat{a}+1 & \widehat{g} \\ 0 & \widehat{a}+\widehat{g}\widehat{f}+1 & 0 \\ 0 & 0 & \widehat{a}+\widehat{g}\widehat{f}+1 \end{bmatrix} \begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} \qquad \begin{bmatrix} \dot{y} \\ \dot{\bar{y}} \\ \dot{\bar{u}} \end{bmatrix} = \begin{bmatrix} a_{\mathbb{P}} + g_{\mathbb{P}}\widehat{f} & 0 & 0 \\ 1 & -1 & 0 \\ \widehat{f} & 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ \bar{y} \\ \bar{u} \end{bmatrix} \tag{1.50}$$

is globally detectable.

1.6 Concluding Remarks

The material in this chapter is taken from [1]. The general idea of a pre-routed search was first used in [2]. The proof of Lemma 3 exploits ideas from [3].

Chapter 2

ACHIEVING DETECTABILITY VIA CERTAINTY EQUIVALENCE

2.1 Linear Multi-Estimators

Let $\mathbb{P}$ be a process with control input u, exeogeneous input w, and measured output y taking values in finite dimensional vector spaces $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{Y}$ respectively. Suppose that the relationship between y and u is modeled by a finite dimensional linear system and that this system is a member of a class of admissible models of the form

$$\mathcal{M} = \bigcup_{p \in \mathcal{P}_M} \mathcal{M}_p$$

where $\mathcal{P}_M$ is either a finite index set or a subset of a real, finite dimensional, normed, linear space. Here each $\mathcal{M}_p$ is a subset of stabilizable and detectable linear systems consisting of a nominal $\mathbb{N}_p$ and norm-bounded perturbations of $\mathbb{N}_p$. All models in $\mathcal{M}$ have the same control input space $\mathcal{U}$ and measured output value space $\mathcal{Y}$ as $\mathbb{P}$. It is of course possible that $\mathbb{P}$ might be exactly modeled by one of $\mathcal{M}$'s nominal systems - say $\mathbb{N}_{p^*}$ - and, in addition, that w might be zero; in the sequel we shall refer to this unlikely situation as the *exact matching case.*

By a *linear multi-estimator* of $\mathbb{P}$ is meant a stable, parameter-dependent, linear dynamical system $\mathbb{E}_{\widehat{p}}$ with parameter space $\mathcal{P} \supset \mathcal{P}_M$, inputs u and y, parameter vector $\widehat{p}$ and output $y_{\widehat{p}}$; $\widehat{p}$ and $y_{\widehat{p}}$ take values in $\mathcal{P}$ and $\mathcal{Y}$ respectively[1] .

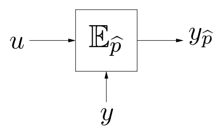

Figure 2.1: Multi-Estimator

To be a multi-estimator, $\mathbb{E}_{\widehat{p}}$ must have the property that, in the exact matching case when $\mathbb{P}$ is exactly modeled by $\mathbb{N}_{p^*}$ and $w = 0$, $y_{\widehat{p}}$ is an assymptotically correct estimate of y assuming $\widehat{p}$ is fixed at p^*. Although there are several different but similar types of parameter-dependent linear systems which meet this requirement, little will be lost by focusing attention on just one. Accordingly, in these notes we shall

[1]The reason for not taking $\mathcal{P} \stackrel{\Delta}{=} \mathcal{P}_M$ will be explained later in the chapter.

deal exclusively with linear multi-estimators of the form

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}} x_{\mathbb{E}} + D_{\mathbb{E}} y + B_{\mathbb{E}} u \tag{2.1}$$

$$y_{\widehat{p}} = C_{\mathbb{E}}(\widehat{p}) x_{\mathbb{E}} + G_{\mathbb{E}}(\widehat{p}) y \tag{2.2}$$

where $A_{\mathbb{E}}, B_{\mathbb{E}}$ and $D_{\mathbb{E}}$ are constant {parameter-independent} matrices with $A_{\mathbb{E}}$ stable. The matrix $I - G_{\mathbb{E}}(p)$ is required to be invertible for all $p \in \mathcal{P}$. For the scalar-output case, this requirement reduces to $G_{\mathbb{E}}(p) \neq 1$ for all $p \in \mathcal{P}$; in this case it is typical to have $G_{\mathbb{E}}(p) = 0,\ p \in \mathcal{P}$. $\mathbb{E}_{\widehat{p}}$ is said to be *continuously parameterized* if $\mathcal{P}$ is a subset of a real, finite dimensional normed linear space and $C_{\mathbb{E}}(p)$ and $G_{\mathbb{E}}(p)$ are continuous functions of p. If, in addition, $C_{\mathbb{E}}(p)$ and $G_{\mathbb{E}}(p)$ are affine linear functions of p, then $\mathbb{E}_{\widehat{p}}$ is *linearly parameterized.*

By fixing $\widehat{p}$ at some value $p \in \mathcal{P}$ and then replacing y with y_p throughout, there results a linear system with coefficient matrix triple

$$\{A_{\mathbb{E}} + D_{\mathbb{E}}(I - G_{\mathbb{E}}(p))^{-1} C_{\mathbb{E}}(p), \quad B_{\mathbb{E}}, \quad (I - G_{\mathbb{E}}(p))^{-1} C_{\mathbb{E}}(p)\}$$

which we call $\mathbb{E}_{\widehat{p}}$'s *pth candidate model.*

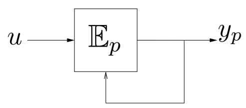

Figure 2.2: $\mathbb{E}_{\widehat{p}}$'s *pth* Candidate Model

We sometimes refer to $\mathbb{E}_{\widehat{p}}$'s $\widehat{p}th$ candidate model as $\mathbb{E}_{\widehat{p}}$'s *estimated model.* Note that because of the stability of $A_{\mathbb{E}}$, the matrix pair $(C_{\mathbb{E}}(p), A_{\mathbb{E}})$ is detectable on $\mathcal{P}$. Since detectability is invariant under the output injection transformation,

$$(C_{\mathbb{E}}(p),\ A_{\mathbb{E}}) \longmapsto ((I - G_{\mathbb{E}}(p))^{-1} C_{\mathbb{E}}(p),\ A_{\mathbb{E}} + D_{\mathbb{E}}(I - G_{\mathbb{E}}(p))^{-1} C_{\mathbb{E}}(p))$$

it follows that each of $\mathbb{E}_{\widehat{p}}$'s candidate models is detectable as well.

For $\mathbb{E}_{\widehat{p}}$'s output $y_{\widehat{p}}$ to be an asymptotically correct estimate of y in the exact matching case when $\mathbb{P}$ is exactly modeled by $\mathbb{N}_{p^*}$ for some $p^* \in \mathcal{P}_M$, $w = 0$, and $\widehat{p}$ is fixed at p^*, it is sufficient that $\mathbb{E}_{\widehat{p}}$'s coefficient matrices be defined so that for each $p \in \mathcal{P}_M$, $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model has the same transfer matrix as $\mathbb{N}_p$. To explain why this is so, let us first note that no matter what the value of $p \in \mathcal{P}_M$, it is always possible to write

$$y = C_p x_{\mathbb{N}} + \mathfrak{e}_p \qquad\qquad \dot{x}_{\mathbb{N}} = A_p x_{\mathbb{N}} + B_p u \tag{2.3}$$

where $\mathfrak{e}_p$ is the *pth modeling error*

$$\mathfrak{e}_p \stackrel{\Delta}{=} y - C_p x_{\mathbb{N}} \tag{2.4}$$

and $\{A_p,\ B_p,\ C_p\}$ is the coefficient matrix triple of a controllable and detectable system with the same transfer matrix the *pth* nominal model $\mathbb{N}_p$. Note that $\mathfrak{e}_p$ would be decaying to zero at a rate determined by $\mathbb{N}_p$'s uncontrollable spectrum in the exact matching case when $\mathbb{P}$'s actual model is $\mathbb{N}_p$ and w is zero. Fix $p \in \mathcal{P}_M$ and suppose that $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model has the same transfer matrix as $\mathbb{N}_p$. Our aim is to show that

$$\boxed{y = y_p + C_{\mathbb{E}}(p) e^{A_{\mathbb{E}} t}(M_q x_{\mathbb{N}}(0) - x_{\mathbb{E}}(0)) + (I - G_{\mathbb{E}}(p))\mathfrak{e}_p + \int_0^t C_{\mathbb{E}}(p) e^{A_{\mathbb{E}}(t-\tau)} D_{\mathbb{E}} \mathfrak{e}_p d\tau} \tag{2.5}$$

where M_p is a suitably defined matrix. Thus we see that $\mathbb{E}_{\widehat{p}}$ is able to generate a signal, namely y_p, which would be an assymptotically correct estimate of y if w were identically zero and if $\mathbb{N}_p$ were the actual model of $\mathbb{P}$. Since $\mathbb{E}_{\widehat{p}}$ is able to generate such an estimate for each possible nominal model in $\mathcal{M}$, it is natural to call $\mathbb{E}_{\widehat{p}}$ a *multi-estimator.*

To explain why (2.5) is true, we first reduce $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model to an observable system with the same transfer matrix. This is acomplished by picking a full rank matrix P, depending on p, whose kernel is the unobservable space of $(C_{\mathbb{E}}(p), A_{\mathbb{E}})$. Next, as in any observability reduction, the equations

$$PA_{\mathbb{E}} = AP \qquad\qquad C_{\mathbb{E}}(p) = CP \tag{2.6}$$

are solved {uniquely} for A and C respectively. The construction implies that the pair (C, A) is observable and consequently that the output-injected pair $((I - G_{\mathbb{E}})^{-1}C, A + PD_{\mathbb{E}}(I - G_{\mathbb{E}})^{-1}C)$ is observable as well. {Here and in what follows we write $G_{\mathbb{E}}$ for $G_{\mathbb{E}}(p)$ to simplify notation.} The construction also implies that that the reduced system matrix triple

$$\{A + PD_{\mathbb{E}}(I - G_{\mathbb{E}})^{-1}C,\;\; PB_{\mathbb{E}},\;\; (I - G_{\mathbb{E}})^{-1}C\} \tag{2.7}$$

has the same transfer matrix as $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model. Therefore the {observable} linear system defined by the coefficient matrix triple in (2.7) has the same transfer matrix as the {controllable} linear system defined by the coefficient matrix triple $\{A_p, B_p, C_p\}$. To relate these coefficient matrix triples to each other, we need the following lemma.

Lemma 4 *Let $\{\bar{A}, \bar{B}, \bar{C}\}$ and $\{A, B, C\}$ be the coefficient matrix triples of two linear systems with the same transfer matrix. Suppose that the former system is $\bar{n}$-dimensional and controllable and that the latter is n-dimensional and observable. Then*

$$\bar{C} = CT \qquad\qquad T\bar{A} = AT \qquad\qquad B = T\bar{B} \tag{2.8}$$

where T the $n \times \bar{n}$ matrix $T \triangleq R\bar{R}^{-1}$, $\bar{R}^{-1}$ being a right inverse of the the controllability matrix of $(\bar{A}, \bar{B})$, and $R \triangleq [\,B \quad AB \quad \cdots \quad A^{\bar{n}-1}B\,]$.

We can apply Lemma 4 to the problem at hand by respectively associating the matrix triples $\{A_p,\ B_p,\ C_p\}$ and $\{A + PD_{\mathbb{E}}(I - G_{\mathbb{E}})^{-1}C,\ PB_{\mathbb{E}},\ (I - G_{\mathbb{E}})^{-1}C\}$ with the matrix triples $\{\bar{A},\ \bar{B},\ \bar{C}\}$ and $\{A,\ B,\ C\}$ in the lemma. Doing this enables us to conclude that there must be a matrix T, depending on p, such that

$$C_p = (I - G_{\mathbb{E}})^{-1}CT \qquad TA_p = (A + PD_{\mathbb{E}}(I - G_{\mathbb{E}})^{-1}C)T \qquad TB_p = PB_{\mathbb{E}} \tag{2.9}$$

These equations and (2.3) enable us to write

$$y = (I - G_{\mathbb{E}})^{-1}CTx_{\mathbb{N}} + \mathfrak{e}_p \qquad\qquad T\dot{x}_{\mathbb{N}} = (A + PD_{\mathbb{E}}(I - G_{\mathbb{E}})^{-1}C)Tx_{\mathbb{N}} + PB_{\mathbb{E}}u$$

or equivalently

$$y = CTx_{\mathbb{N}} + G_{\mathbb{E}}y + (I - G_{\mathbb{E}})\mathfrak{e}_p \qquad\qquad T\dot{x}_{\mathbb{N}} = ATx_{\mathbb{E}} + PD_{\mathbb{E}}(y - \mathfrak{e}_p) + PB_{\mathbb{E}}u$$

Meanwhile, applying (2.6) to multi-estimator equations (3.19) and (2.2) we obtain

$$y_p = CPx_{\mathbb{E}} + G_{\mathbb{E}}y \qquad\qquad P\dot{x}_{\mathbb{E}} = APx_{\mathbb{E}} + PD_{\mathbb{E}}y + PB_{\mathbb{E}}u$$

Setting $z \triangleq Tx_{\mathbb{N}} - Px_{\mathbb{E}}$ and making use of the preceding, we can write

$$y = y_p + Cz + (I - G_{\mathbb{E}})\mathfrak{e}_p \qquad\qquad \dot{z} = Az - PD_{\mathbb{E}}\mathfrak{e}_p + PB_{\mathbb{E}}u$$

Therefore

$$y = y_p + Ce^{At}(Tx_{\mathbb{N}}(0) - Px_{\mathbb{E}}(0)) + (I - G_{\mathbb{E}})\mathfrak{e}_p + \int_0^t Ce^{A(t-\tau)}PD_{\mathbb{E}}\mathfrak{e}_p d\tau \tag{2.10}$$

Using the fact that $Ce^{At}P = C_{\mathbb{E}}e^{A_{\mathbb{E}}t}$ {which is a consequence of (2.6)}, we can rewrite (2.10) as (2.5) where $M_p \triangleq P^{-1}T$ and P^{-1} is a right inverse for P.

Proof of Lemma 4: Equality of the transfer matrices of $\{\bar{A}, \bar{B}, \bar{C}\}$ and $\{A, B, C\}$ implies that

$$\bar{C}\bar{A}^i\bar{B} = CA^iB, \quad i \geq 0$$

From the relations there follow

$$\bar{N}\bar{R} = NR \qquad \bar{C}\bar{R} = CR \qquad \bar{N}\bar{A}\bar{R} = NAR \qquad \bar{N}\bar{B} = NB$$

where N is the observability matrix of (C, A) and

$$\bar{N} \triangleq \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix}$$

These equations respectively imply that

$$N^{-1}\bar{N} = R\bar{R}^{-1} \qquad \bar{C} = CR\bar{R}^{-1} \qquad N^{-1}\bar{N}\bar{A} = AR\bar{R}^{-1} \qquad B = N^{-1}\bar{N}\bar{B}$$

where N^{-1} is a left inverse of N. Since $T \triangleq R\bar{R}^{-1}$, it follows from the preceding that the lemma is true. ■

2.1.1 Multi-Estimator Construction when $\mathcal{P}_M$ is a Finite Set

As has just been shown, (2.5) will hold for a given process $\mathbb{P}$ and admissible model class $\mathcal{M}$ provided $\mathbb{E}_{\widehat{p}}$ and $\mathcal{P}$ are defined in such a way so that $\mathcal{P} \supset \mathcal{P}_M$ and for each $p \in \mathcal{P}_M$, $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model has the same transfer matrix as that of $\mathbb{N}_p$. There remains the problem of constructing such a multi-estimator for a given $\mathcal{M}$. In this section we explain how to do this for the cases when $\mathcal{P}_M$ is a finite set. For this case there is no reason not to assume that that

$$\mathcal{P} \triangleq \mathcal{P}_M, \tag{2.11}$$

We henceforth make this assumption.

An easy way to realize a multi-estimator for the case at hand, is simply to use a parallel implementation of a family of "output-estimating observers," one for each nominal model in $\mathcal{M}$. By an *output-estimating observer* for a linear system $y = Cx, \quad \dot{x} = Ax + Bu$ is meant a time-invariant linear system of the form $\widehat{y} = Mz + Ny, \quad \dot{z} = Hz + Ky + VBu$ where H is a stability matrix, $(I - N)$ is nonsingular, and V is a matrix for which the *observer design equations*

$$VA = HV + KC \qquad C = MV + NC$$

hold. These equations imply that

$$z - Vx = e^{Ht}(z(0) - Vx(0))$$

and that

$$y = \widehat{y} + Me^{At}(Vx(0) - z(0))$$

Thus one way to realize a multi-estimator for a finite set of nominal models $\mathcal{N} = \{\mathbb{N}_p : p \in \mathcal{P}\}$ is to construct an output-estimating observer for each $\mathbb{N}_p$, and then implement all such observers in parallel. Specifically, for each $p \in \mathcal{P}$, let

$$y_p = M_p z_p + N_p y \qquad \dot{z}_p = H_p z_p + K_p y + V_p B_p u$$

be an output-estimating observer for $\mathbb{N}_p$ in that

$$V_p A_p = H_p V_p + K_p C_p \qquad C_p = M_p V_p + N_p C_p$$

where H_p is a stability matrix, $(I - N_p)$ is nonsingular, and $\{A_p,\ B_p,\ C_p\}$ is the coefficient matrix triple of $\mathbb{N}_p$. Assuming $\mathcal{P}$ contains m elements, $\mathbb{E}_{\widehat{p}}$ is then simply the system

$$y_{\widehat{p}} = C_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}} + G_{\mathbb{E}}y \qquad \dot{x}_{\mathbb{E}} = \begin{bmatrix} H_{p_1} & 0 & \cdots & 0 \\ 0 & H_{p_2} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & H_{p_m} \end{bmatrix} x_{\mathbb{E}} + \begin{bmatrix} K_{p1} \\ K_{p_2} \\ \vdots \\ K_{p_m} \end{bmatrix} y + \begin{bmatrix} V_{p_1}B_{p1} \\ V_{p_2}B_{p_2} \\ \vdots \\ V_{p_m}B_{p_m} \end{bmatrix} u$$

where $x_{\mathbb{E}}$ is the *stacked state* $x_{\mathbb{E}} \triangleq [\, z'_{p_1} \quad z'_{p_2} \quad \cdots \quad z'_{p_m} \,]'$, $G_{\mathbb{E}}(p) \triangleq N_p,\ p \in \mathcal{P}$, and

$$C_{\mathbb{E}}(p) \triangleq [\, \delta_{pp_1}M_{p_1} \quad \cdots \quad \delta_{pp_i}M_{p_i} \quad \cdots \quad \delta_{pp_m}M_{p_m} \,], \quad p \in \mathcal{P},$$

δ_{pq} being the Kroneker delta.

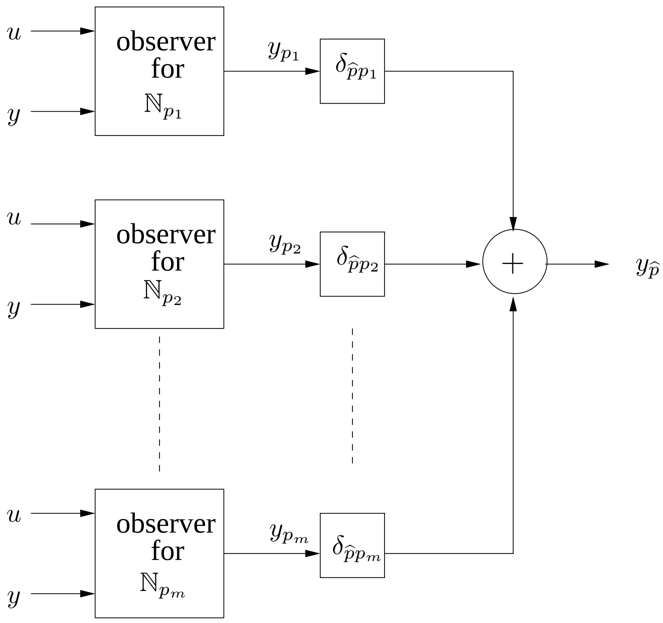

Figure 2.3: Multi-Estimator for Finite $\mathcal{P}$

There are many different ways to define each of the observers making up $\mathbb{E}_{\widehat{p}}$. For example, for a given p one could take $N_p = 0$, V_p any full rank matrix whose kernel is the unobservable space of (C_p, A_p), $M_p = C_p$, and $H_p = \bar{A}_p - K_p\bar{C}_p$ where $\bar{C}_p$ and $\bar{A}_p$ are the unique solutions to $C_p = \bar{C}_pV_p$ and $V_pA_p = \bar{A}_pV_p$ respectively and K_p is any matrix which stabilizes $\bar{A}_p - K_p\bar{C}_p$. Other constuctions using reduced dimensional observers are also possible.

The construction of a multi-estimator for the case when $\mathcal{Q}$ is not a finite set presents challenges not encountered in the finite case. In the first place, it is obvious that the parallel observer architecture just described is not applicable if $\mathcal{P}_M$ is not finite. We defer discussion of this case until later in these notes.

2.2 Certainty Equivalence Control

In this section we study the properties of an estimator-based self-adjusting control system for the case when $\mathbb{E}_{\widehat{p}}$ is a linear multi-estimator, $\mathbb{C}_{\widehat{q}}$ is a linear multi-controller with parameter space $\mathcal{Q}$ and $\mathbb{P}$ is a process admitting a linear stabilizable and detectable model of the form

$$y = C_{\mathbb{P}} x_{\mathbb{P}} \qquad\qquad \dot{x}_{\mathbb{P}} = A_{\mathbb{P}} x_{\mathbb{P}} + B_{\mathbb{P}} u \tag{2.12}$$

In this chapter we deal exclusively with the case when $\mathcal{Q} = \mathcal{P}$ and $\widehat{q} \stackrel{\Delta}{=} \widehat{p}$. The system of interest is shown in Figure 2.4.

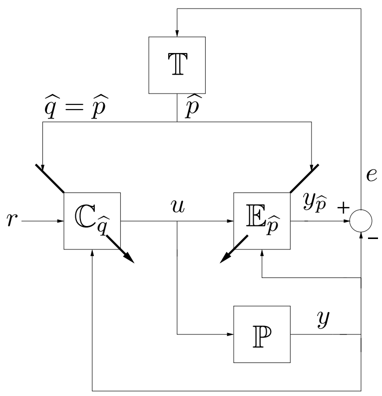

Figure 2.4: Error-Driven, Estimator-Based Adaptive Control System

As we've already pointed out, a necessary condition for $\mathbb{T}$ to be a stabilizing tuner for this system, is that the feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in Figure 2.5 be detectable through e for each fixed value of $\widehat{p}$ in the $\mathbb{T}$'s equlibrium set[2]. The question of interest is thus the following. How do we choose the $\mathbb{C}_{\widehat{p}}$ so as to insure

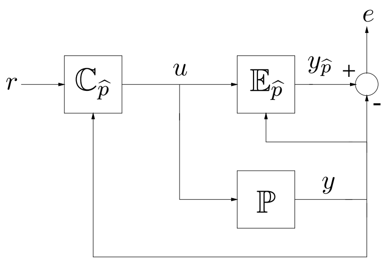

Figure 2.5: Feedback Sub-System $\mathbb{F}_{\widehat{p}\widehat{p}}$

that for each fixed value of $\widehat{p}$ in a given subset $\mathcal{E} \subset \mathcal{P}$, $\mathbb{F}_{\widehat{p}\widehat{p}}$ is detectable through e? As a first step towards answering this question, we shall investigate the consequences of choosing the $\mathbb{C}_{\widehat{p}}$ is accordance with the idea of "certainty equivalence." Informally speaking, *certainty equivalence* is a heuristic idea which advocates that, in an adaptive context, the feedback control to an imprecisely modeled process should, at each instant of time, be designed on the basis of a current estimate of what the process model is, with the understanding that each such estimate is to be viewed as correct even though it may not be. On the surface justification for certainty equivalence seems self-evident: if process model estimates converge to the "true" process model, then a certainty equivalence based controller ought to converge to the nonadaptive controller which would have been implemented had there been no process uncertainty. The problem with this justification is that because of noise and unmodelled dynamics, process model estimates don't typically converge to the true process model – even in those instances where certainty equivalence controls can be shown to perform in a satisfactory manner. In the sequel it will become clear that there is another, more conviencing reason for using certainty equivalence control. But first we must make precise just what we mean by a certainty equivalence control.

[2]The double subscript notation for $\mathbb{F}_{\widehat{p}\widehat{q}}$ will be used in the next chapter where we do not set $\widehat{q} = \widehat{p}$.

Let $\mathcal{P}$ and $\mathbb{E}_{\widehat{p}}$ be given and let $\mathcal{E}$ be a given, nonempty subset of $\mathcal{P}$. We say that multi-controller $\mathbb{C}_{\widehat{p}}$ is a *certainty equivalence stabilizing controller* on $\mathcal{E}$ if for each $p \in \mathcal{E}$, $\mathbb{C}_p$ stabilizes $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model; in other words, for $\mathbb{C}_{\widehat{p}}$ to be a certainty equivalence stabilizing controller on $\mathcal{E}$, it must be true that for each $p \in \mathcal{E}$ the system shown in Figure 2.6 and denoted by $\mathbb{E}_p \oplus \mathbb{C}_p$, is a stable linear system.

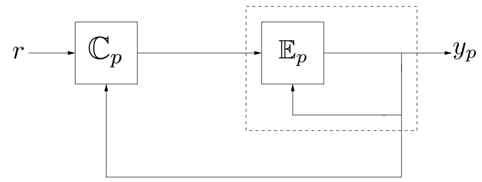

Figure 2.6: *pth* Controlled Candidate Model $\mathbb{E}_p \oplus \mathbb{C}_p$

To understand the consequences of this requirement, first recall that

$$e \triangleq y_{\widehat{p}} - y$$

Next note that by replacing y by $y_{\widehat{p}} - e$ as the input to both $\mathbb{C}_{\widehat{p}}$ and $\mathbb{E}_{\widehat{p}}$ in Figure 2.5, and then fixing $\widehat{p}$ at some value $p \in \mathcal{P}$, we obtain the sub-system shown in Figure 2.7, with inputs e and r and output y_p.

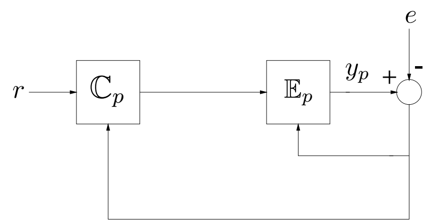

Figure 2.7: *pp*th Injected System $\mathbb{I}_p$

We call this system $\mathbb{F}_{\widehat{p}\widehat{p}}$'s pth *injected subsystem* and we denote it by $\mathbb{I}_p$. Note that the only difference between $\mathbb{I}_p$ and $\mathbb{E}_p \oplus \mathbb{C}_p$ is that e acts as an additional input to $\mathbb{I}_p$. The spectrums of $\mathbb{I}_p$ and $\mathbb{E}_p \oplus \mathbb{C}_p$ are thus one and the same. Clearly then, requiring $\mathbb{C}_{\widehat{p}}$ to be a stabilizing certainty equivalence controller on $\mathcal{E}$ means that $\mathbb{F}_{\widehat{p}\widehat{p}}$'s injected subsystems are stable for each value of p in $\mathcal{E}$[3]. It will soon become evident, if it is not already, that stability of $\mathbb{I}_{pp}$ or equivalently $\mathbb{E}_p \oplus \mathbb{C}_p$ for each $p \in \mathcal{P}$ together with the assumed detectability of the model of $\mathbb{P}$ in (2.12), imply that for each fixed value of $\widehat{p}$ in $\mathcal{E}$, the feedback sub-system shown in Figure 2.5 must be detectable through e. This simple observation proves to be of paramount importance, and so we summarize.

Theorem 3 (Certainty-Equivalence Stabilization Theorem) *If $\mathbb{C}_{\widehat{p}}$ is a certainty equivalence stabilizing controller on $\mathcal{E}$, then the feedback sub- system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in Figure 2.5 is detectable through e for each frozen value of $\widehat{p}$ in $\mathcal{E}$.*

The theorem makes clear one of the central implications of certainty equivalence control: If $\mathbb{C}_{\widehat{p}}$ is chosen to stabilize $\mathbb{E}_{\widehat{p}}$'s estimated model for each fixed $\widehat{p}$ in $\mathcal{E}$, and is then implemented as a component subsystem of the adaptive control system shown in Figure 2.4 then the associated feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in

[3]In a later chapter, when we extend these ideas to nonlinear systems, we will re-define $\mathbb{C}_{\widehat{p}}$ to be a certainty equivalence stabilizing controller on $\mathcal{E}$ if it "input-to-state" stabilizes $\mathbb{I}_{pp}$ on $\mathcal{E}$ with respect to its inputs r and e. For the linear case, input-to-state stability and stability are equivalent properties, so for the linear case the new and old definitions of a certainthy equivalence stabilizing control are equivalent.

Figure 2.5 is detectable through e for each value of $\widehat{p}$ in $\mathcal{E}$. For this to be true it is *not* necessary for process model defined by (2.12) to have a transfer matrix in the class of transfer matrices equal to the transfer matrix of any of $\mathbb{E}_{\widehat{p}}$'s candidate models. In this context the notion of "true process model parameters" is neither necessary nor relevant.

Unfortunately when $\mathcal{P}$ is not a finite set, it is usually not possible to make use of the preceding theorem to achieve detectability on all of $\mathcal{P}$ because, as we shall see, for this case $\mathcal{P}$ typically contains points at which $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model is not stabilizable. There is a way to deal with this problem. which will be discussed later in the chapter. However for the case when $\mathbb{P}$ is "minimum phase", the problem can often often be avoided all together. To understand why, we need a few more ideas.

Let us agree to call linear multivariable system $\{A, B, C\}$ *minimum phase* if its transfer matrix is left invertible and its transmission zeros[4] are all stable; as defined, minimum phase systems are necessarily detectable. Call a matrix pair (C, A) {or a system (A, B, C)} *output stabilized* if $Ce^{At} \to 0$ as $t \to \infty$. Note that (C, A) is output stabilized if and only if each of its unstable modes is unobservable. On the other hand, (C, A) is detectable if and only if each of its unobservable modes is stable. It follows that (C, A) is *both* output stabilized and detectable if and only if A has no unstable modes; i.e., if and only if A is a stability matrix.

Let $\mathcal{P}$ and $\mathbb{E}_{\widehat{p}}$ be fixed and let $\mathcal{E}$ be a given, nonempty subset of $\mathcal{P}$. We say that multi-controller $\mathbb{C}_{\widehat{p}}$ is a *certainty equivalence output stabilizing controller* on $\mathcal{E}$ if for each $p \in \mathcal{E}$, $\mathbb{C}_p$ output stabilizes $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model; in other words, for $\mathbb{C}_{\widehat{p}}$ to be a certainty equivalence output stabilizing controller on $\mathcal{E}$, it must be true that for each $p \in \mathcal{E}$ the system $\mathbb{E}_p \oplus \mathbb{C}_p$ shown in Figure 2.6 is an output-stable linear system. We can now state the

Theorem 4 (Certainty Equivalence Output Stabilization Theorem) *If $\mathbb{P}$ is minimum phase and $\mathbb{C}_{\widehat{p}}$ is a certainty equivalence, output stabilizing controller on $\mathcal{E}$, then the feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in Figure 2.5 is detectable through e for each fixed value of $\widehat{p}$ in $\mathcal{E}$.*

The theorem's importance rests on the fact that output stabilization of a linear system is a less stringent requirement of a controller than is internal stabilization. As we shall see, there are models which are not internally stabilizable for all values of their parameters, which nevertheless can be output stabilized for all such values. Theorems 3 and 4 are direct consequences of the following lemma.

Lemma 5 *Suppose each $\mathbb{C}_p$ is detectable.*

1. *If $\mathbb{P}$ is detectable and $\mathbb{F}_{\widehat{p}\widehat{p}}$'s qth injected sub-system $\mathbb{I}_q$ is internally stable, then $\mathbb{F}_{\widehat{p}\widehat{p}}$ is detectable through e at $\widehat{p} = q$.*
2. *If $\mathbb{P}$ is minimum phase and $\mathbb{F}_{\widehat{p}\widehat{p}}$'s qth injected sub-system $\mathbb{I}_q$ is output stabilized, then $\mathbb{F}_{\widehat{p}\widehat{p}}$ detectable through e at $\widehat{p} = q$.*

Proof of Lemma 5:

To prove the lemma it is enough to show that for each unstable complex number λ, the equations

$$C(q)x = 0 \tag{2.13}$$

$$A(q)x = \lambda x \tag{2.14}$$

[4]For a system $\{A, B, C\}$ with left-invertible transfer matrix $T(s) = C(sI - A)^{-1}B$, a complex number λ is a *transmission zero* if there exists a nonzero vector $[x', u']'$ such that $Cx = 0$ and $(\lambda I - A)x + Bu = 0$.

imply $x = 0$, where $A(\widehat{p})$ and $C(\widehat{p})$ are the coefficient matrices of the feedback subsystem $\mathbb{F}_{\widehat{p}}$ determined by the detectable process model

$$\begin{aligned} y &= C_{\mathbb{P}} x_{\mathbb{P}} \\ \dot{x}_{\mathbb{P}} &= A_{\mathbb{P}} x_{\mathbb{P}} + B_P u, \end{aligned}$$

the multi-estimator

$$\begin{aligned} y_{\widehat{p}} &= C_{\mathbb{E}}(\widehat{p}) x_{\mathbb{E}} + G_{\mathbb{E}}(\widehat{p}) y \\ \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}} x_{\mathbb{E}} + D_{\mathbb{E}} y + B_{\mathbb{E}} u, \end{aligned}$$

the detectable multi-controller

$$\begin{aligned} u &= C_{\mathbb{C}}(\widehat{p}) x_{\mathbb{C}} + G_{\mathbb{C}}(\widehat{p}) r + M_{\mathbb{C}}(\widehat{p}) y \\ \dot{x}_{\mathbb{C}} &= A_{\mathbb{C}}(\widehat{p}) x_{\mathbb{C}} + B_{\mathbb{C}}(\widehat{p}) r + N_{\mathbb{C}}(\widehat{p}) y \end{aligned}$$

and the output estimation error

$$e = y_{\widehat{p}} - y$$

With λ so defined and

$$x \triangleq \begin{bmatrix} x_{\mathbb{P}} \\ x_{\mathbb{E}} \\ x_{\mathbb{C}} \end{bmatrix},$$

equations (2.13)-(2.14) can be written out as

$$-(I - G_{\mathbb{E}}) C_{\mathbb{P}} x_{\mathbb{P}} + C_{\mathbb{E}} x_{\mathbb{E}} = 0 \tag{2.15}$$
$$(A_{\mathbb{P}} + B_{\mathbb{P}} M_{\mathbb{C}} C_{\mathbb{P}}) x_{\mathbb{P}} + B_{\mathbb{P}} C_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{P}} \tag{2.16}$$
$$(B_{\mathbb{E}} M_{\mathbb{C}} C_{\mathbb{P}} + D_{\mathbb{E}} C_{\mathbb{P}}) x_{\mathbb{P}} + A_{\mathbb{E}} x_{\mathbb{E}} + B_{\mathbb{E}} C_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{E}} \tag{2.17}$$
$$N_{\mathbb{C}} C_{\mathbb{P}} x_{\mathbb{P}} + A_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{C}} \tag{2.18}$$

where, for clarity, we have deleted the notation indicating the dependence of each matrix on q. Using (2.15) to eliminate $C_{\mathbb{P}} x_{\mathbb{P}}$ from (2.16) - (2.18), we obtain

$$-C_{\mathbb{P}} x_{\mathbb{P}} + \bar{C}_{\mathbb{E}} x_{\mathbb{E}} = 0 \tag{2.19}$$
$$A_{\mathbb{P}} x_{\mathbb{P}} + B_{\mathbb{P}} M_{\mathbb{C}} \bar{C}_{\mathbb{E}} x_{\mathbb{E}} + B_{\mathbb{P}} C_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{P}} \tag{2.20}$$
$$(A_{\mathbb{E}} + D_{\mathbb{E}} \bar{C}_{\mathbb{E}} + B_{\mathbb{E}} M_{\mathbb{C}} \bar{C}_{\mathbb{E}}) x_{\mathbb{E}} + B_{\mathbb{E}} C_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{E}} \tag{2.21}$$
$$N_{\mathbb{C}} \bar{C}_{\mathbb{E}} x_{\mathbb{E}} + A_{\mathbb{C}} x_{\mathbb{C}} = \lambda x_{\mathbb{C}} \tag{2.22}$$

where

$$\bar{C}_{\mathbb{E}} \triangleq I - G_{\mathbb{E}})^{-1} C_{\mathbb{E}}$$

Set $x_{\mathbb{I}} = [\, x'_{\mathbb{E}} \quad x'_{\mathbb{C}} \,]'$ and let $y_{\widehat{p}} = C_{\mathbb{I}} x_{\mathbb{I}} + G_{\mathbb{I}} e$ and $\dot{x}_{\mathbb{I}} = A_{\mathbb{I}} x_{\mathbb{I}} + B_{\mathbb{I}} r + D_{\mathbb{I}} e$ where $\mathbb{I}_q$ denotes $\mathbb{F}_{\widehat{p}}$'s qth injected sub-system ; then

$$C_{\mathbb{I}} = [\, \bar{C}_{\mathbb{E}} \quad 0 \,] \qquad\qquad A_{\mathbb{I}} = \begin{bmatrix} A_{\mathbb{E}} + D_{\mathbb{E}} \bar{C}_{\mathbb{E}} + B_{\mathbb{E}} M_{\mathbb{C}} \bar{C}_{\mathbb{E}} & B_{\mathbb{E}} C_{\mathbb{C}} \\ N_{\mathbb{C}} \bar{C}_{\mathbb{E}} & A_{\mathbb{C}} \end{bmatrix} \tag{2.23}$$

To prove the lemma's first claim, suppose that $A_{\mathbb{I}}$ is a stability matrix. Since λ unstable, it follows from (2.21) and (2.22) that both $x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are zero. Hence (2.19) and (2.20) become $C_{\mathbb{P}} x_{\mathbb{P}} = 0$ and $A_{\mathbb{P}} x_{\mathbb{P}} = \lambda x_{\mathbb{P}}$ respectively; but by assumption $\mathbb{P}$ is a detectable model, so $x_{\mathbb{P}} = 0$. Therefore the lemma's first claim is true.

To prove the lemma's second claim, suppose that $\mathbb{F}_{\widehat{p}}$'s qth injected system is output stabilized Since λ is unstable it therefore follows from (2.21), (2.22) that $C_{\mathbb{I}} x_{\mathbb{I}} = 0$. Hence by (2.23)

$$\bar{C}_{\mathbb{E}} x_{\mathbb{E}} = 0 \tag{2.24}$$

In view of this, (2.19) and (2.20) can be written as $C_{\mathbb{P}}x_{\mathbb{P}} = 0$ and $A_{\mathbb{P}}x_{\mathbb{P}} + B_{\mathbb{P}}u_{\mathbb{P}} = \lambda x_{\mathbb{P}}$ respectively where $u_{\mathbb{P}} = C_{\mathbb{C}}x_{\mathbb{C}}$. By assumption, $\mathbb{P}$ is minimum phase so λ cannot be a transmission zero; therefore both $x_{\mathbb{P}}$ and $u_{\mathbb{P}}$ must be zero. In particular,

$$C_{\mathbb{C}}x_{\mathbb{C}} = 0 \tag{2.25}$$

Using (2.24) and (2.25), (2.21) can be written as $A_{\mathbb{E}}x_{\mathbb{E}} = \lambda x_{\mathbb{E}}$; from this it follows that $x_{\mathbb{E}} = 0$ since $A_{\mathbb{E}}$ is a stability matrix. Similarly (2.22) and (2.25) imply that $A_{\mathbb{C}}x_{\mathbb{C}} = \lambda x_{\mathbb{C}}$ and $C_{\mathbb{C}}x_{\mathbb{C}} = 0$ respectively; therefore $x_{\mathbb{C}} = 0$ since by assumption $(C_{\mathbb{C}}, A_{\mathbb{C}})$ is a detectable pair. ■

2.2.1 Multi-Estimator State Feedback

All the results just discussed regarding certainty equivalence extend in a straightforward manner to the more general case when there is multi-estimator state feedback to the multi-controller as shown in Figure 2.8.

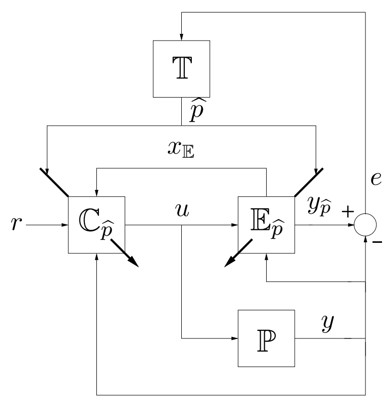

Figure 2.8: Adaptive System with Estimator State Feedback

In this case, $\mathbb{F}_{\widehat{p}\widehat{p}}$ and $\mathbb{E}_p \oplus \mathbb{C}_p$ would be as shown in Figures 2.9a and 2.9b respectively.

Figure 2.9: $\mathbb{F}_{\widehat{p}\widehat{p}}$ and $\mathbb{E}_p \oplus \mathbb{C}_p$ with Estimator State Feedback

We leave it to the reader to verify that both the Certainty Equivalence Stabilization Theorem and the Certainty Equivalence Output Stabilization Theorem apply to this case.

2.3 Design Models

In the sequel we again consider the problem of constructing a multi-estimator for a class of admissible process models of the form

$$\mathcal{M} = \bigcup_{p \in \mathcal{P}_M} \mathcal{M}_p$$

but now with $\mathcal{P}_M$ and the set of nominals $\mathcal{N} = \{\mathbb{N}_p : p \in \mathcal{P}_M\}$ both containing infinitely many members. As before, our goal is to explain how to construct $\mathbb{E}_{\widehat{p}}$ and $\mathcal{P}$ in such a way so that $\mathcal{P} \supset \mathcal{P}_M$ and for each $p \in \mathcal{P}_M$, $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model has the same transfer matrix as that of $\mathbb{N}_p$. Since $\mathcal{P}_M$ is not a finite set, it is obvious that the observer stacking construction used earlier in §2.1.1 for the finite $\mathcal{Q}$ case, can not be used in this case.

We shall approach the problem studying various specially structured parameterized families of linear systems called "design models." All members of any one such family will have the same state dimension and all will be at least detectable linear systems. The space within which a design model family's parameter takes values will be $\mathcal{P}$. For a given design model family $\mathcal{D} \stackrel{\Delta}{=} \{\mathbb{D}_p : p \in \mathcal{P}\}$ with parameter space $\mathcal{P}$ we will specify a subset $\mathcal{P}_M \subset \mathcal{P}$ in such a way so that the members of $\{\mathbb{D}_p : p \in \mathcal{P}_M\}$ are at least stabilizable and detectable. Accordingly, we shall view the members of $\{\mathbb{D}_p : p \in \mathcal{P}_M\}$ as nominal process models. In other words, the way we will come up with a parameterized family of nominals $\{\mathbb{N}_p : p \in \mathcal{P}_M\}$ is by first defining a larger specially structured parameterized family of linear systems $\mathcal{D} = \{\mathbb{D}_p : p \in \mathcal{P}\}$, and then picking for $\{\mathbb{N}_p : p \in \mathcal{P}_M\}$ a subset of $\mathcal{D}$ whose members are both stabilizabe and detectable. Naturally for this program to work, we have to make sure that $\mathcal{D}$ is sufficiently large in the first place. To make these ideas more concrete, we consider first the SISO indirect design model family.

2.3.1 SISO Indirect Control Design Model Families

Perhaps the simplest example of a design model family $\mathcal{D}$ is the set of n-dimensional, linear systems $\{\mathbb{D}_p : p \in \mathcal{P}\}$ whose *pth* element $\mathbb{D}_p$ is of the form

$$y_{\mathbb{D}} = cx_{\mathbb{D}} \qquad \dot{x}_{\mathbb{D}} = (A + d(p)c)x_{\mathbb{D}} + b(p)u_{\mathbb{D}} \tag{2.26}$$

where p is a parameter vector taking values in a parameter space $\mathcal{P}$. Here $d(p)$ and $b(p)$ are parameter-dependent $n-$vectors, and (c, A) is a single output, n-dimensional, parameter-independent observable pair chosen so that A is a stability matrix. Any set of systems of this form is called an n-dimensional *SISO indirect control design model family.* Such a family is said to be *linearly parameterized* if $\mathcal{P}$ is a subset of a finite dimensional, normed linear space and $d(p)$ and $b(p)$ are affine linear functions of p.

Let us note that since $(c, A + d(p)c)$ is an output injection away from the observable pair (c, A), $\mathbb{D}_p$ is observable, no matter what the value of $p \in \mathcal{P}$; i.e., $\mathcal{D}$ is a *globally observable family.* Next observe that $\mathbb{D}_p$'s transfer function is

$$t_p(s) = \frac{v(s)b(p)}{\alpha(s) - v(s)d(p)}$$

where $\alpha(s)$ is the characteristic polynomial of A and $v(s)$ is a row vector $v(s) = [\,\beta_1(s) \quad \beta_2(s) \quad \cdots \quad \beta_n(s)\,]$ defined by the formula

$$v(s) = \alpha(s)c(sI - A)^{-1}$$

Because (c, A) is observable, the set of entries of $v(s)$ form a basis for the linear space of polynomials of degree less than n. This means that if $p \longmapsto [\,d'(p) \quad b'(p)\,]'$ is a surjective mapping onto $\mathbb{R}^{2n}$, then for any strictly proper rational function

$$\frac{\pi_2(s)}{\alpha - \pi_1(s)}$$

with polynomials $\pi_1(s)$ and π_2 each of degree less than n, there is a parameter value $\bar{p} \in \mathcal{P}$ such that $\pi_1(s) = v(s)d_{\bar{p}}$ and $\pi_2(s) = v(s)b_{\bar{p}}$. Thus if $p \longmapsto [\, d'(p) \quad b'(p)\,]'$ is a surjective, the class of reduced transfer functions which $\mathcal{D}$ can *match* {i.e., be made equal to} as p ranges over $\mathcal{P}$ is all strictly proper transfer functions of McMillan degree not exceeding n. This means that if we let $\mathcal{Q}$ denote the set of values of p for which $\mathbb{D}_p$ is stabilizable, then $\{\mathbb{D}_p : p \in \mathcal{Q}\}$ too can match all strictly proper transfer functions of McMillan degree not exceeding n; moreover the elements of this subset, being both stabilizable and detectable, can all be taken as a et of candidate nominals. Note however, that the geometry of $\mathcal{Q}$ is extremely complicated [**?**] and that because of this, the construction of a tuner which restricts $\widehat{p}$ to take values only in $\mathcal{Q}$, is typically not a tractable problem.

2.3.2 The Multi-Estimator of a SISO Indirect Control Design Model Family

Each n-dimensional siso indirect control parameterized design model family $\mathcal{D}$ uniquely determines a specific multi-estimator $\mathbb{E}_{\widehat{p}}$ with the same parameter space as $\mathcal{D}$, whose *pth* candidate model has the same transfer function as $\mathbb{D}_p$. $\mathbb{E}_{\widehat{p}}$ is a $2n$-dimensional parameter-dependent linear system described by the equations

$$y_{\widehat{p}} = c_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}} \qquad\qquad \dot{x}_{\mathbb{E}} = A_{\mathbb{E}}x_{\mathbb{E}} + d_{\mathbb{E}}y + b_{\mathbb{E}}u \tag{2.27}$$

where

$$A_{\mathbb{E}} = \begin{bmatrix} A' & 0 \\ 0 & A' \end{bmatrix} \tag{2.28}$$

$$d_{\mathbb{E}} = \begin{bmatrix} c' \\ 0 \end{bmatrix} \tag{2.29}$$

$$b_{\mathbb{E}} = \begin{bmatrix} 0 \\ c' \end{bmatrix} \tag{2.30}$$

$$c_{\mathbb{E}}(p) = [\, d'(p) \quad b'(p)\,] \tag{2.31}$$

In order to explain the reason for defining $\mathbb{E}_{\widehat{p}}$ in this particular way, we will need the following equations which relate the coefficient matrices of $\mathbb{E}_p$ to those of the $\mathbb{D}_p$.

Linking Equations:

$$L_{\mathbb{E}}(p)A_{\mathbb{E}} = AL_{\mathbb{E}}(p) \tag{2.32}$$

$$L_{\mathbb{E}}(p)b_{\mathbb{E}} = b(p) \tag{2.33}$$

$$L_{\mathbb{E}}(p)d_{\mathbb{E}} = d(p) \tag{2.34}$$

$$c_{\mathbb{E}}(p) = cL_{\mathbb{E}}(p) \tag{2.35}$$

Here

$$L_{\mathbb{E}}(p) = [\, N^{-1}R'(d(p)) \quad N^{-1}R'(b(p))\,]\,, \tag{2.36}$$

where N is the observability matrix of (c, A) and $R(d(p))$ and $R(b(p))$ are the controllability matrices of $(A, d(p))$ and $(A, b(p))$ respectively; i.e.,

$$N = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{-1} \end{bmatrix}, \quad R(d(p)) = [\, d(p) \quad Ad(p) \quad \cdots \quad A^{n-1}d(p)\,]\,, \quad R(b(p)) = [\, b(p) \quad Ab(p) \quad \cdots \quad A^{n-1}b(p)\,]$$

In the sequel it will be shown that with $L_{\mathbb{E}}(p)$ so defined, the linking equations hold for all $p \in \mathcal{P}$.

Suppose we fix $\widehat{p}$ arbitrarily at some $q \in \mathcal{P}_M$. As noted before, no matter what the value of q we can always relate u to y by a system of equations of the form

$$y = cx + \mathfrak{e}_q \qquad\qquad \dot{x} = (A + d(q)c)x + b(q)u \tag{2.37}$$

where $\mathfrak{e}_q$ is the qth modeling error $\mathfrak{e}_q \stackrel{\Delta}{=} y - cx$. Define

$$x_q \stackrel{\Delta}{=} L_{\mathbb{E}}(q)x_{\mathbb{E}} \tag{2.38}$$

Using (2.26) and (2.32)-(2.34) we can write

$$\begin{aligned} \dot{x}_q &= L_{\mathbb{E}}(q)\dot{x}_{\mathbb{E}} \\ &= L_{\mathbb{E}}(q)(A_{\mathbb{E}}x_{\mathbb{E}} + d_{\mathbb{E}}y + b_{\mathbb{E}}u) \\ &= AL_{\mathbb{E}}(q)x_{\mathbb{E}} + d(q)y + b(q)u \\ &= Ax_q + d(q)y + b(q)u \end{aligned}$$

Meanwhile from (2.37), $\dot{x} = Ax + d(q)(y - \mathfrak{e}_q) + b(q)u$. Therefore $\dot{x} - \dot{x}_q = A(x - x_q) + d(q)\mathfrak{e}_q$. From this and (2.38) there follows

$$x = L_{\mathbb{E}}(q)x_{\mathbb{E}} + \int_0^t e^{A(t-\tau)}d(q)\mathfrak{e}_q d\tau + \epsilon \tag{2.39}$$

where ϵ is the exponentially decaying time function

$$\epsilon = e^{At}(x(0) - L_{\mathbb{E}}(q)x_{\mathbb{E}}(0))$$

If we multiply both sides of this expression by c, and then use (2.37), we obtain

$$y = cL_{\mathbb{E}}(q)x_{\mathbb{E}} + c\epsilon + \bar{\mathfrak{e}}_q \tag{2.40}$$

where

$$\bar{\mathfrak{e}}_q = \mathfrak{e}_q + \int_0^t ce^{A(t-\tau)}d(q)\mathfrak{e}_q d\tau$$

But $y_{\widehat{p}} = c_{\mathbb{E}}x_{\mathbb{E}}$ and $c_{\mathbb{E}}(p) = cL_{\mathbb{E}}(p),\ p \in \mathcal{P}$ because of (2.27) and(2.35) respectively. Thus (2.40) can also be written as

$$y = y_{\widehat{p}} + (c_{\mathbb{E}}(q) - c_{\mathbb{E}}(\widehat{p}))x_{\mathbb{E}} + c\epsilon + \bar{\mathfrak{e}}_q \tag{2.41}$$

Since this relation is true for all $q \in \mathcal{P}$, we see that in the exact matching case when $\mathfrak{e}_{q^*} = 0$

$$y = y_{\widehat{p}} + (c_{\mathbb{E}}(q^*) - c_{\mathbb{E}}(\widehat{p}))x_{\mathbb{E}} + c\epsilon$$

This clearly shows that in the exact matching case, y_{q^*} will in fact be an asymptotically correct estmate of y as claimed.

We remind the reader that $\mathfrak{e}_{q^*}$ may well not equal zero, except in the exact matching case, even if $\mathbb{P}$ admits a model in $\mathcal{M}_{q^*}$. This is because the model in question may not be $\mathbb{N}_{q^*}$ itself or because w might not equal zero or both. In any event, the *output estimation error* $e_{\widehat{p}} \stackrel{\Delta}{=} y_{\widehat{p}} - y$ can always be written as

$$e_{\widehat{p}} = (c_{\mathbb{E}}(\widehat{p}) - c_{\mathbb{E}}(q^*))x_{\mathbb{E}} - c\epsilon - \bar{\mathfrak{e}}_{q^*}$$

because (2.41) holds in general for all $q \in \mathcal{P}$. In view of (2.31), this formula can be rewritten as

$$e_{\widehat{p}} = [\, d'(\widehat{p}) - d'(q^*) \quad b'(\widehat{p}) - b'(q^*)\,]\, x_{\mathbb{E}} - c\epsilon - \bar{\mathfrak{e}}_{q^*} \tag{2.42}$$

Thus in the case when $\mathcal{D}$ is linearly parameterized {i.e., $[\, d'(p) \quad b'(p)\,] = p'M + a$ for some parameter-independent matrix M and row-vector a}, $e_{\widehat{p}}$ can be written directly in terms of the *parameter error* $\widehat{p} - q^*$ as

$$e_{\widehat{p}} = (\widehat{p} - q^*)'Mx_{\mathbb{E}} - c\epsilon - \bar{\mathfrak{e}}_{q^*} \tag{2.43}$$

The most familar version of this equation is when M is the identity. What results is the error equation

$$e_{\widehat{p}} = (\widehat{p} - q^*)' x_{\mathbb{E}} - c\epsilon - \bar{\mathfrak{e}}_{q^*}$$

This particular equation marks the starting point for the analysis of tuning algorithms of all types. See §1.5 of these notes for an example of the way in which the equation is used in the exact matching case when $\mathfrak{e}_{q^*} = 0$.

To summarize, we see that the structure of $\mathcal{D}$ enables us to uniquely specify a multi-estimator $\mathbb{E}_{\widehat{p}}$ whose inputs are u and y and whose output $y_{\widehat{p}}$ is asymptotically correct estimate of y in the exact matching case when $\mathbb{D}_{q^*}$ exactly models $\mathbb{P}$, $w = 0$ and $\widehat{p}$ is held fixed at q^*. More generally, for the exact matching case or not, the output estimation error $e_{\widehat{p}} = y_{\widehat{p}} - y$ can be written as (2.42) or, if $\mathcal{D}$ is linearly parameterized, as (2.43). The linking equations provide relationships between the coefficient matrices of the $\mathbb{D}_p$ and those of $\mathbb{E}_p$ which enables us to deduce that these properties are true.

Before concluding this section it will be shown that the linking equations hold. To do this we need the following result.

Lemma 6 *Let (c, A) be any n-dimensional single output observable pair. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ denote the linear function*

$$q \longmapsto N^{-1} R'(q) \tag{2.44}$$

where N and $R(q)$ are the observability and controllability matrices matrices of (c, A) and (A, q) respectively, and N^{-1} is a left inverse of N. For each $q \in \mathbb{R}$

$$\begin{aligned} cS(q) &= q' && (2.45)\\ AS(q) &= S(q)A' && (2.46)\\ S(q)c' &= q && (2.47) \end{aligned}$$

Proof of Lemma 6: Fix $q \in \mathbb{R}^n$. Then $\{A', c', q\}$ and $\{A, q, c\}$ are the coefficient matrix triples of two n-dimensional linear systems with the same transfer function. Moreover the former system is controllable and the latter is observable. Hence if we respectively identify $\{A', c', q'\}$ and $\{A, q, c\}$ with the the matrix triples $\{\bar{A}, \bar{B}, \bar{C}\}$ and $\{A, B, C\}$ in Lemma 4, we can conclude that the matrix $T(q) \triangleq R(q)N'^{-1}$ satisfies $q' = cT(q), T(q)A' = AT(q)$, and $T(q)c' = q$. Taking transposes and setting $S(p) = T'(p)$ then yields the desired result.■

Observe that linking equation (2.32) is an immediate consequence of (2.46), and the definitions of $A_{\mathbb{E}}$ and $L_{\mathbb{E}}$ in (2.28) and (2.36) respectively. Linking equation (2.33) follows from (2.47), and the definitions of $b_{\mathbb{E}}$ and $L_{\mathbb{E}}$ in (2.29) and (2.36) respectively. In the same way linking equation (2.34) follows from (2.47), (2.30) and (2.36). Linking equation (2.35) can be viewed as the definition of $c_{\mathbb{E}}$. With $c_{\mathbb{E}}$ so defined, (2.31) holds because of (2.45) and the definition of $L_{\mathbb{E}}$ in (2.36).

2.3.3 Design Model Families: General Form

The siso indirect control design model family just discussed is not the only type of design model family of use for dealing with siso systems. Other types of siso design model families as well as mimo design model families will be discussed later in the chapter. Each such family $\mathcal{D} = \{\mathbb{D}_p : p \in \mathcal{P}\}$ will consist of models of the form

$$\begin{aligned} \dot{x}_{\mathbb{D}} &= (A_{\mathbb{D}} + D_{\mathbb{D}}(p)(I - G_{\mathbb{D}}(p))^{-1} C_{\mathbb{D}}(p)) x_{\mathbb{D}} + B_{\mathbb{D}}(p) u_{\mathbb{D}} && (2.48)\\ y_{\mathbb{D}} &= (I - G_{\mathbb{D}}(p))^{-1} C_{\mathbb{D}}(p) x_{\mathbb{D}} && (2.49) \end{aligned}$$

where p takes values in a parameter space $\mathcal{P}$, $B_{\mathbb{D}} : \mathcal{P} \to \mathbb{R}^{n_{\mathbb{D}} \times n_u}$, $C_{\mathbb{D}} : \mathcal{P} \to \mathbb{R}^{n_y \times n_{\mathbb{D}}}$, $D_{\mathbb{D}} : \mathcal{P} \to \mathbb{R}^{n_{\mathbb{D}} \times n_y}$, and $G_{\mathbb{D}} : \mathcal{P} \to \mathbb{R}^{n_y \times n_y}$ are matrix valued functions, $A_{\mathbb{D}}$ is a constant $n_{\mathbb{D}} \times n_{\mathbb{D}}$ stability matrix, and, for each $p \in \mathcal{P}$, $(I - G_{\mathbb{D}}(p))$ is invertible.

There are a number of reasons for restricting attention to models of this type. One is that such models are detectable for all values of their parameters. This can be seen by observing that $A_{\mathbb{D}} + D_{\mathbb{D}}(p)(I - G_{\mathbb{D}}(p))^{-1}C_{\mathbb{D}}(p)$ is an output injection away from the constant stability matrix $A_{\mathbb{D}}$. Another reason is that each such design model family $\mathcal{D}$ of this form determines an associated multi-estimator $\mathbb{E}_{\widehat{p}}$ whose coefficient matrices are related to those of the design models by "linking equations" of much the same form as those we have already encountered in the preceding section. In particular the structure of the $\mathcal{D}_p$ enables one to construct a matrix-valued function $L_{\mathbb{E}} : \mathcal{P} \to \mathbb{R}^{n_{\mathbb{D}} \times n_{\mathbb{E}}}$, with $n_{\mathbb{E}}$ typically larger than $n_{\mathbb{D}}$, and constant matrices $A_{\mathbb{E}}, B_{\mathbb{E}}$, and $D_{\mathbb{E}}$, with $A_{\mathbb{E}}$ stable, so that the following equations hold.

Linking Equations:

$$\begin{aligned} L_{\mathbb{E}}(p)A_{\mathbb{E}} &= A_{\mathbb{D}}L_{\mathbb{E}}(p) \\ L_{\mathbb{E}}(p)B_{\mathbb{E}} &= B_{\mathbb{D}}(p) \\ L_{\mathbb{E}}(p)D_{\mathbb{E}} &= D_{\mathbb{D}}(p) \qquad p \in \mathcal{P} \end{aligned} \tag{2.50}$$

These matrices together with the matrices

$$\begin{aligned} G_{\mathbb{E}}(p) &\triangleq G_{\mathbb{D}}(p) \\ C_{\mathbb{E}}(p) &\triangleq C_{\mathbb{D}}(p)L_{\mathbb{E}}(p) \qquad p \in \mathcal{P} \end{aligned} \tag{2.51}$$

determine an $n_{\mathbb{E}}$-dimensional multi-estimator $\mathbb{E}_{\widehat{p}}$ of the standard form

$$y_{\widehat{p}} = C_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}} + G_{\mathbb{E}}(\widehat{p})y \qquad \dot{x}_{\mathbb{E}} = A_{\mathbb{E}}x_{\mathbb{E}} + D_{\mathbb{E}}y + B_{\mathbb{E}}u \tag{2.52}$$

Procedures for constructing matrices $A_{\mathbb{E}}$, $B_{\mathbb{E}}$, $D_{\mathbb{E}}$ and $L_{\mathbb{E}}(p)$ for specific design model families will be developed later in this chapter. For the present, we assume that these matrices have been defined for the design model family of interest, that $A_{\mathbb{E}}$ is a constant stability matrix, and that the linking equations hold.

One implication of the linking equations is that for each $p \in \mathcal{P}$, the transfer matrix of $\mathbb{E}_{\widehat{p}}$'s *pth* candidate model is the same as that of $\mathbb{D}_p$. Another is that for each $q \in \mathcal{P}_M$, it is possible to write the output estimation error $e_{\widehat{p}} = y_{\widehat{p}} - y$ as

$$\boxed{e_{\widehat{p}} = (C_{\mathbb{E}}(\widehat{p}) - C_{\mathbb{E}}(q))x_{\mathbb{E}} + (G_{\mathbb{E}}(\widehat{p}) - G_{\mathbb{E}}(q))y + C_{\mathbb{E}}(q)e^{A_{\mathbb{D}}t}(L_{\mathbb{E}}(q)x_{\mathbb{E}}(0) - x(0)) - \bar{\mathfrak{e}}_q} \tag{2.53}$$

where x is a solution to

$$\dot{x} = (A_{\mathbb{D}} + D_{\mathbb{D}}(q)(I - G_{\mathbb{D}}(q))^{-1}C_{\mathbb{D}}(q))x + B_{\mathbb{D}}(q)u,$$

$\mathfrak{e}_q$ is the qth modeling error

$$\mathfrak{e}_q \triangleq y - (I - G_{\mathbb{D}}(q))^{-1}C_{\mathbb{D}}(q)x$$

and

$$\bar{\mathfrak{e}}_q = (I - G_{\mathbb{E}}(q))\mathfrak{e}_q + \int_0^t C_{\mathbb{E}}(q)e^{A_{\mathbb{E}}(t-\tau)}D_{\mathbb{E}}\mathfrak{e}_q d\tau$$

These conclusions can be deduced using reasoning along much the same lines as was used earlier for the siso indirect control design model.

2.4 Design Models and Certainty Equivalence

Consider again the estimator-based adaptive control system shown in Figure 2.4. The Certainty Equivalence Stabilization Theorem states that feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in Figure 2.10 will automatically be detectable for some range of values of $\widehat{p}$ provided the controlled candidate model system $\mathbb{E}_p \oplus \mathbb{C}_p$ shown in Figure 2.11 is stable for the same range of values of p.

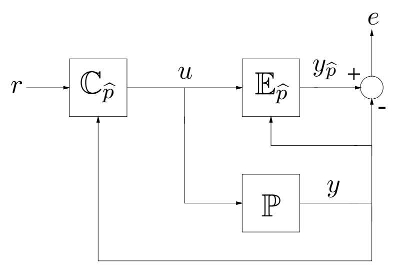

Figure 2.10: Feedback Sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$

If the multi-estimator under discussion is derived from a corresponding design model family $\mathbb{D}_p,\ p \in \mathcal{P}$, then detectability of $\mathbb{F}_{\widehat{p}\widehat{p}}$ for the given range of values of $\widehat{p}$ can also be achieved by designing $\mathbb{C}_{\widehat{p}}$ to stabilize $\mathbb{D}_{\widehat{p}}$ for the same range of values of $\widehat{p}$. The aim of this section is to prove that this is in fact so. Basing the design of $\mathbb{C}_{\widehat{p}}$ on design models rather than on a multi-estimators's candidate models has several advantages. Design models typically have state dimensions lower than corresponding mutli-estimator candidate models. More importantly, at least in the indirect control case, design models are generally observable, while corresponding multi-estimator candidate models are generally are not.

We begin our analysis with the following lemma which establishes important relationships between the closed-loop eigenvalues of $\mathbb{E}_p \oplus \mathbb{C}_p$ and the closed-loop eigenvalues of $\mathbb{D}_p \oplus \mathbb{C}_p$.

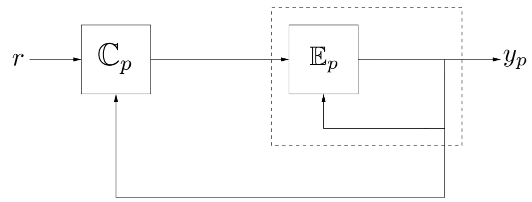

Figure 2.11: *pth* Controlled Candidate Model $\mathbb{E}_p \oplus \mathbb{C}_p$

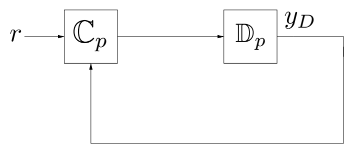

Figure 2.12: *pth* Controlled Design Model $\mathbb{D}_p \oplus \mathbb{C}_p$

Lemma 7 *Let $\widehat{p}$ be fixed at p. Then*

i. Each eigenvalue of $\mathbb{E}_p \oplus \mathbb{C}_p$ is either an eigenvalue of $\mathbb{D}_p \oplus \mathbb{C}_p$ or an eigenvalue of $A_{\mathbb{E}}$.

ii. Each unobservable eigenvalue of $\mathbb{E}_p \oplus \mathbb{C}_p$ is either an unobservable eigenvalue of $\mathbb{D}_p \oplus \mathbb{C}_p$ or an eigenvalue of $A_{\mathbb{E}}$.

The first statement of Lemma 7 implies that if $\mathbb{C}_p$ stabilizes $\mathbb{D}_p \oplus \mathbb{C}_p$ all values of p in some subset $\mathcal{E}$, then $\mathbb{E}_p \oplus \mathbb{C}_p$ is stable for the same range of values of p. In addition, the Certainty Equivalence Stabilization

Theorem tells us that if $\mathbb{C}_p$ stabilizes $\mathbb{E}_p \oplus \mathbb{C}_p$ for all $p \in \mathcal{E}$, then $\mathbb{F}_{\widehat{p}}$ is detectable through e for all $\widehat{p} \in \mathcal{E}$. Thus stability of $\mathbb{D}_p \oplus \mathbb{C}_p$ on $\mathcal{E}$ implies detectability of $\mathbb{F}_{\widehat{p}}$ on $\mathcal{E}$. We are led to the following corollary to the Certainty Equivalence Stabilization Theorem.

Corollary 1 *If $\mathbb{C}_p$ stabilizes $\mathbb{D}_p \oplus \mathbb{C}_p$ for all values of p in some subset $\mathcal{E} \subset \mathcal{P}$, then $\mathbb{F}_{\widehat{p}\widehat{p}}$ is detectable through e for the same range of values of $\widehat{p}$.*

Corollary 1 says that for $\mathbb{F}_{\widehat{p}\widehat{p}}$ to be detectable on $\mathcal{E}$ it is sufficient that $\mathbb{D}_p \oplus \mathbb{C}_p$ be stable on $\mathcal{E}$. It can be shown that stability of $\mathbb{D}_p \oplus \mathbb{C}_p$ on $\mathcal{E}$ also *necessary* in order for $\mathbb{F}_{\widehat{p}\widehat{p}}$ to be detectable on $\mathcal{E}$ {see Exercise **??**}.

Similar reasoning applies to the case when $\mathbb{P}$ is minimum phase and $\mathbb{D}_p \oplus \mathbb{C}_p$ is only output stable on $\mathcal{E}$. In particular, output stability of $\mathbb{D}_p \oplus \mathbb{C}_p$ on $\mathcal{E}$ imples output stability of $\mathbb{E}_p \oplus \mathbb{C}_p$ on $\mathcal{E}$ because of Lemma 7 - and if $\mathbb{P}$ is minimum phase, output stability of $\mathbb{E}_p \oplus \mathbb{C}_p$ on $\mathcal{E}$ imples detectablilty of $\mathbb{F}_{\widehat{p}\widehat{p}}$ on $\mathcal{E}$, because of the Certainty Equivalence Output Stabilization Theorem. We summarize.

Corollary 2 *If $\mathbb{P}$ is minimum phase and $\mathbb{C}_p$ output stabilizes $\mathbb{D}_p \oplus \mathbb{C}_p$ for all values of p in some subset $\mathcal{E} \subset \mathcal{P}$, then $\mathbb{F}_{\widehat{p}\widehat{p}}$ is detectable through e for the same range of values of $\widehat{p}$.*

Proof of Lemma 7: To establish the lemma's first claim, let λ be an eigenvalue of $\mathbb{E}_p \oplus \mathbb{C}_p$. Then for some non-zero vector $[\, x'_{\mathbb{E}} \quad x'_{\mathbb{C}} \,]'$

$$\begin{aligned} (A_{\mathbb{E}} + D_{\mathbb{E}}\bar{C}_{\mathbb{E}} + B_{\mathbb{E}}M_{\mathbb{C}}\bar{C}_{\mathbb{E}})x_{\mathbb{E}} + B_{\mathbb{E}}C_{\mathbb{C}}x_{\mathbb{C}} &= \lambda x_{\mathbb{E}} && (2.54)\\ N_{\mathbb{C}}\bar{C}_{\mathbb{E}}x_{\mathbb{E}} + A_{\mathbb{C}}x_{\mathbb{C}} &= \lambda x_{\mathbb{C}} && (2.55)\end{aligned}$$

where for clarity, we have deleted the notation indicating the dependence of each matrix on p and $\bar{C}_{\mathbb{E}} \stackrel{\Delta}{=} (I - G_{\mathbb{E}})^{-1}C_{\mathbb{E}}$.

Application of the linking equations (2.50) and (2.51) thus yields

$$\begin{aligned} (A_{\mathbb{D}} + D_{\mathbb{D}}\bar{C}_{\mathbb{D}} + B_{\mathbb{D}}M_{\mathbb{C}}\bar{C}_{\mathbb{D}})x_{\mathbb{D}} + B_{\mathbb{D}}C_{\mathbb{C}}x_{\mathbb{C}} &= \lambda x_{\mathbb{D}} && (2.56)\\ N_{\mathbb{C}}\bar{C}_{\mathbb{D}}x_{\mathbb{D}} + A_{\mathbb{C}}x_{\mathbb{C}} &= \lambda x_{\mathbb{C}} && (2.57)\end{aligned}$$

and

$$\bar{C}_{\mathbb{E}} = \bar{C}_{\mathbb{D}}L_{\mathbb{E}} \tag{2.58}$$

where $x_{\mathbb{D}} \stackrel{\Delta}{=} L_{\mathbb{E}}x_{\mathbb{E}}$ and

$$\bar{C}_{\mathbb{D}} \stackrel{\Delta}{=} (I - \bar{G}_{\mathbb{E}})^{-1}C_{\mathbb{D}}$$

Now if $L_{\mathbb{E}}x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are both zero, then $A_{\mathbb{E}}x_{\mathbb{E}} = \lambda x_{\mathbb{E}}$ because of (2.54) and (2.58); thus in this case λ is an eigenvalue of $A_{\mathbb{E}}$. Now suppose that $L_{\mathbb{E}}x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are not both zero. Then $[\, x'_{\mathbb{D}} \quad x'_{\mathbb{C}} \,]'$ is nonzero and thus an eigenvector of $\mathbb{D}_p \oplus \mathbb{C}_p$. Hence in this case λ is an eigenvalue of $\mathbb{D}_p \oplus \mathbb{C}_p$. Therefore claim i. of the lemma is true.

Now suppose that the eigenvector $[\, x'_{\mathbb{E}} \quad x'_{\mathbb{C}} \,]'$ just defined is in the unobservable space of $\mathbb{E}_p \oplus \mathbb{C}_p$. In other words suppose that

$$\bar{C}_{\mathbb{E}}x_{\mathbb{E}} = 0 \tag{2.59}$$

Then because of (2.58), $\bar{C}_{\mathbb{D}}x_{\mathbb{D}} = 0$ where $x_{\mathbb{D}} = L_{\mathbb{E}}x_{\mathbb{E}}$. If $L_{\mathbb{E}}x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are both zero, then λ must be an eigenvalue of $A_{\mathbb{E}}$ for the same reasons as before. On the other hand, if $L_{\mathbb{E}}x_{\mathbb{E}}$ and $x_{\mathbb{C}}$ are not both zero, then $[\, x'_{\mathbb{D}} \quad x'_{\mathbb{C}} \,]'$ is nonzero and thus an eigenvector of $\mathbb{D}_p \oplus \mathbb{C}_p$. Since $\bar{C}_{\mathbb{D}}x_{\mathbb{D}} = 0$, $[\, x'_{\mathbb{D}} \quad x'_{\mathbb{C}} \,]'$ is also in the unobservable space of $\mathbb{D}_p \oplus \mathbb{C}_p$. Therefore in this case, λ must be one of $\mathbb{D}_p \oplus \mathbb{C}_p$'s unobservable eigenvalues. Thus claim ii. of the lemma is true. ■

2.4.1 Design Model State Feedback

All the results just discussed regarding certainty equivalence and design models extend in a straightforward manner to the more general case when there is design model state feedback to the multi-controller designed using the feedback system shown in Figure 2.13.

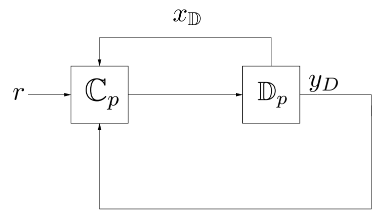

Figure 2.13: *pth* Controlled Design Model $\mathbb{D}_p \oplus \mathbb{C}_p$ with State-Feedback

Actual implementation of a multi-controller designed in this way would be accomplished by using in place of $x_{\mathbb{D}}$ the *surrogate design model state*

$$\bar{x}_{\mathbb{D}} \triangleq L_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}}$$

as shown in Figures 2.14a and 2.14b.

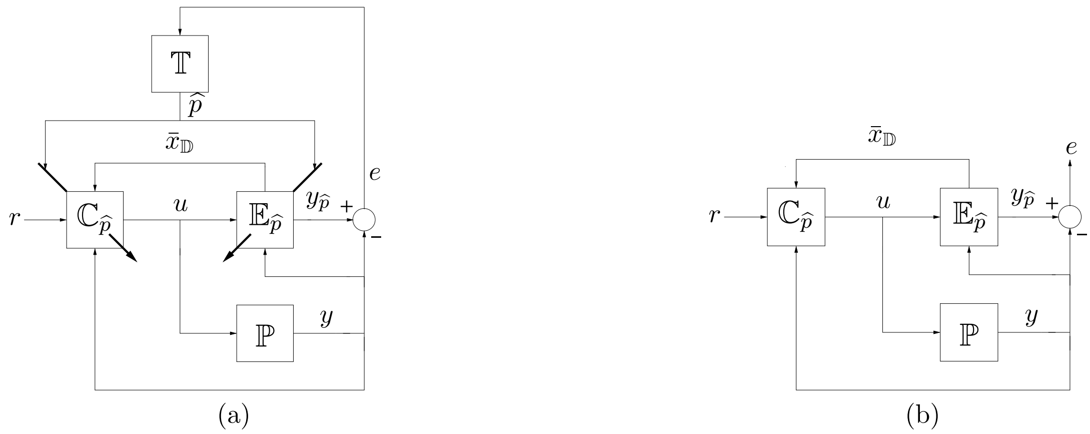

Figure 2.14: Adaptive System and $\mathbb{F}_{\widehat{p}\widehat{p}}$ with Surogate Design Model State Feedback

We leave it to the reader to verify that Corollaries 1 and 2 hold in this case as well.

2.5 SISO Direct Control Design Model

Let n and m be fixed positive integers with $m \leq n$. A useful parameterized family of design models, which finds application in model reference adaptive control, is the set of n-dimensional, linear systems $\mathcal{D} = \{\mathbb{D}_p : p \in \mathcal{P}\}$ whose *pth* element $\mathbb{D}_p$ is of the form

$$y_{\mathbb{D}} = [\,0 \quad g(p)\bar{c}\,]\begin{bmatrix} x \\ \bar{x} \end{bmatrix} \qquad \begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} A & d(p)g(p)\bar{c} \\ \bar{b}c & \bar{A} \end{bmatrix}\begin{bmatrix} x \\ \bar{x} \end{bmatrix} + \begin{bmatrix} b(p) \\ \bar{b} \end{bmatrix} u_{\mathbb{D}} \tag{2.60}$$

where p is a parameter vector taking values in a parameter space $\mathcal{P}$, $d(p)$ and $b(p)$ are parameter-dependent n-vectors, and $g(p)$ is a parameter dependent scalar. Here (c, A) is a single-output, n-dimensional, parameter-independent observable pair with A stable, and $(\bar{A}, \bar{b}, \bar{c})$ is the coefficient matrix triple of a parameter-independent, m-dimensional, stable linear system with transfer function of the form $\frac{1}{\bar{\alpha}(s)}$, where $\bar{\alpha}$ is a monic, stable polynomial of degree m. Any set of systems of this type is called a n-dimensional, relative degree *m direct control design model family.* We will say that this family is *bilinearly paramaterized* if $\mathcal{P}$ is a subset of a finite dimensional normed linear space and $d(p), b(p)$ and $g(p)$ are affine linear functions of p. As we shall see, each bilinearly parameterized design model family of this type lead to a multi-estimator whose dependence on p, while not linear, is nevertheless tractable enough to allow for the devising of a provably correct parameter tuner.

Note that in some cases there may well be values of p {e.g., any value of p for which $g(p) = 0$} for which $\mathbb{D}_p$ model is not observable. On the other hand, since

$$\begin{bmatrix} A & d(p)g(p)\bar{c} \\ \bar{b}c & \bar{A} \end{bmatrix}$$

is an output injection away from the stability matrix

$$\begin{bmatrix} A & 0 \\ \bar{b}c & \bar{A} \end{bmatrix},$$

$\mathbb{D}_p$ is detectable for every value of $p \in \mathcal{P}$; i.e., like all design model families, $\mathcal{D}$ is *globally detectable.*

The utility of this design model family rests on the fact that each of its members is output stabilizable. In particular, the control law

$$u_{\mathbb{D}} = -cx$$

output stabilizes each and every $\mathbb{D}_p$ as can easily be seen by looking at $\mathbb{D}_p$'s structure in (2.60).

Next observe that $\mathbb{D}_p$'s transfer function is

$$t_{\mathbb{D}}(s, p) = \frac{g(p)(\alpha(s) + v(s)b(p))}{\bar{\alpha}(s)\alpha(s) - v(s)d(p)g(p)}$$

where $\alpha(s)$ and $\bar{\alpha}(s)$ are the characteristic polynomials of A and $\bar{A}$ respectively, and $v(s)$ is a row vector $v(s) = [\, \beta_1(s) \quad \beta_2(s) \quad \cdots \quad \beta_n(s) \,]$ defined by the formula

$$v(s) = \alpha(s)c(sI - A)^{-1}$$

Because (c, A) is observable, the set of entries of $v(s)$ form a basis for the linear space of polynomials of degree less than n. This means that if $p \longmapsto [\, g(p) \quad d'(p) \quad b'(p) \,]'$ is a surjective mapping onto $\mathbb{R}^{2n+1}$, then for any scalar π_0 and polynomials $\pi_1(s)$ and $\pi_2(s)$ each of degree less than n, there is a parameter value $\bar{p} \in \mathcal{P}$ such that $\pi_0 = g(\bar{p})$, $\pi_1(s) = v(s)d(\bar{p})$ and $\pi_2(s) = v(s)b(\bar{p})$.

Let h be a scalar and let $\gamma(s)$ and $\delta(s)$ be any pair of monic polynominals of degrees n and m respectively. We claim that if $p \longmapsto [\, g(p) \quad d'(p) \quad b'(p) \,]'$ is a surjective mapping, then there must be a value $\bar{p} \in \mathcal{P}$ such that

$$t_{\mathbb{D}}(s, \bar{p}) = h\frac{\gamma(s)}{\delta(s)} \tag{2.61}$$

In other words, if $p \longmapsto [\, g(p) \quad d'(p) \quad b'(p) \,]'$ is a surjective mapping, then the class of transfer functions which $\mathbb{D}_p$ can match as p ranges over $\mathcal{P}$, contains all {reduced} transfer functions of relative degree m and McMilian degree not exceeding n. To show that this is so, let $\beta(s)$ and $\rho(s)$ be the unique quotient and remainder of $\bar{\alpha}(s)\alpha(s)$ divided by $\gamma(s)$. Thus $\bar{\alpha}(s)\alpha(s) = \beta(s)\gamma(s) + \rho(s)$ where $\deg \rho(s) < \deg \gamma(s)$. These equations imply that $\deg \rho(s) < n$ and that $\beta(s)$ is monic and of degree m. From the latter it follows that

$\beta(s)\gamma(q)$ is monic and of degree n, and therefore that $\beta(s)\gamma(s) - \alpha(s)$ is of degree less than n. Thus if $p \longmapsto [\, g(p) \quad d'(p) \quad b(p) \,]$ is surjective, is possible to find a value $\bar{p} \in \mathcal{P}$ such that

$$g(\bar{p}) = h \qquad\qquad v(s)d(\bar{p}) = \rho(s) \qquad\qquad v(s)b(\bar{p}) = \beta(s)\gamma(s) - \alpha(s)$$

With $\bar{p}$ so defined

$$t_{\mathbb{D}}(s,p) = h\frac{\beta(s)\gamma(s)}{\bar{\alpha}(s)\alpha(s) - \rho(s)} = h\frac{\beta(s)\gamma(s)}{\beta(s)\delta(s)}$$

from which (2.61) follows at once.

2.6 The Multi-Estimator of a SISO Direct Control Design Model Family

Our aim here is to develop the equations of a multi-estimator for the n-dimensional, relative degree m direct control design model family $\mathcal{D}$ defined by (2.60). Towards this end, let us first note that these equations can be written in more detail as

$$\begin{aligned} y_{\mathbb{D}} &= g\bar{c}\bar{x} && (2.62)\\ \dot{x} &= Ax + dy_{\mathbb{D}} + bu_{\mathbb{D}} && (2.63)\\ \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{b}(u_{\mathbb{D}} + cx) && (2.64) \end{aligned}$$

where, for clarity, we've suppressed the notation indicating the dependence of this system's coefficient matrices on p. Let us note that (2.63) has the same structure as the indirect control design model studied earlier in these notes. Thus we can write

$$cx \approx [\, d' \quad b' \,]\, z_{\mathbb{D}} \qquad\qquad \dot{z}_{\mathbb{D}} = \begin{bmatrix} A' & 0 \\ 0 & A' \end{bmatrix} z_{\mathbb{D}} + \begin{bmatrix} c' \\ 0 \end{bmatrix} y_{\mathbb{D}} + \begin{bmatrix} 0 \\ c' \end{bmatrix} u_{\mathbb{D}} \qquad (2.65)$$

where by $\approx$ we mean "is asymptotically equal to." Therefore, in view of (2.62) and (2.64)

$$y_{\mathbb{D}} \approx g\bar{c}\bar{x}_{\mathbb{D}} \qquad\qquad \dot{\bar{x}}_{\mathbb{D}} = \bar{A}\bar{x} + \bar{b}(u_{\mathbb{D}} + [\, d' \quad b' \,]'\, z_{\mathbb{D}}) \qquad (2.66)$$

Moreover if we introduce the subsystem

$$\dot{H}_{\mathbb{D}} = \bar{A}H_{\mathbb{D}} + \bar{b}\,[\, u_{\mathbb{D}} \quad z'_{\mathbb{D}} \,]$$

by direct calculation

$$\bar{x}_{\mathbb{D}} \approx H_{\mathbb{D}} \begin{bmatrix} 1 \\ d \\ b \end{bmatrix}$$

so

$$y_{\mathbb{D}} \approx g\bar{c}H_{\mathbb{D}} \begin{bmatrix} 1 \\ d \\ b \end{bmatrix}$$

In the light of these relations, it is now clear that the linking equations will hold if we define $\mathbb{E}$ as the $2n + m(2n+1)$-dimensional system parameter-dependent linear system

$$y_{\widehat{p}} = g(\widehat{p})\bar{c}H \begin{bmatrix} 1 \\ d(\widehat{p}) \\ b(\widehat{p}) \end{bmatrix} \qquad\qquad \dot{H} = \bar{A}H + \bar{b}\,[\, u \quad z' \,] \qquad\qquad \dot{z} = \begin{bmatrix} A' & 0 \\ 0 & A' \end{bmatrix} z + \begin{bmatrix} c' \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ c' \end{bmatrix} u \qquad (2.67)$$

This multi-estimator can also be written in the more standard form by defining $x_{\mathbb{E}}$ to be the column vector,

$$x_{\mathbb{E}} = \begin{bmatrix} z \\ h_0 \\ h_1 \\ \vdots \\ h_{2n+1} \end{bmatrix}$$

where h_{i-1} is the ith column of the $m \times (2n+1)$ matrix H. Calculation of matrices $A_{\mathbb{E}}, B_{\mathbb{E}}, D_{\mathbb{E}}$ and $L_{\mathbb{E}}(p)$ for which the linking equations holds is a straightforward, task. For example, $L_{\mathbb{E}}(p)$ is the block diagonal matrix

$$L_{\mathbb{E}}(p) = \begin{bmatrix} N^{-1}\,[\,R'(d(p)) \quad R'(b(p))\,] & 0 \\ 0 & [\,I \quad d_1 I \quad d_2 I \quad \cdots \quad d_n I \quad b_1 I \quad b_2 I \quad \cdots \quad b_n I\,] \end{bmatrix}$$

where N is the observability matrix of (c, A), $R(d(p))$ and $R(b(p))$ are the controllability matrices of $(A, d(p))$ and $(A, b(p))$ respectively, and d_i and b_i are the *ith* entries of $d(p)$ and $b(p)$ respectively.

Since the control law

$$u_{\mathbb{D}} = cx_1$$

output stabilizes the direct control design model, the multi-estimator state-feedback control law

$$u = [\,c \quad 0\,]\, L_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}}$$

is a certainty equivalence output stabilizing control. In view of the definition of $x_{\mathbb{E}}$ and structure of $L_{\mathbb{E}}$, this control can be written more simply as

$$u = [\,d'(\widehat{p}) \quad b'(\widehat{p})\,]\, z \tag{2.68}$$

The term "direct control" stems from the fact that this multi-estimator contains coefficient vectors, namely $d(\widehat{p})$ and $b(\widehat{p})$ which at the same time comprise $\mathbb{C}_{\widehat{p}}$'s "gains." In other words, in the direct control case $\mathbb{C}_{\widehat{p}}$'s gains are directly tuned, whereas with the indirect control case, $\mathbb{C}_{\widehat{\ }}$'s gains are determined in an indirect manner. Although somewhat artificial, the terminology seems to have caught on and does serve to distinguish two parameterization with different properties.

The Certainty Equivalence Output Stabilization Theorem does not require $\mathbb{P}$'s model to have a transfer matrix which can be matched by $\mathbb{D}_p$ for some value of $p \in \mathcal{P}$. It is interesting to note however, that in the case when exact matching is possible, to achieve detectability it is not necessary for the design model to be stabilizable, even at the "true design model parameter value" at which matching takes place! For example, the process model

$$\dot{y} = -4y + u$$

has the same transfer function as the direct control design model

$$\begin{aligned} \dot{x} &= -2x + dg\bar{x} + bu_{\mathbb{D}} \\ \dot{\bar{x}} &= x - \bar{x} + u_{\mathbb{D}} \\ y_{\mathbb{D}} &= g\bar{x} \end{aligned}$$

at

$$p = \begin{bmatrix} d \\ b \\ g \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix}$$

and for this value of p, the design model has an uncontrollable eigenvalue at $s = 1$.

Since for obvious reasons a process model for $\mathbb{P}$ must always be assumed to be at least stabilizable and detectable, this example shows that a design model with its parameter vector fixed cannot necessarily be viewed merely as a "nominal" process model.

2.7 MIMO Indirect Control Design Models

For given positive integers m, n and k with $m \leq n$, let $\mathcal{C}_P$ denote the set of all k input, n-dimensional, m-output, controllable, observable linear systems whose coefficient matrix triples (C_P, A_P, B_P) satisfy rank $C_P = m$. Our aim is to explain how to construct a finite collection of parameterized design model families whose class of matchable transfer matrices contains the transfer matrix of each system in $\mathcal{C}_P$. In order to do this, it is helpful to recall certain basic algebraic properties of observable matrix pairs.

With n and m as above, let $\mathcal{O}^{(m,n)}$ denote the set of all real, m-output, n-dimensional, observable matrix pairs $(C_P^{m\times n}, A_P^{n\times n})$ with rank $C_P = m$. The following theorem summarizes several well-known results [4, 5, 6]

Theorem 5 *For each pair $(C_P, A_P) \in \mathcal{O}^{(m,n)}$ there is exactly one list of positive integers $\{n_1, n_2, \ldots, n_m\}$ with the following properties:*

1. $n_1 + n_2 + \cdots n_m = n$
2. *There exist real matrices $N^{m\times m}$, $K^{n\times m}$ and $T^{n\times n}$, with N strictly lower triangular and T nonsingular, such that*

$$\begin{aligned}(I+N)C_PT^{-1} &= \bar{C}\\ T(A_P+KC_P)T^{-1} &= \bar{A}\end{aligned}$$

where

$$\begin{aligned}\bar{C} &= \text{block diagonal } \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m\}\\ \bar{A} &= \text{block diagonal } \{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_m\}\end{aligned}$$

and

$$\begin{aligned}\bar{c}_i &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times n_i}\\ \bar{A}_i &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n_i\times n_i}\end{aligned}$$

The integers $n_1, n_2, \ldots, n_m$ are known as the *observability indices* of (C_P, A_P) and the matrix pair $(\bar{C}, \bar{A})$ is (C_P, A_P)'s *output-injection* canonical form[5]. The $n_1, n_2, \ldots, n_m$ are also known as the *controllability indices* of the transposed, controllable pair (A'_P, C'_P) and $(\bar{A}', \bar{C}')$ is sometimes called (A'_P, C'_P)'s *state-feedback* canonical form. The fact that (C_P, A_P) can be mapped by means of the transformation $(C_P, A_P) \longmapsto ((I+N)C_PT^{-1}, T(A_P+KC_P)T^{-1})$ into output-injection canonical form is a direct consequence of the work of (Luenberger). It is known (Brunovsky) that the *unordered* set of observability indices of (C_P, A_P) is invariant with respect to transformations of the type $(C_P, A_P) \longmapsto (HC_PT^{-1}, T(A_P+KC_P)T^{-1})$ where H is *any* nonsingular $m \times m$ matrix. Since $(I+N)$ is nonsingular, it follows that (C_P, A_P)'s observability indices are also invariant with respect to the type of transformation of interest here, namely $(C_P, A_P) \longmapsto ((I+N)C_PT^{-1}, T(A_P+KC_P)T^{-1})$. It turns out that the *order* of the list $\{n_1, n_2, \ldots, n_m\}$ is also invariant with respect to such transformations. This can easily be deduced from the cited work.

[5]Note that $(\bar{C}, \bar{A})$ is uniquely determined by the list $\{n_1, n_2, \ldots, n_m\}$.

Call two pairs (C_P, A_P) and $(\tilde{C}_P, \tilde{A}_P)$ in $\mathcal{O}^{(m,n)}$, *output-injection equivalent* if there exists a strictly lower triangular matrix N, a nonsingular matrix T and a "output-injection" matrix K such that

$$(C_P,\ A_P) = ((I+N)\tilde{C}_P T^{-1},\ T(\tilde{A}_P + K\tilde{C}_P)T^{-1})$$

In the light of the preceding discussion it is easy to see that output-injection equivalence is an equivalence relation on $\mathcal{O}^{(m,n)}$ and also that the list of observability indices of each pair $(C_P, A_P) \in \mathcal{O}^{(m,n)}$ uniquely characterizes the equivalence class of (C_P, A_P) determined by this relation. The number of such equivalence classes is the same as the number of distinct lists of m positive integers $n_1, n_2, \ldots, n_m$ satisfying $n_1 + n_2 + \cdots + n_m = n$; this number is known (Hall) to equal the binomial coefficient

$$\binom{n-1}{m-1} = \frac{(n-1)!}{(n-m)!(m-1)!}$$

It is natural to call two systems $\Sigma_P = (C_P, A_P, B_P)$ and $\tilde{\Sigma}_P = (\tilde{C}_P, \tilde{A}_P, \tilde{B}_P)$ in $\mathcal{C}_P$, *output-injection equivalent* if (C_P, A_P) and $(\tilde{C}_P, \tilde{A}_P)$ are output-injection equivalent. Output-injection equivalence is an equivalence relation on $\mathcal{C}_P$ which partitions $\mathcal{C}_P$ into $\binom{n-1}{m-1}$ {disjoint} equivalence classes $\mathcal{C}_P\{list_1\}, \mathcal{C}_P\{list_2\}, \ldots$, one for each list of m positive integers whose sum is n. In particular, for any such list $l = \{n_1, n_2, \ldots, n_m\}$, $\mathcal{C}_P\{l\}$ is the subset of all systems in $\mathcal{C}_P$ whose coefficient matrix triples (C_P, A_P, B_P) have $\{n_1, n_2, \ldots, n_m\}$ as their list of observability indices.

In [7] it was suggested that parameter identification might be approached in the state space by using suitably defined parameterizations of specific output-injection equivalence classes. A parameterized design model family $\mathcal{D}\{l\}$ which achieves this for a given equivalence class $\mathcal{C}_P\{l\}$ would then be a design model family whose set of matchable transfer matrices contains the transfer matrices of each systems in $\mathcal{C}_P\{l\}$. Such a design model family can be constructed as follows.

Let n_0 denote the value of the largest integer in l. Pick any monic stable polynomial $\alpha(s)$ of degree n_0, in such a way that for each $i \in \{1, 2, \ldots, m\}$, $\alpha(s)$ has a real monic factor $\alpha_i(s)$ of degree n_i[6]. For each $i \in \{1, 2, \ldots, m\}$, let (c_i, A_i) be any single-output, n_i-dimensional observable pair for which characteristic polynomial $A_i = \alpha_i(s)$. Next define

$$\begin{aligned} C_{\mathbb{D}} &= \text{block diagonal } \{c_1, c_2, \ldots, c_m\} \\ A_{\mathbb{D}} &= \text{block diagonal } \{A_1, A_2, \ldots, A_m\} \end{aligned}$$

and note that $(C_{\mathbb{D}}, A_{\mathbb{D}})$'s list of observability indices is l. For fixed l, $\mathbb{D}_p^l$ is then defined by the equations

$$\begin{aligned} \dot{x}_{\mathbb{D}} &= (A_{\mathbb{D}} + D(I-G)^{-1}C_{\mathbb{D}})x_{\mathbb{D}} + Bu_{\mathbb{D}} \\ y_{\mathbb{D}} &= (I-G)^{-1}C_{\mathbb{D}}x_{\mathbb{D}} \end{aligned} \tag{2.69}$$

where D, B, and G are parameter matrices taking values in $\mathbb{R}^{n\times m}$, $\mathbb{R}^{n\times k}$ and $\mathcal{G}$ respectively, $\mathcal{G}$ being the linear space of strictly lower triangular $m \times m$ matrices. Thus $p = \{D, B, G\}$ and $\mathbb{D}_p^l$'s parameter space is $\mathcal{P} = \mathbb{R}^{m\times n} \oplus \mathbb{R}^{n\times k} \oplus \mathcal{G}$. As defined, $\mathbb{D}_p^l$ has the following properties:

1. *Global Observability:* For each $p \in \mathcal{P}$, $\mathbb{D}_p^l$ in observable.

2. *Match-Point Controllability:* $\mathbb{D}_p^l$ is controllable at any point $q \in \mathcal{P}$ for which there is a process model $\Sigma_P \in \mathcal{C}_P$ whose transfer matrix is matched by $\mathbb{D}_q^l$'s.

3. *Matchability:* For each $\Sigma_P \in \mathcal{C}_P\{l\}$, there is a point $p_0 \in \mathcal{P}$ at which $\mathbb{D}_{p_0}^l$'s transfer matrix equals Σ_P's.

[6] While each of the $\alpha_i(s)$ must be stable and of degree n_i, for this construction to lead to a design model with observability index list l, it is not necessary for the $\alpha_i(s)$ to be factors of $\alpha(s)$. However, if the $\alpha_i(s)$ are chosen in this way, the dimension of the multi-estimator $\mathbb{E}$ determined by $\mathcal{D}\{l\}$, will be smaller than that which would result, if the $\alpha_i(s)$ were not so constrained.

Property 1 is obvious since observability is invariant under output injection and $((I-G)^{-1}C_{\mathbb{D}}),\ A_{\mathbb{D}} + D(I-G)^{-1}C_{\mathbb{D}})$ is an output injection away from the observable pair $(C_{\mathbb{D}},\ A_{\mathbb{D}})$. Property 2 is a consequence of the definition of $\mathcal{C}_P\{l\}$ since each system in this class is controllable, observable and of the same dimension as $\mathbb{D}_p^l$. To establish property 3, fix $\Sigma_P \in \mathcal{C}_P\{l\}$ and let (C_P, A_P, B_P) denote its coefficient matrix triple. Since (C_P, A_P) and $(C_{\mathbb{D}}, A_{\mathbb{D}})$ both have the same list of observability indices, namely l, they must be output-injection equivalent pairs. Thus there must exist matrices T, N, and K, with T nonsingular and $N \in \mathcal{G}$ such that

$$(C_P,\ A_P) = ((I+N)C_{\mathbb{D}}T^{-1},\ T(A_{\mathbb{D}} + KC_{\mathbb{D}})T^{-1})$$

In addition, since matrices of the form $(I+N)$ constitute a multiplicative group, there must be a matrix $\bar{N} \in \mathcal{G}$ for which $(I+N)^{-1} = (I+\bar{N})$. Set $G_0 = -\bar{N}$, $D_0 = K(I+N)^{-1}$ and $B_0 = T^{-1}B_P$. These definitions imply that

$$(C_P, A_P, B_P) = ((I-G_0)^{-1}C_{\mathbb{D}}T^{-1},\ T(A_{\mathbb{D}} + D_0(I-G_0)^{-1}C_{\mathbb{D}}T^{-1},\ TB_0)$$

and therefore that $\mathbb{D}_p^l$'s transfer matrix matches Σ_P's at $p = p_0 = \{D_0, B_0, G_0\}$.

Remark 1 Although not widely appreciated, match-point controllability {or more generally, "match-point stabilizability" [7]} is an key property of design models used in conjunction with adaptive control algorithms of all types which employ certainty equivalence stabilizing controls. This is because design models which are not at least match-point stabilizable, can cause nominal process model overmodeling problems and consequential numerical ill-conditioning.

Remark 2 There is an alternative globally observable design model of the same general form as (2.69) but without the parameter matrix G, which is capable of matching any process model in the full class $\mathcal{C}_P$. This model, which does not require the specification of a list of observability indices, is constructed by picking any single-output, n-dimensional observable, stable pair $(\bar{c}, \bar{A})$ and simply defining $C_{\mathbb{D}}$ and $A_{\mathbb{D}}$ to be $m \times nm$ and $nm \times nm$ block diagonal matrices, whose ith diagonal blocks are $\bar{c}$ and $\bar{A}$ respectively. The problem with this design model {not to mention its increased dimensionality}, is that it is not match-point stabilizable for the given class of process models.

2.8 The Multi-Estimator of a MIMO Indirect Control Design Model

Because of the structure of $\mathbb{D}_p$ in (2.69), and the particular way in which $C_{\mathbb{D}}$ and $A_{\mathbb{D}}$ are defined, it is possible to construct a linear, matrix-valued function $L_{\mathbb{E}} : \mathcal{P} \rightarrow \mathbb{R}^{n \times n_0(m+k)}$ and constant matrices $A_{\mathbb{E}}, B_{\mathbb{E}}$, and $D_{\mathbb{E}}$, with $A_{\mathbb{E}}$ stable, so that the linking equations hold. As noted previously, these matrices together with the matrices defined by (**??**) determine an $n_0(m+k)$-dimensional multi-estimator $\mathbb{E}$. We now outline how to go about constructing these matrices.

To motivate what comes next, first note that (2.69) can also be written as

$$\begin{aligned} \dot{x}_{\mathbb{D}} &= A_{\mathbb{D}}x_{\mathbb{D}} + Dy_{\mathbb{D}} + Bu_{\mathbb{D}} \\ y_{\mathbb{D}} &= Gy_{\mathbb{D}} + C_{\mathbb{D}}x_{\mathbb{D}} \end{aligned} \tag{2.70}$$

Next observe that by partitioning $x_{\mathbb{D}}$ as

$$x_{\mathbb{D}} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \tag{2.71}$$

[7] A design model $\mathbb{D}_p$ is *match-point stabilizable* {resp. *match-point controllable*} with respect to a class $\mathcal{C}_P$ of process models if $\mathbb{D}_p$ is stabilizable {resp. controllable} at any point $p_0 \in \mathcal{P}$ for which there is a process model $\Sigma_P \in \mathcal{C}_P$ whose transfer matrix is matched by $\mathbb{D}_{p_0}$'s.

where $x_i \in \mathbb{R}^{n_i}$, $\mathbb{D}_p$ can be decomposed into the m subsystems

$$\dot{x}_i = A_i x_i + [\, D_i \quad B_i \,] \begin{bmatrix} y_{\mathbb{D}} \\ u_{\mathbb{D}} \end{bmatrix} \qquad i \in \{1, 2 \ldots, m\} \tag{2.72}$$

where $[\, D_i \quad B_i \,]_{n_i \times (m+k)}$ is a the ith submatrix of $[\, D \quad B \,]$, the latter being partitioned in the obvious way.

Pick any n_0-dimensional observable pair (c, A) so that A's characteristic polynomial is $\alpha(s)$. Next recall that each A_i has been chosen so that its characteristic polynomial is a factor of $\alpha(s)$. Because of this and the observability of the pairs (c_i, A_i), realization theory tells us that it is possible to find real-valued, left-invertible matrices $M_1, M_2, \ldots, M_m$ for which the equations

$$c_i(sI - A_i)^{-1} = c(sI - A)^{-1} M_i, \quad i \in \{1, 2, \ldots, m\}$$

hold[8]. These equations imply that for $i \in \{1, 2, \ldots, m\}$,

$$\begin{aligned} c_i &= cM_i & (2.73) \\ M_i A_i &= A M_i & (2.74) \end{aligned}$$

Thus if we define $\bar{x}_i = M_i x_i$, then (2.74) together with (2.72) enable us to write

$$\begin{aligned} \dot{\bar{x}}_i &= A\bar{x}_i + M_i [\, D_i \quad B_i \,] \begin{bmatrix} y_{\mathbb{D}} \\ u_{\mathbb{D}} \end{bmatrix} \qquad i \in \{1, 2 \ldots, m\} \\ x_i &= R_i \bar{x}_i \end{aligned} \tag{2.75}$$

where R_i is a left inverse of M_i.

Our aim at this point is to develop a single, $n_0(m+k)$-dimensional linear system with inputs $u_{\mathbb{D}}$ and $y_{\mathbb{D}}$, whose state equation does not depend on B and D, and whose outputs asymptotically estimate each of the $\bar{x}_i$. For this, we need the following notation: Let $S : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0 \times n_0}$ denote the linear transformation

$$q \longmapsto N^{-1} R'(q) \tag{2.76}$$

where N and $R(q)$ are the observability and controllability matrices of (c, A) and (A, q) respectively. Next define the linear transformation $H : \mathbb{R}^{n_0 \times (m+k)} \to \mathbb{R}^{n_0 \times (n_0(m+k))}$ by

$$Q \longmapsto [\, S(q_1) \quad S(q_2) \quad \cdots \quad S(q_{(m+k)}) \,] \tag{2.77}$$

where q_i is the ith column of Q. Finally, let *stack* denote the linear transformation $\mathbb{R}^{(m+k) \times n_0} \to \mathbb{R}^{(m+k)n_0}$ which assigns to each $(m+k) \times n_0$ matrix Z, the "stacked" $(m+k)n_0$-vector $[\, z_1 \quad z_2 \quad \cdots \quad z_{(m+k)} \,]'$, z_i being the ith row of Z.

Lemma 8 *If $X_{\mathbb{D}}$ satisfies*

$$\dot{X}_{\mathbb{D}} = X_{\mathbb{D}} A + \begin{bmatrix} y_{\mathbb{D}} \\ u_{\mathbb{D}} \end{bmatrix} c \tag{2.78}$$

then for $i \in \{1, 2 \ldots, m\}$

$$\bar{x}_i = H\{M_i [\, D_i \quad B_i \,]\} \operatorname{stack}\{X_{\mathbb{D}}\} + e^{A_t} v_i \tag{2.79}$$

where $v_i = \bar{x}_i(0) - H\{M_i [\, D_i \quad B_i \,]\} \operatorname{stack}\{X_{\mathbb{D}}(0)\}$

In the light of Lemma 8, (2.71) and (2.75), it is clear that with $p = \{D, B, G\}$,

$$x_{\mathbb{D}} = L_{\mathbb{E}}(p) \operatorname{stack}\{X_{\mathbb{D}}\} + \epsilon_{\mathbb{D}} \tag{2.80}$$

[8] M_i can be defined explicitly by the formula $M_i = H_i N_i$ where N_i is the observability matrix of (c_i, A_i) and H_i is a right inverse of the matrix whose rows are the first n_i rows of the observability matrix of (c, A).

where $L_{\mathbb{E}}(p)$ is the $n \times n_0(m+k)$ matrix

$$L_{\mathbb{E}}(p) = \begin{bmatrix} R_1 H\{M_1 [\, D_1 \quad B_1 \,]\} \\ R_2 H\{M_2 [\, D_2 \quad B_2 \,]\} \\ \vdots \\ R_m H\{M_m [\, D_m \quad B_m \,]\} \end{bmatrix}, \tag{2.81}$$

and

$$\epsilon_{\mathbb{D}} = \text{column}\,\{R_1 e^{At} v_1, \ldots, R_m e^{At} v_m\}$$

It is also clear from Lemma 1 and (2.70), that

$$y_{\mathbb{D}} = G y_{\mathbb{D}} + C_{\mathbb{D}} L_{\mathbb{E}}(p)\,\text{stack}\,\{X_{\mathbb{D}}\} + C_{\mathbb{D}} \epsilon_{\mathbb{D}} \tag{2.82}$$

Equations (2.78), (2.80), and(2.82) suggest that to generate $e_{\mathbb{E}}$ and $\widehat{x}_{\mathbb{D}}$, one might use as a multi-estimator $\mathbb{E}$ the system

$$\dot{X}_{\mathbb{E}} = X_{\mathbb{E}} A + \begin{bmatrix} y \\ u \end{bmatrix} c \tag{2.83}$$

$$\widehat{x}_{\mathbb{D}} = L_{\mathbb{E}}(\widehat{p})\,\text{stack}\,\{X_{\mathbb{E}}\} \tag{2.84}$$

$$e_{\mathbb{E}} = C_{\mathbb{D}} L_{\mathbb{E}}(\widehat{p})\,\text{stack}\,\{X_{\mathbb{E}}\} + \widehat{G} y - y \tag{2.85}$$

where $\widehat{p} = \{\widehat{D}, \widehat{B}, \widehat{G}\} \in \mathcal{P}$ is the parameter to be tuned[9] . In fact, if $A_{\mathbb{E}}, B_{\mathbb{E}}$, and $D_{\mathbb{E}}$ are defined by

$$\begin{aligned} A_{\mathbb{E}} &= \text{block diagonal}\,\{\overbrace{A', \ldots, A'}^{m \text{ times}}, \overbrace{A', \ldots, A'}^{k \text{ times}}\} \\ B_{\mathbb{E}} &= \text{block diagonal}\,\{\overbrace{0, \ldots, 0}^{m \text{ times}}, \overbrace{c', \ldots, c'}^{k \text{ times}}\} \\ D_{\mathbb{E}} &= \text{block diagonal}\,\{\overbrace{c', \ldots, c'}^{m \text{ times}}, \overbrace{0, \ldots, 0}^{k \text{ times}}\} \end{aligned}$$

then a simply calculation shows that the linking equations (2.50) hold and the stacked identifier state $x_{\mathbb{E}} = \text{stack}\,\{X_{\mathbb{E}}\}$ satisfies identifier equations

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}} x_{\mathbb{E}} + D_{\mathbb{E}} y + B_{\mathbb{E}} u \\ \widehat{x}_{\mathbb{D}} &= L_{\mathbb{E}}(\widehat{p}) x_{\mathbb{E}} \\ e_{\mathbb{E}} &= C_{\mathbb{E}}(\widehat{p}) x_{\mathbb{E}} + G_{\mathbb{E}}(\widehat{p}) y - y \end{aligned}$$

where $C_{\mathbb{E}}(p) = C_{\mathbb{D}} L_{\mathbb{E}}(p)$.

Example 4 Suppose $m = 3,\ n = 4,\ k = 1,\ n_1 = 1,\ n_2 = 2$ and $n_3 = 1$. Pick $\alpha = (s+1)(s+2) = s^2+3s+2$. Choose polynomials $\alpha_1 = s+1$, $\alpha_2 = s^2+3s+2$, and $\alpha_3 = s+2$. Note that α_i divides α and has degree n_i as required. Accordingly, define the observable matrix pairs,

$$\begin{aligned} c &= [\,0 \quad 1\,] \\ A &= \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \end{aligned}$$

[9] Using the easily derived identity, $cH(Q)\,\text{stack}\,\{Z\} = \text{trace}\,\{ZQ\}$, which holds for all $Q \in \mathbb{R}^{n_0 \times (m+k)}$ and all $Z \in \mathbb{R}^{(m+k) \times n_0}$ it is not difficult to show that $C_{\mathbb{D}} L_{\mathbb{E}}(\widehat{p})\,\text{stack}\,\{X_{\mathbb{E}}\} = \text{column}\,\{\text{trace}\,\{X_{\mathbb{E}} M_1 [\, \widehat{D}_1 \quad \widehat{B}_1 \,]\}, \ldots, \text{trace}\,\{X_{\mathbb{E}} M_m [\, \widehat{D}_m \quad \widehat{B}_m \,]\}\}$ which simplifies the expression for $e_{\mathbb{E}}$ in (2.85).

$(c_1,\ A_1) = (1,\ -1), (c_2,\ A_2) = (c,\ A), (c_3,\ A_3) = (1,\ -2).$ and design model matrices $C_{\mathbb{D}} =$ block diag. $\{c_1, c_2, c_3\}$ and $A_{\mathbb{D}} =$ block diag. $\{A_1, A_2, A_3\}$. $\Sigma_{\mathbb{D}}$ is thus a four-dimensional system with parameter matrices $D = \{d_{ij}\}_{4\times 3}$, $B = [\,b_1 \quad b_2 \quad b_3 \quad b_4\,]'_{1\times 4}$ and $G_{3\times 3}$, the latter being a strictly lower triangular matrix with three parameters. For this design model, $\mathcal{P}$ is therefore a 19-dimensional linear space.

For this example, $\mathbb{E}$ is an eight-dimensional system described by (2.83)-(2.85). To construct $L_{\mathbb{E}}$ it is necessary to first compute M_1, M_2, M_3 and S. These matrices turn out to be $M_1 = [\,2 \quad 1\,]'$, $M_2 = I_{2\times 2}$, $M_3 = [\,1 \quad 1\,]'$ and

$$S\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) = \begin{bmatrix} 3q_1 - 2q_2 & q_1 \\ q_1 & q_2 \end{bmatrix}$$

respectively. Next pick left inverses $R_1 = [\,0 \quad 1\,]$, $R_2 = I_{2\times 2}$ and $R_3 = [\,0 \quad 1\,]$ of M_1, M_2 and M_3 respectively. $L_{\mathbb{E}}$ can now be written out as

$$L_{\mathbb{E}} = \begin{bmatrix} 2d_{11} & d_{11} & 2d_{12} & d_{12} & 2d_{31} & d_{31} & 2b_1 & b_1 \\ 3d_{21} - 2d_{31} & d_{21} & 3d_{22} - 2d_{32} & d_{22} & 3d_{23} - 2d_{33} & d_{23} & 3b_2 - 2b_3 & b_2 \\ d_{21} & d_{31} & d_{22} & d_{32} & d_{23} & d_{33} & b_2 & b_3 \\ d_{41} & d_{41} & d_{42} & d_{42} & d_{43} & d_{43} & b_4 & b_4 \end{bmatrix}$$

and $C_{\mathbb{E}}$ can easily be computed using (2.85).♠

The methods presented here and in the last secction explain how to construct both a parameterized design model and an associated multi-estimator for a class of process models whose elements all have the same list of observability indices. To cover the full class $\mathcal{C}_P$ of m-output, n-dimensional, k-input controllable observable linear systems, one needs to construct a design model and a multi-estimator for each output-injection equivalence class.

Proof of Lemma 8: The rule defining H in (2.77) together with (2.45) imply that for all $Q \in \mathbb{R}^{n_0\times(m+k)}$ and all $z \in \mathbb{R}^{m+k}$

$$H(Q)\text{stack}\,\{zc\} = Qz \tag{2.86}$$

Similarly, (2.77) and (2.47)imply that

$$H(Q)\text{stack}\,\{ZA\} = AH(Q)\text{stack}\,\{Z\} \tag{2.87}$$

for all $Z \in \mathbb{R}^{(m+k)\times n_0}$ and all $Q \in \mathbb{R}^{n_0\times(m+k)}$. Define $w = H(M_i\,[\,D_i \quad B_i\,])\text{stack}\,\{Z\}$. A direct calculation employing (2.78),(2.86), and (2.87) shows that w satisfies the same differential equation as $\bar{x}_i$, namely (2.75). Thus (2.79) must hold. ■

Chapter 3

ACHIEVING DETECTABILITY VIA CYCLIC SWITCHING

By the *stabilizability problem* for linear estimator-based adaptive control systems is meant multi-control synthesis problem which arises in the design of estimator-based adaptive controllers in those cases when there are design models in *scrD* which are not stabilizable. In this chapter w will explain why this problem occurs and how to deal with it using generalized for of certainty equivalence control which employs "cyclic switching." A cyclicly-switched multi-controller is a bona fide example of a hybrid control system which is interesting in its own right because it is both intended for a specific purpose and provably correct. Cyclic switching can be used to deal with the stabilizability problem with or without process excitation. In particular, for the technique to work it is not necessary for there to be a mechanism for tuning $\widehat{p}$ to avoiding the singularities in parameter space at which there are unstabilzable design models.

3.1 Global Detectability Problem

In the sequel we further consider the problem of constructing an estimator-based adaptive control system for a class of admissible process models of the form

$$\mathcal{M} = \bigcup_{p \in \mathcal{P}_M} \mathcal{M}_p$$

for the case when the set of nominals $\mathcal{N} = \{\mathbb{N}_p : p \in \mathcal{P}_M\}$ contains a continuum of members. As before we assume that all models in $\mathcal{M}$ are stabilizable and detectable linear systems with the same control input space $\mathcal{U}$ and measured output value space $\mathcal{Y}$. We will take as given a parameter space $\mathcal{P} \supset \mathcal{P}_M$, a family of design models $\mathcal{D} = \{\mathbb{D}_p : p \in \mathcal{P}\} \supset \mathcal{N}$ and an associated linear multi-estimator $\mathbb{E}_{\widehat{p}}$ whose coeffiicient matrices are related to those of the design models by linking equations.

There is an important and subtle point here: Although it is clearly reasonable to assume that all models in $\mathcal{M}$ are at least stabilizable and detectable, the same assumption cannot be made about $\mathcal{D}$ with out considerable loss of generality. The reason for this is as follows. At some point we will have to deal with the problem of devising a tuner or switching logic for adjusting $\widehat{p}$ which seeks to minimize some suitably defined norm of the output estimation error $y_{\widehat{p}} - y$. In order for this problem to be tractable, $\mathcal{D}$ will typically have to be linearly parameterized and $\mathcal{P}$ will have to be a convex set {or more generally a finite union of disjoint convex sets.} These constraints on $\mathcal{D}$ and $\mathcal{P}$ together with the requirement that all of the design models in $\mathcal{D}$ be stabilizable and detectable, severely restrict the size of process model class $\mathcal{M}$ to which estimator-based adaptive control can be applied. The following simple example, illustrates the problem.

Example 5 Suppose that the class of admissible process models $\mathcal{M}$ consists of controllable and observable realizations of the transfer functions

$$t_\omega(s) = \frac{s - \frac{1}{6}(\omega + 2)}{s^2 + \omega s - \frac{2}{9}\omega(\omega + 2)}, \qquad \omega \in \{-1 \leq \omega \leq 1\}$$

Since this transfer function has no poles and zeros in common for any value of ω, all models in $\mathcal{M}$ will be of dimension 2. The coefficient vector of $t_\omega(s)$,, namely

$$[\, -\tfrac{1}{6}(\omega + 2) \quad \omega \quad -\tfrac{2}{9}\omega(\omega + 2)\,]'$$

sweeps out a curve $\mathcal{C}$ as ω varies between -1 and 1 which is a segment of one-dimensional parabolic curve in $\mathbb{R}^3$. Because of this the smallest convex set in $\mathbb{R}^3$ which contains $\mathcal{C}$ is the two-dimensional surface of points in $\mathbb{R}^3$ bounded by $\mathcal{C}$ and by the straight line connecting $\mathcal{C}$'s two end-points. This set is given by

$$\{[\, -\tfrac{1}{6}(p_1 + 2) \quad p_1 \quad p_2\,]' : [\, p_1 \quad p_2\,]' \in \mathcal{P}\}$$

where $\mathcal{P}$ is the convex parameter space

$$\mathcal{P} \triangleq \{[\, p_1 \quad p_2\,]' : -\frac{4}{9}(p_1 + \frac{1}{2}) \leq p_2 \leq -\frac{2}{9}p_1(p_1 + 2)\}$$

Thus one could choose for $\mathcal{D}$ any family of two-dimensional SISO indirect control design models whose transfer functions, as $p \triangleq [\, p_1 \quad p_2\,]$ ranges over $\mathcal{P}$, are the rational functions

$$h_p(s) = \frac{s - \frac{1}{6}(p_1 + 2)}{s^2 + p_1 s + p_2)}$$

Note that $h_p(s)$ has an unstable pole-sero cancellation at $p = [\, 0 \quad -\frac{1}{9}\,]'$, so the design model $\mathbb{D}_p$ at this value of p cannot be stabilizable. What this all means that there does not exist a linearly parameterized family of design models $\mathcal{D}$ on a convex parameter space $\mathcal{P}$ whose members are all stabilizable and whose class of matchable transfer functions includes all of those defined by $t_\omega(s)$ as ω ranges over its admisable values.

The fact that $\mathcal{D}$ may contain design models which are not stabilizable has important implications. In particular, recall that in carrying out a certainty equivalence synthesis, one tries to define a multi-controller $\mathbb{C}_{\widehat{p}}$ so that for each $p \in \mathcal{P}$, the controlled design system $\mathbb{D}_p \oplus \mathbb{C}_p$ shown in Figure 3.1, is at least internally stable. Corollary 1 to the Certainty Equivalence Stabilization Theorem states that if $\mathbb{C}_q$ achieves this with

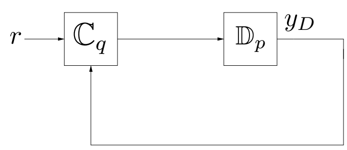

Figure 3.1: Controlled Design Model $\mathbb{D}_p \oplus \mathbb{C}_q$

$q \triangleq p$ for all values of p in some subset $\mathcal{E} \subset \mathcal{P}$, then the feedback subsystem $\mathbb{F}_{\widehat{p}\widehat{q}}$ shown in Figure 3.2 is detectable with $\widehat{q} \triangleq \widehat{p}$ for each fixed $\widehat{p} \in \mathcal{E}$. Since $\widehat{p}$ can take on any value in $\mathcal{P}$, one would like $\mathbb{C}_{\widehat{p}}$ to be chosen to make $\mathbb{F}_{\widehat{p}\widehat{p}}$ detectable on all of $\mathcal{P}$. It is clearly not possible to make use of Corollary 1 to achieve this if $\mathcal{D}$ contains design models which are not stabilizable. Various ideas have been proposed to deal with this problem. One is to choose $\mathbb{C}_p$ to stabilize $\mathbb{D}_p$ only at points outside of some region $\mathcal{S} \subset \mathcal{P}$ which contains the points at which $\mathbb{D}_p$ is not stabilizable; since detectability via design model stabilization cannot be achieved

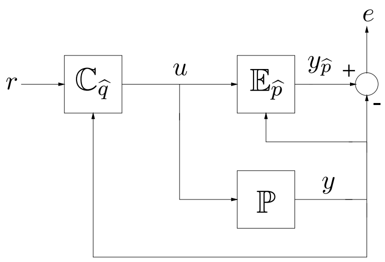

Figure 3.2: Feedback Sub-system $\mathbb{F}_{\widehat{p}\widehat{q}}$

at points inside of $\mathcal{S}$, in order to avoid instability, either some form of system excitation must be used to make certain that $\widehat{p}$ does not get stuck in $\mathcal{S}$ or, to account for the possibility that $\widehat{p}$ might remain in $\mathcal{S}$, another method must be used to define $\mathbb{C}_{\widehat{p}}$ to insure that misbehavior cannot occur. This line of reasoning leads naturally to the

Global Detectability Problem: *For a given parameterized design model family $\mathcal{D}$ with parameter space $\mathcal{P}$ and a given class of stabilizable and detectable process models $\mathcal{M}$ whose nominals are contained in $\mathcal{D}$, find if possible, a multi-controller $\mathbb{C}_{\widehat{p}}$ which makes the feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ shown in Figure 3.2 detectable for each frozen value of $\widehat{p} \in \mathcal{P}$.*

The challenge of course, is to come up with a solution to this problem for those cases when the Corollary 1 to the Certainty Equivalence Stabilization Theorem cannot be applied on all of $\mathcal{P}$ {eg., when $\mathbb{D}_p$ is not stabilizable on $\mathcal{P}$}. The following example illustrates some of the issues involved in achieving global detectability.

3.1.1 Global Detectability of One-Dimensional SISO Systems

Let $\mathcal{M}$ denote the class of all SISO process models of the form

$$\dot{y} = ay + bu \tag{3.1}$$

where a and b are constants safisfying

$$|a| \leq \bar{a} \qquad\qquad \tilde{b} \leq |b| \leq \bar{b}$$

Here $\bar{a}, \bar{b}$ and $\tilde{b}$ are positive constants such that $\tilde{b} < \bar{b}$. The preceding defines a class of controllable and observable process models with a compact but non-convex parameter space

$$\mathcal{P}_M = \{[\,a \quad b\,]' : |a| \leq \bar{a}, \tilde{b} \leq |b| \leq \bar{b}\} \tag{3.2}$$

Our first objective is to embed $\mathcal{M}$ in a family of linearly parameterized design models whose parameter space $\mathcal{P}$ is convex[1]. Toward this end, pick $\lambda > 0$ and let $\mathcal{P} \triangleq \{p : |p_1| \leq \bar{a}, |p_2| \leq \bar{b}\}$ where $p \triangleq [\,p_1 \quad p_2\,]'$. Let $\mathcal{D}$ denote the family of two-dimensional, linearly parameterized indirect control design models $\mathbb{D}_p$ of the form

$$\dot{y}_D = -\lambda y_D + (p_1 + \lambda) y_D + p_2 u_D$$

Note that $\mathcal{M} \subset \mathcal{D}$, that $\mathcal{D}$ is linearly parameterized, and that $\mathcal{P}$ is convex. $\mathcal{D}$'s multi-estimator $\mathbb{E}_{\widehat{p}}$ is described by the equations

$$\begin{aligned} \dot{x_1} &= -\lambda x_1 + y \\ \dot{x_2} &= -\lambda x_2 + u \\ y_{\widehat{p}} &= (\widehat{p}_1 + \lambda) x_1 + \widehat{p}_2 x_2 \end{aligned} \tag{3.3}$$

[1]Although for this example $\mathcal{P}_M$ is actually the union of two convex sets, we will not exploit this property here because we are trying to illustrate concepts which might be applied in more general situations where $\mathcal{P}_M$ might not have this structure.

Note that the coefficient matrics of $\mathbb{D}_p$ and $\mathbb{E}_p$ are linked by the equations

$$\begin{aligned} L_p \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} &= -\lambda L_p \\ L_p \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= p_2 \\ L_p \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= p_1 + \lambda \end{aligned}$$

where

$$L_p = [\, p_1 + \lambda \quad p_2 \,]$$

As a multi-controller $\mathbb{C}_{\widehat{p}}$, let us consider a simple output-feedback law of the form

$$u = f_{\widehat{p}} y \tag{3.4}$$

where f_p is parameter dependent feedback gain.

Our objective is to explain how to define f_p so that for each value of $[\, a \quad b \,]' \in \mathcal{P}_M$, the closed-loop feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{p}}$ defined by

$$\begin{aligned} \dot{y} &= (a + b f_{\widehat{p}}) y \\ \dot{x}_1 &= -\lambda x_1 + y \\ \dot{x}_2 &= -\lambda x_2 + f_{\widehat{p}} y \\ e &= (\widehat{p}_1 + \lambda) x_1 + \widehat{p}_2 x_2 - y \end{aligned} \tag{3.5}$$

is detectable for every fixed value of $\widehat{p}$ in $\mathcal{P}$.

As a first step toward this end, let us note that the preceding system will be globally detectable if the matrix pair

$$c_p = [\, -1 \quad p_1 + \lambda \quad p_2 \,] \tag{3.6}$$

$$\tag{3.7}$$

$$A_p = \begin{bmatrix} a + b f_p & 0 & 0 \\ 1 & -\lambda & 0 \\ f_p & 0 & -\lambda \end{bmatrix} \tag{3.8}$$

is detectable for each $p \in \mathcal{P}$. Next observe that the of the vector

$$z = \begin{bmatrix} 0 \\ -p_2 \\ p_1 + \lambda \end{bmatrix}$$

satisfies $A_p z = -\lambda z$ and $c_p z = 0$. Thus $e^{-\lambda} z$ is a stable unobservable mode of the pair (c_p, A_p). Moreover

$$Q_p z = 0 \qquad Q_p A_p = \bar{A}_p Q_p \qquad c_p = \bar{c}_p Q_p$$

where

$$Q_p \triangleq \begin{bmatrix} 1 & 0 \\ 0 & L_p \end{bmatrix} \qquad \bar{A}_p = \begin{bmatrix} a & b f_p \\ & \\ 0 & p_1 + p_2 f_p \end{bmatrix} \qquad \bar{c}_p = [\, -1 \quad 1 \,]$$

Because of this it is easy to see that the matrix pair (c_p, A_p) will be detectable at p just in case the reduced matrix pair $(\bar{C}_p, \bar{A}_p)$ is. Thus to make $\mathbb{F}_{\widehat{p}\widehat{p}}$ detectable for each fixed $\widehat{p} \in \mathcal{P}$ and each $[\, a \quad b \,]' \in \mathcal{P}_M$, it is sufficient to define f_p so that for each such $[\, a \quad b \,]'$ and each $p \in \mathcal{P}$, the matrix pair $(\bar{c}_p, \bar{A}_p)$ is detectable.

At this point it is especially useful to recognize is that the preceding can be accomplished by choosing f_p so that for each such $[\, a \quad b \,]'$ and p, either

1. $p_1 + +p_2 f_p$ is stable, or

2. $(\bar{c}_p, \bar{A}_p)$ is an *observable* matrix pair.

Condition 2 is equivalent to the inequality $p_1 + p_2 f_p \neq a + b f_p$. Thus our problem reduces to trying to define f_p on $\mathcal{P}$ so that for each admissible $[\,a \quad b\,]' \in \mathcal{P}_M$ and each $p \in \mathcal{P}$, f_p satisfies at least one of the following two

Detectability Conditions:

i. $p_1 + p_2 f_p < 0$

ii. $p_1 + p_2 f_p \neq a + b f_p$

These conditions are both necessary and sufficient for (c_p, A_p), as well as $(\bar{c}_p, \bar{A}_p)$ to be a detectable matrix pair.

Let $[\,a \quad b\,]$ be any fixed value in $\mathcal{P}_M$. Suppose that for some locally bounded function $p \mapsto f_p$, the pair $(\bar{c}_p, \bar{A}_p)$ is detectable on $\mathcal{P}$. Suppose, in addition, that for each fixed $p_1 \geq 0$, f_p is at least piecewise-continuous in p_2; then for each such p_1, there must be range of small values of p_2 for which detectability condition i fails to hold. It is clear that for such values of p_1 and p_2, condition ii above must necessarily hold. In other words, to attain detectability on $\mathcal{P}$, f_p *must* be chosen so that whenever $p_1 \geq 0$ and p_2 is in some neighborhood of the number 0, detectability condition ii holds.

It will now be shown that global detectability of $\mathbb{F}_{\widehat{p}\widehat{p}}$ {i.e., detectability of $(\bar{c}_p, \bar{A}_p)$ on $\mathcal{P}$} cannot be attained with any continuous feedback function.

Proposition 1: *There is no continuous feedback function $f_p : \mathcal{P} \rightarrow \mathbb{R}$ which makes $\mathbb{F}_{\widehat{p}\widehat{p}}$ detectable on $\mathcal{P}$ for all values of $[\,a \quad b\,]' \in \mathcal{P}_M$.*

Proof: Let $[\,a^* \quad b^*\,]'$ be any point in $\mathcal{P}_M$ at which number $a^* \geq 0$. Let $f_p : \mathcal{P} \rightarrow \mathbb{R}$ be any continuous feedback function. Suppose that with this particular function, $\mathbb{F}_{\widehat{p}}$ is detectable on $\mathcal{P}$ at the admissible points $[\,a^* \quad b^*\,]'$ and $[\,a^* \quad -b^*\,]'$. Since detectability condition ii above can't possibly hold when p and $[\,a \quad b\,]'$ both equal $[\,a^* \quad b^*\,]'$, it must be true that

$$a^* + b^* f_{p^*} < 0 \tag{3.9}$$

where $p^* \triangleq [\,a^* \quad b^*\,]'$. By similar reasoning applied to the situation when both p and $[\,a \quad b\,]'$ equal $[\,a^* \quad -b^*\,]'$, we obtain the inequality

$$a^* - b^* f_{\bar{p}^*} < 0 \tag{3.10}$$

where $\bar{p}^* \triangleq [\,a^* \quad -b^*\,]'$. These inequalities imply that

$$\begin{array}{lcl} b^* f_{p^*} & < & 0 \\ b^* f_{\bar{p}^*} & > & 0 \end{array} \tag{3.11}$$

Therefore, since f_p is continuous and $b^* \neq 0$, there must be a value $b^{**} \in (-b^*, b^*)$ at which $f_{p^{**}} = 0$ where $p^{**} \triangleq [\,a^* \quad b^{**}\,]'$. Clearly, at the point $p^{**} \in \mathcal{P}$ neither detectability condition i nor ii can hold when $a = a^*$. Therefore $\mathbb{F}_{\widehat{p}\widehat{p}}$ cannot be detectable on $\mathcal{P}$ when a takes on the admissible value a^*. Since this is a contradiction, the proposition is true. ∎

We now explain how to choose a *discontinuous* feedback function f_p which achieves global detectability for all $[\,a \quad b\,]' \in \mathcal{P}_M$ provided $\bar{a} < \infty$.

Case 1 We begin by first considering a simplified version of the problem in which the set of admissible process models $\bar{\mathcal{M}} \subset \mathcal{M}$ is limited to systems of the form

$$\dot{y} = \bar{a}y + bu \tag{3.12}$$

where $\bar{a}$ is a given positive constant and as before, b is a constant safisfying

$$\tilde{b} \leq |b| \leq \bar{b}$$

The preceding thus defines a one-parameter class of controllable and observable process models with a compact, convex parameter space

$$\bar{\mathcal{P}}_M = \{[\,\bar{a} \quad b\,]' : \tilde{b} \leq |b| \leq \bar{b}\} \tag{3.13}$$

Let $\bar{\mathcal{P}} \triangleq \{\bar{p} : |p_2| \leq \bar{b}\}$ where $\bar{p} \triangleq [\,\bar{a} \quad p_2\,]'$ and write $\bar{\mathcal{D}}$ for the one-parameter sub- family of linearly parameterized indirect control design models $\mathbb{D}_p$

$$\dot{y}_D = -\lambda y_D + (\bar{a} + \lambda)y_D + p_2 u_D$$

In view of the preceding discussion it is clear that to attain global detectability, it is enough to choose $f_{\bar{p}}$ so that $\bar{a} + p_2 f \bar{p} \neq \bar{a} + b f_{\bar{p}}$ for all p_2 satisfying $|p_2| < \tilde{b}$ and all $\tilde{b} \leq b \leq \bar{b}$, and also so that $\bar{a} + p_2 f_{\bar{p}} < 0$ for $\bar{b}|p_2| \geq \tilde{b}$. One particular function which satisfies these requirements is

$$f_{\bar{p}} = -\frac{2\bar{a} + d}{p_2 + \tilde{b}\mathrm{sign}(p_2)}$$

where d is an arbitrary but fixed positive number and $\mathrm{sign}(0) = 1$. This is, in fact, the feedback function proposed in [8].

Case 2 We now consider the case when $\mathbb{P}$ admits a model in the two-parameter faimly of process models $\mathcal{M}$ defined earlier. As already noted, for $(\bar{c}_p, \bar{A}_p)$ to be detectable for values of p_2 close to zero, it is not only sufficient but also necessary that $(\bar{c}_p, \bar{A}_p)$ be observable at such values. The following lemma serves to characterize the family of feedback functions f_p for which the matrix pair $(\bar{c}_p, \bar{A}_p)$ is observable on the subset $\{p : |p_1| \leq \bar{a}, |p_2| \leq \tilde{b}\}$ for every $[\,a \quad b\,]'$ in the set $\bar{\mathcal{P}}_M \triangleq \{[\,a \quad b\,]' : |a| \leq \bar{a}, |b| \geq \tilde{b}\} \supset \mathcal{P}_M$.

Lemma 9 *Let $f^*, q_1,$ and q_2 be a real number satisfying $|q_1| \leq \bar{a}$ and $|q_2| \leq \tilde{b}$ Then*

$$q_1 + q_2 f^* \neq a + bf^* \qquad \forall\,[\,a \quad b\,]' \in \bar{\mathcal{P}}_M \tag{3.14}$$

if and only if one of the following two conditions holds:

1. $f^* > \max\left\{\frac{\bar{a}+q_1}{\tilde{b}-q_2}, \frac{\bar{a}-q_1}{\tilde{b}+q_2}\right\}$
2. $f^* < \min\left\{-\frac{\bar{a}+q_1}{\tilde{b}+q_2}, -\frac{\bar{a}-q_1}{\tilde{b}-q_2}\right\}$

Proof: Let us note first that (3.14) implies $f^* \neq 0$; for if $f^* = 0$, then (3.14) would fail to hold at any point $[\,a \quad b\,]' \in \bar{\mathcal{P}}_M$ at which $a = q_1$. Suppose that $f^* \neq 0$. We claim that (3.14) is equivalent to

$$|q_1 + q_2 f^*| < -\bar{a} + \tilde{b}|f^*| \tag{3.15}$$

To show this, note first that if (3.15) holds, then

$$|q_1 + q_2 f^*| < -\bar{a} + \tilde{b}|f^*| \leq -|a| + |bf^*|$$

from which (3.14) follows. For the reverse implication, observe that if (3.14) is true and (3.15) fails to hold, then $|q_1 + q_2 f^*| \geq -\bar{a} + \tilde{b}|f^*|$; this implies that the point $[\, a \quad b\,]'$ defined by

$$a = -\bar{a}(\text{sign}(q_1 + q_2 f^*)), \qquad\qquad b = \frac{q_1 + q_2 f^* - a}{f^*}$$

is in $\bar{\mathcal{P}}_M$, since $|b| = (|q_1 + q_2 f^*| + \bar{a})/|f^*| \geq \tilde{b}$ and $|a| = \bar{a}$. But with a and b so defined, $q_1 + q_2 f^* = a + bf^*$ which contradicts (3.14). Therefore the claim is correct.

To complete the proof it is enough to show that (3.15) is equivalent to condition 1 of the lemma if $f^* > 0$ or to condition 2 if $f^* < 0$. To establish the former, note that when $f^* > 0$, (3.15) is equivalent to the requirement that the inequalities

$$\begin{array}{rcl} q_1 + q_2 f^* & < & -\bar{a} + \tilde{b} f^* \\ -(q_1 + q_2 f^*) & < & -\bar{a} + \tilde{b} f^* \end{array}$$

both hold. Since $|q_2| < \tilde{b}$, these inequalities are respectively equivalent to the inequalities

$$f^* > \frac{\bar{a} + q_1}{\tilde{b} - q_2} \tag{3.16}$$

$$f^* > \frac{\bar{a} - q_1}{\tilde{b} + q_2} \tag{3.17}$$

Therefore (3.15) must hold just in case f^* satisfies both (3.16) and (3.17). From this and the equivalence of (3.14) and (3.15) it follows that for $f^* > 0$, (3.14) holds if and only if condition 1 does . The equivalence of (3.14) and condition 2 for the case $f^* < 0$ is established in essentially the same way. ■

Note that if $\bar{a} = \infty$, then it is impossible to find a number f^* for which either of the conditions of Lemma 9 hold. Note, in addition, that if $\bar{b} = \infty$, then $\mathcal{P}_M = \bar{\mathcal{P}}_M$. Thus if f_p is a function for which $\mathbb{F}_{\widehat{p}}$ is global detectable for all $a \in \mathbb{R}$ and all $b \in \mathbb{R}$ satisfying $|b| \geq \tilde{b}$, then f_p would have to satisfy detectability condition i for all $p \in \mathcal{P} = \mathbb{R}^2$. Since this condition can't possibly hold when $p_1 > 0$ and $p_2 = 0$, there can be no such function f_p. We summarize:

Proposition 2: *There does not exist a locally bounded feedback function $f_p : \mathcal{P} \rightarrow \mathbb{R}$ which makes $\mathbb{F}_{\widehat{p}\widehat{p}}$ detectable on $\mathcal{P} = \mathbb{R}^2$ for all $a \in \mathbb{R}$ and all $b \in \mathbb{R}$ satisfying $|b| \geq \tilde{b}$.*

Assume that $\bar{a}$ and $\bar{b}$ are finite numbers. We shall define f_p to satisfy detectability condition i for $|p_2| \geq \frac{1}{2}\tilde{b}$ and detectability condition ii for $|p_2| < \frac{1}{2}\tilde{b}$. In particular, let

$$f_p = \begin{cases} -\frac{\bar{a}+|p_1|+1}{p_2} & \text{if} \quad |p_2| \geq \frac{1}{2}\tilde{b} \\ -\frac{\bar{a}+|p_1|+1}{\tilde{b}-|p_2|}\text{sign}(p_2) & \text{if} \quad |p_2| < \frac{1}{2}\tilde{b} \end{cases} \tag{3.18}$$

Note that f_p's only points of discontinuity occur on the line $p_2 = 0$. Using Lemma 9 and the detectability conditions, it is easy to verify that this choice of f_p has what is required for $\mathbb{F}_{\widehat{p}\widehat{p}}$ to be detectable on $\mathcal{P}$ for each $[\, a \quad b\,]' \in \mathcal{P}_M$.

Despite the discontinuous nature of f_p, it is possible to devise a switching logic for $\widehat{p}$, which together with the pair $(\mathbb{E}_{\widehat{p}}, \mathbb{C}_{\widehat{p}})$, constitutes a provably correct switched adaptive control system for siso processes admitting models in $\mathcal{M}$. Such switching logics will be discussed in the last two lectures today.

3.2 Generalization

Although the preceding approach to the underlying stabilizability problem will yield a solution whenever the associated global detectability problem can be constructively solved, versions of the latter for more general classes of proces models prove to be untractable. Thus a somewhat different approach is needed to settle the stabilizability question for more general process model classes. Such an approach will be presented in the next section. The objective of this section is to introduce several concepts which are needed to make the transition from the solution to the one dimensional global detectability problem treated in the last section to the solution to the corresponding stabilizability problem for more general linear systems. In the first place, we will no longer be able to simply define the multi-controller parameter space $\mathcal{Q}$ to be $\mathcal{P}$, and there be values of hp for which $\mathbb{C}_{\widehat{p}}$ is not chosen in accordance with certainty equivalence.

In order to proceed and also to lay the foundation for subsequent discussions of switched adaptive control systems, we need to slightly reinterpret what is meant by a linear multi-estimator. Recall that in Chapter 2 we defined a linear multi-estimator $\mathbb{E}_{\widehat{p}}$ for a parameterized process model class $\mathcal{M}$, to be a stable, parameter-dependent, linear dynamical system of the general form

$$\dot{x}_{\mathbb{E}} = A_{\mathbb{E}}x_{\mathbb{E}} + D_{\mathbb{E}}y + B_{\mathbb{E}}u \tag{3.19}$$
$$y_{\widehat{p}} = C_{\mathbb{E}}(\widehat{p})x_{\mathbb{E}} + G_{\mathbb{E}}(\widehat{p})y \tag{3.20}$$

where $A_{\mathbb{E}}, B_{\mathbb{E}}$ and $D_{\mathbb{E}}$ are constant {parameter-independent} matrices with $A_{\mathbb{E}}$ stable and $\widehat{p}$ is a parameter vector taking values in $\mathcal{P}$. We continue to take this as the definition of $\mathbb{E}_{\widehat{p}}$, but now broaden our interpretation

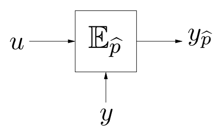

Figure 3.3: Multi-Estimator $\mathbb{E}_{\widehat{p}}$

of multi-estimator to include as outputs, all signals

$$y_p = C_{\mathbb{E}}(p)x_{\mathbb{E}} + G_{\mathbb{E}}(p)y, \quad p \in \mathcal{P} \tag{3.21}$$

We denote this slightly broader interpretation of a multi-estimator by the unsubscripted symbol $\mathbb{E}$ and we refer to the y_p, $p \in \mathcal{P}$ as $\mathbb{E}$'s *latent outputs.* Of course it is only possible to actually generate at most a finite

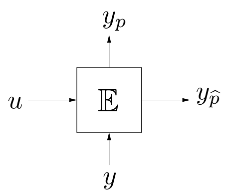

Figure 3.4: Multi-Estimator $\mathbb{E}$ with Latent Output y_p

number of $\mathbb{E}$'s latent outputs.

The introduction of latent outputs provides a convenient means of eliminating $\mathbb{P}$ from the diagram for the feedback sub-system $\mathbb{F}_{\widehat{p}\widehat{q}}$ shown in Figure 3.2. Towards this end, define *output estimation errors*

$$e_p \triangleq y_p - y, \quad p \in \mathcal{P} \tag{3.22}$$

Fix l at any value in $\mathcal{P}_M$ and repace y by $y_l - e_l$ where ever y appears in Figure 3.2. What results is the *injected subsystem* $\mathbb{I}_{\widehat{p}\widehat{q}l}$ shown in Figure 3.5.

We regard $\mathbb{I}_{\widehat{p}\widehat{q}l}$ as a parameter-dependent linear system with inputs r and e_l and output e. It should be noted that this apparent slight-of-hand is actually completely legitimate and the diagram correctly represents the relationships betwen the actual signals u, e, y, r, etc. For analysis purposes,, we will assume that l is

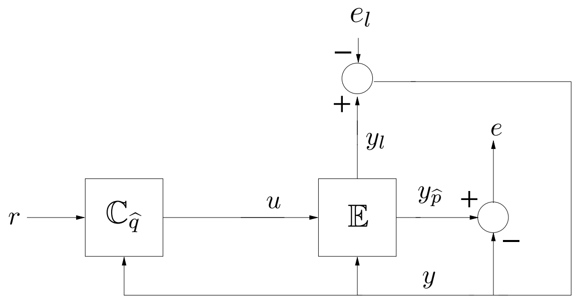

Figure 3.5: Injected Sub-System $\mathbb{I}_{\widehat{p}\widehat{q}l}$

the "right" index to go with $\mathbb{P}$, in the sense that $\mathbb{P}$'s model is in $\mathcal{M}_l$; this will allow us to assume that *lth* modeling error $\mathfrak{e}_l$ defined in Chapter 2 is small, and consequently that e_l is small as well. Of course, since l is not know at the design stage, what ever system properties we achieve will have to hold for all $l \in \mathcal{P}_M$. This plus our objective not to be limited to setting $\widehat{q} = \widehat{p}$, , justifies the need for the triple subscript notation.

What we would like to do if we could, much like before, is to choose $\widehat{q} = \widehat{p}$ and define $\mathbb{C}_{\widehat{p}}$ so that $\mathbb{IF}_{\widehat{p}\widehat{p}l}$ is detectable through e for each $l \in \mathcal{P}_M$ and each fixed $\widehat{p} \in \mathcal{P}$. But as we've already noted, this may be impossible to achieve via certainty equivalence because of points p in $\mathcal{P}$ at which $\mathbb{E}$'s *pth* candidate model {or design model} is not stabilizable. Destpite this, our ultimate goal is to make sure that smallness of e implies smallness of the state of $\mathbb{I}_{\widehat{p}\widehat{p}l}$ {i.e., the state of $\mathbb{F}_{\widehat{p}\widehat{q}}$} under every posible condition. To accomplish this we will define $\mathcal{Q}$ and the $\mathbb{C}_q$, $q \in \mathcal{Q}$ somewhat differently than before and we will generate $\widehat{q}$ as the output of a suitably defined *hybrid switching logic* $\mathbb{S}$ driven by $\widehat{p}$. With this modification included, the injected system shown in Figure 3.5 will become the injected system shown in Figure 3.6

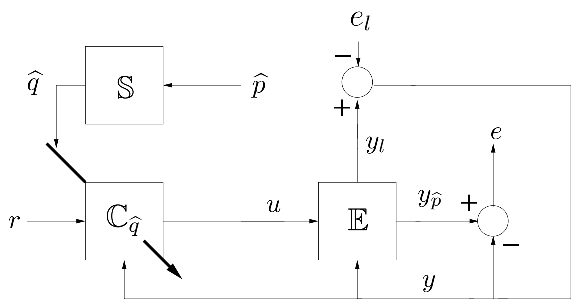

Figure 3.6: Cyclicly-Switched Injected Sub-System $\mathbb{I}_{\widehat{p}l}$

Similarly, addition of cyclic switching to the feedback subsystem shown in Figure 3.2 with result in the feedback sub-system shown in Figure 3.7.

3.3 Cyclic Switching

In the sequel be are going to assume that the process model class $\mathcal{M}$ consists of SISO controllable and observable systems. We further assume that $\mathcal{M}$'s subclass of nominals $\mathcal{N}$ is a subset of a familiy of n-dimensional, SISO, indirect control linear-parameterized design models $\mathcal{D}$; recall that all such models are observable. We assume that $\mathcal{D}$'s parameter space $\mathcal{P}$ is compact with a nonempty interior, and that its geometry is simple enough so that the problem of constucting a switching supervisor is tractable. For example, were $\mathcal{P}$ to be a finite union of compact convex subsets, this would be enough. As before, we assume that *scrM*'s parameter space $\mathcal{P}_M$ is a subset of $\mathcal{P}$. We shall assume that all points in $\mathcal{P}$ at which $\mathcal{D}$'s models

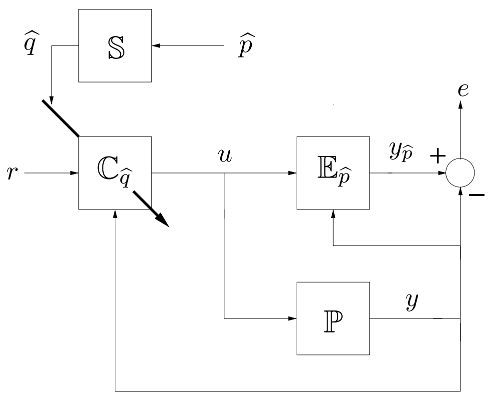

Figure 3.7: Cyclicly-Switched Feedback Sub-System $\mathbb{F}_{\widehat{p}}$

lose controllability are contained in the interior of a closed region $\mathcal{S} \subset \mathcal{P}$ chosen so the sets of transfer functions of the models in $\mathcal{N}$ and $\{\mathbb{D}_p : p \in \mathcal{S}\}$ respectively, are disjoint. It is always possible to so define $\mathcal{S}$ because all models in $\mathcal{M}$ are controllable. In the sequel we write $\mathcal{R}$ for the closure of the complement of $\mathcal{S}$ in $\mathcal{P}$. Note that for all $p \in \mathcal{R}$, $\mathbb{D}_p$ is necessarily both controllable and observable.

Example 6 More often than not, the selection of $\mathcal{P}$ is likely to precede that of $\mathcal{P}_M$ and $\mathcal{S}$. Consider for example, the two-parameter family of transfer functions

$$\nu_p = \frac{s - p_1}{s^2 + (p_1 + p_2)s + 1}$$

defined on the rectangle

$$\mathcal{P} = \left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} : 0.1 \leq p_i \leq 2, \;\; i \in \{1, 2\} \right\}$$

in $\mathbb{R}^2$. Let $\mathcal{D}$ be a family of two-dimensional, linearly parameterized design models with parameter space $\mathcal{P}$, which is been defined so that for each $p \in \mathcal{P}$, ν_p is the transfer function of $\mathbb{D}_p$. Since ν_p has an unstable pole-zero cancellation an each point in $\mathcal{P}$ on the algebraic curve $p_1 p_2 = 1$, this algebraic curve is exactly the set of points in $\mathcal{P}$ on which $\mathbb{D}_p$ is not controlable. Thus one might define $\mathcal{S}$ to be the band of points $\mathcal{S} \triangleq \{p : |p_1 p_2 - 1| \leq \epsilon, p \in \mathcal{P}\}$. where ϵ is small positive number. In this case $\mathcal{P}_M$ could be taken as $\mathcal{P}_M \triangleq \{p : |p_1 p_2 - 1| \geq \bar{\epsilon}, p \in \mathcal{P}\}$ where $\bar{\epsilon}$ is a number greater than ϵ.

3.3.1 Synthesis

We now turn to the main objective of this section, which is to explain how to pick $\mathcal{Q}$, a multi-controller family $\{\mathbb{C}_q : q \in \mathcal{Q}$ and a switching logic $\mathbb{S}$ achieve a form of "detectability" through e of the feedback sub-system $\mathbb{F}_{\widehat{p}}$ shown in Figure 3.7 for each frozed value of $\widehat{p}$, despite the existence of singular surface in $\mathcal{P}$ on which some of $\mathbb{D}$'s models may not be stabilizable. We begin with the design of $\{\mathbb{C}_q : q \in \mathcal{Q}$.

Multi-Controller Definition

The family of multi-controllers to be considered, depends, in part, on a set of n_S distinct scalar gains $\{g_1, g_2, \ldots, g_{n_S}\}$ which fulfil the following requirement.

Observation Requirement: For each $l \in \mathcal{P}_M$ and each $p \in \mathcal{S}$ there is a value of $q \in \mathcal{Q}_S$ for which the system shown in Figure 3.9 is observable through e_{ff}.

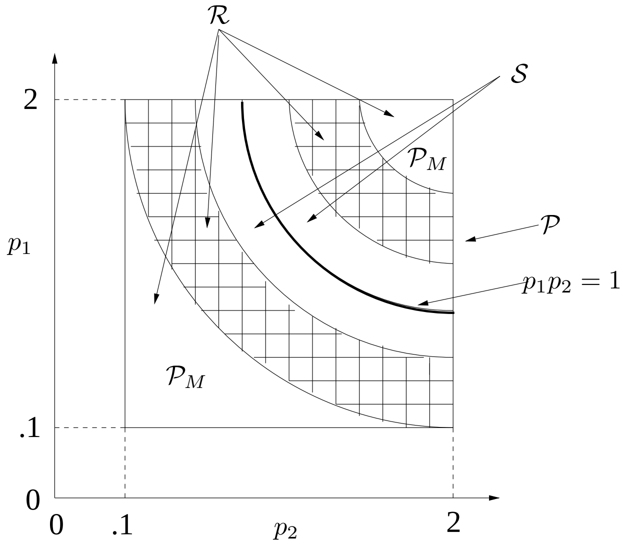

Figure 3.8: Singular Curve $p_1p_2 = 1$ in $\mathcal{P}$

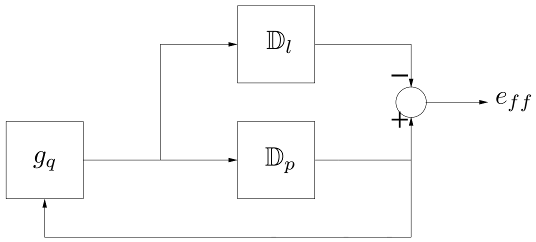

Figure 3.9: Feedforward Interconection

The following result establishes the existence of such a set of gains.

Proposition 2 *The observation requirement is satisfied by any set of n_S distinct gains provided $n_S > 2n$, where n is the dimension of each of the design models in $\mathcal{D}$.*

Proposition 2 is a simple consequence of the following lemma.

Lemma 10 *Let $\{A, b, c\}$ and $\{\bar{A}, \bar{b}, (\bar{c}\}$ be the coefficient matrices of observable, siso linear systems with unreduced transfer functions $\frac{\beta(s)}{\alpha(s)}$ and $\frac{\bar{\beta}(s)}{\bar{\alpha}(s)}$ respectively. Let Λ denote the set of zeros of the polynomial $\gamma(s) = \beta(s)\bar{\alpha}(s) - \bar{\beta}(s)\alpha(s)$. If $f \in \mathbb{R}$ is a number which makes the matrix pair*

$$\begin{bmatrix} -c & \bar{c} \end{bmatrix} \tag{3.23}$$

$$\begin{bmatrix} A & bf\bar{c} \\ 0 & \bar{A} + \bar{b}f\bar{c} \end{bmatrix} \tag{3.24}$$

unobservable, then for some $\lambda \in \Lambda$

$$\alpha(\lambda) - f\beta(\lambda) = 0 \tag{3.25}$$

Proof : Since f makes the above matrix pair unobservable, there must exist a complex number λ and a nonzero vector $[\,x' \quad \bar{x}'\,]'$ for which

$$\begin{aligned} cx &= \bar{c}\bar{x} && (3.26)\\ Ax + bf\bar{c}\bar{x} &= \lambda x && (3.27)\\ \bar{A}\bar{x} + \bar{b}f\bar{c}\bar{x} &= \lambda\bar{x} && (3.28) \end{aligned}$$

In addition, it must be true that

$$\bar{c}\bar{x} \neq 0 \qquad (3.29)$$

For if this were not so, then (3.26) to (3.28) together with the assumed observability of (c, A) and $(\bar{c}, \bar{A})$, would imply $[\,x' \quad \bar{x}'\,]' = 0$ which is not so.

Let $D(\lambda)$ and $\bar{D}(\lambda)$ denote the cofactor matrices of $\lambda I - A$ and $\lambda I - \bar{A}$ respectively. Multiplying (3.27) and (3.28) by $cD(\lambda)$ and $\bar{c}\bar{D}(\lambda)$ respectively, replacing cx by $\bar{c}\bar{x}$ in what results, and then canceling out $\bar{c}\bar{x}$ which by (3.29) we can do, yields the equations

$$\begin{aligned} \alpha(\lambda) - f\beta(\lambda) &= 0 && (3.30)\\ \bar{\alpha}(\lambda) - f\bar{\beta}(\lambda) &= 0 \end{aligned}$$

Since these equations can hold only if $\gamma(\lambda)$ vanishes, λ must be in Λ. Therefore, since (3.30) and (3.25) are the same equation, the lemma is true. ■

Proof of Proposition 2: Fix $l \in \mathcal{P}_M$ and $p \in \mathcal{S}$, let $\{A, b, c\}$ and $\{\bar{A}, \bar{b}, (\bar{c}\}$ be the coefficient matrices of $\mathbb{D}_l$ and $\mathbb{D}_p$ respectively. Then (3.23) and (3.24) are respectively the read-out and state matrices of the feedforward connection shown in Figure 3.9. Since the transfer functions of $\mathbb{D}_l$ and $\mathbb{D}_p$ are necessarily different, the polynomial $\gamma(s)$ referred to in the statement of Lemma 10 is a nonzero polynomial; thus Λ must be a finite set consisting of at most $2n - 1$ complex numbers. Since $\mathbb{D}_l \in \mathcal{N}$, it must be a controllable, observable system. Therefore, $\alpha(s)$ and $\beta(s)$ must be coprime. This in turn means that (3.25) cannot hold if $\beta(\lambda) = 0$. Taken together, these observations thus imply that there can be at most $2n - 1$ values of f for which (3.25) holds for some $\lambda \in \Lambda$. In other words, since $\mathbb{D}_l$ is is controllable, since $\mathbb{D}_l$ and $\mathbb{D}_p$ have different transfer functions and since $\{f_1, f_2, \ldots, f_m\}$ is any set of $2n$ or more distinct numbers, there must be at least one value for f in this set for which the matrix pair (3.23), (3.24) is observable. ■

We are now in a position to define $\mathcal{Q}$ and the multi-controller family $\{\mathbb{C}_q : q \in \mathcal{Q}\}$. As a first step we define $\mathcal{Q} \triangleq (\mathcal{P} - \mathcal{S}) \cup \mathcal{Q}_S$ where $\mathcal{P} - \mathcal{S}$ denotes the complement of $\mathcal{S}$ in $\mathcal{P}$. Next, define on $\mathcal{R}$ {i.e., the closure of $\mathcal{P} - \mathcal{S}$}, a continuously parameterized family of controllers $\mathbb{C}_p$, all of the same dimension {say n_C}, such that for each $q \in \mathcal{R}$, $\mathbb{C}_q$ stabilizes the design model $\mathbb{C}_q \oplus \mathbb{D}_q$. This is possible because $\mathbb{D}_q$ is continuous, controllable and observable on $\mathcal{R}$. Doing this makes $\mathbb{F}_{\widehat{p}}$ detectable through e if $\widehat{q} = \widehat{p} \in \mathcal{P} - \mathcal{S}$, because of certaintly equivalence. The definition also insures that the function $q \longmapsto \mathbb{C}_q$ is bounded, because $\mathbb{C}_q$ is continuous on $\mathcal{R}$ and because $\mathcal{R}$ is compact. For values in of q in $\mathcal{Q}_S$, define $\mathbb{C}_{\widehat{q}}$ by the equations

$$\begin{aligned} u &= g_{\widehat{q}}\, y\\ \dot{x}_{\mathbb{C}} &= A_0 x_{\mathbb{C}} \end{aligned}$$

where A_0 is a parameter-independent $n \times n$ stability matrix. Thus from an input-output of view, each $\mathbb{C}_q, q \in \mathcal{Q}_S$ is simply a constant gain output feedback law.

Cyclic Switching Logic Definition

We now turn to the definition of a cyclic switching logic. Toward this end, we first pick a positive number T_S which we will call a *cycle period*. By a *cyclic switching logic* is meant a hybrid dynamical system with input $\widehat{p}$, state $\{\tau, \phi\}$ and output $\widehat{q}$.

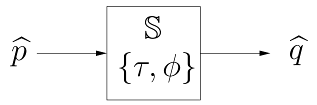

Figure 3.10: Cyclic Switching Logic

Here ϕ is a logical variable taking values in the finite set $\{0, 1\}$ and τ is a continuous timing signal taking values in the closed interval $[0, T_S]$; the evolution of τ over time is governed by the differential equation

$$\dot{\tau} = \phi \tag{3.31}$$

together with appropriate initializations. The relationship between $\widehat{q}$ and $\widehat{p}, \phi$ and τ is given by the formula

$$\widehat{q} = C_S(\widehat{p}, \phi, \tau) \tag{3.32}$$

where $C_S : \mathcal{P} \times \{0, 1\} \times [0, T_S] \rightarrow \mathcal{Q}$ is a readout function defined so that

$$\widehat{q} = \begin{cases} \widehat{p} & \text{if } \widehat{p} \in \mathcal{P} - \mathcal{S} \text{ and } \phi = 0 \\ \rho(\tau) & \text{if } \widehat{p} \in \mathcal{S} \text{ or } \phi = 1 \end{cases}$$

Here $\rho : [0, T] \rightarrow \mathcal{Q}_S$ is a *routing function* defined so that

$$\rho(\mu) = q, \quad \mu \in \left[(q-1)\frac{T_S}{n_S},\ q\frac{T_S}{n_S}\right) \qquad q \in \mathcal{Q}_S$$

the way $\mathbb{S}$ works is to do one of two possible things, depending on past history. On the one hand, whenever $\widehat{p}$ first enters $\mathcal{S}$, $\mathbb{S}$ outputs a "switching cycle" where by a *switching cycle* is meant a signal of duration T_S of the form

$$\rho(\tau),\ 0 \leq \tau \leq T_S$$

On the other hand, $\mathbb{S}$ outputs $\widehat{p}$ whenever $\widehat{p} \notin \mathcal{S}$, unless $\mathbb{S}$ is in the midst of outputting a switching cycle. The logic which accomplishes this is described by the computer flow diagram shown in Figure 3.11. $\mathbb{S}$ is thus

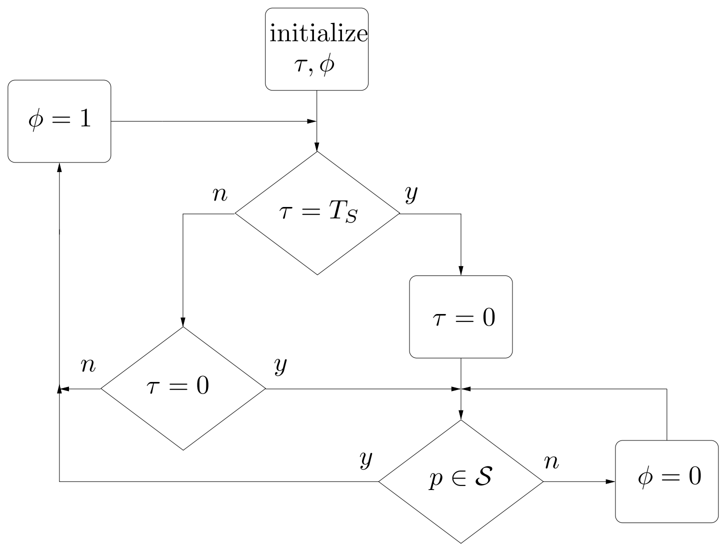

Figure 3.11: Cyclic Switching Computer Flow Diagram

completely defined by (3.31), (3.32) and the flow diagram shown in Figure (3.11).

The behavior of $\mathbb{S}$ can be explained roughly as follows. Suppose that τ and ϕ are both initialized at 0 and initially that $\widehat{p} \in \mathcal{P} - \mathcal{S}$. Then $\mathbb{S}$'s output is set equal to $\widehat{p}$ and both τ and ϕ remain at their initial values. If and when $\widehat{p}$ first enters $\mathcal{S}$, ϕ is switched to the value 1, τ starts to increase linearly from 0 and $\mathbb{S}$'s output takes on the values in a switching cycle. At the end of the cycle, after T_S time units have elapsed, $\mathbb{S}$ checks to see whether or not $\widehat{p}$ is in $\mathcal{S}$. If $\widehat{p}$ is in $\mathcal{S}$, then another switching cycle in executed. On the other hand, if $\widehat{p}$ is not in $\mathcal{S}$, $\widehat{q}$ is set equal to $\widehat{p}$ until the next time $\widehat{p}$ enters $\mathcal{S}$.

3.3.2 Achieving Detectability

For the SISO problem under consideration, the equations describing the multi-controller and multi-estimator in the cyclicaly switched feedback subsystem $\mathbb{F}_{\widehat{p}}$ shown in Figure (3.7) are

$$\begin{aligned} \dot{x}_{\mathbb{E}} &= A_{\mathbb{E}} x_{\mathbb{E}} + d_{\mathbb{E}} y + b_{\mathbb{E}} u \\ y_p &= c_{\mathbb{E}}(p) x_{\mathbb{E}}, \qquad p \in \mathcal{P} \\ \dot{x}_{\mathbb{C}} &= A_{\mathbb{C}}(q) x_{\mathbb{C}} + N_{\mathbb{C}}(q) y + b_{\mathbb{C}}(q) r \\ u &= c_{\mathbb{C}}(q) x_{\mathbb{C}} + M_{\mathbb{C}}(q) y + G_{\mathbb{C}}(q) r \end{aligned}$$

If we define the output estimation errors

$$e_p = y_p - y, \quad p \in \mathcal{P} \tag{3.33}$$

and fix l arbitrarily in $\mathcal{P}_M$, then we can substitute $c_l x_{\mathbb{E}} - e_l$ for y throughout. What results are the equations for the cyclicly switched injected system $\mathbb{I}_{\widehat{p}l}$ shown in Figure 3.6. These equations are of the form

$$e = c_{\widehat{p}l} x + e_l \qquad\qquad \dot{x} = A_{\widehat{q}l} x + d_{\widehat{q}} e_l + b_{\widehat{q}} r \tag{3.34}$$

where $x \triangleq [\, x'_{\mathbb{E}} \quad x'_{\mathbb{C}} \,]'$,

$$c_{pl} = [\, c_{\mathbb{E}}(p) - c_{\mathbb{E}}(l) \quad 0 \,] \qquad A_{ql} = \begin{bmatrix} A_{\mathbb{E}} + (d_{\mathbb{E}} + b_{\mathbb{E}} M_{\mathbb{C}}(q)) c_{\mathbb{E}}(l) & b_{\mathbb{E}} c_{\mathbb{C}}(q) \\ N_{\mathbb{C}}(q) c_{\mathbb{E}}(l) & A_{\mathbb{C}}(q) \end{bmatrix} \quad p \in \mathcal{P},\ q \in \mathcal{Q},\ l \in \mathcal{P}_M \tag{3.35}$$

and d_q and b_q are apprpriately defined vectors depending on q.

We already know from the Certainty Equivalence Stabilization Theorem that if in $p \in \mathcal{P} - \mathcal{S}$ then $(c_{pl},\ A_{ql})$ is a detectable matrix pair no matter what the value of $l \in \mathcal{P}_M$. This, in turn, is equivalent to there existing a family of output injection matrices $K_{ppl},\ p \in \mathcal{P} - \mathcal{S},\ l \in \mathcal{P}_M$ which exponentially stabilize the matrices $A_{ql} + K_{ppl} c_{pl}$. In this regard, a similar statement can be made about the situation when $p \in \mathcal{S}$.

Theorem 6 *For each $p \in \mathcal{S}$ and each $l \in \mathcal{P}_M$, there exists a family of n_S matrices $\{K_{pql},\ q \in \mathcal{Q}_S\}$ with the following property. For each $p \in \mathcal{S}$, each $l \in \mathcal{P}_M$ and each initialization of $\mathbb{S}$, $K_{p\widehat{q}_p l}$ uniformly exponentially stabilizes the time-varying matrix $A_{\widehat{q}_p l} + K_{p\widehat{q}_p l} c_{pl}$, where $\widehat{q}_p$ is the response of $\mathbb{S}$ to the constant input $\widehat{p} = p$.*

Using these properties it can be shown that the time varying matrix $A_{\widehat{q}_{\widehat{p}} l} + K_{\widehat{p}\widehat{q}_{\widehat{p}} l} c_{\widehat{p}l}$ will also be exponentially stable provided $\widehat{p}$ is slowly varying in a suitably defined sense. For example, if $\widehat{p}$ is adjusted using dwell-time switching {to be discussed later in this course}, then $\widehat{p}$ will be piece-wise constant with at least a prespecified dwell time $\tau_D > 0$ between any two switching times. In this case, slow enough variation of $\widehat{p}$ can be assured by simply choosing τ_D larger than the time it takes for the state transition matrices of all of the $A_{\widehat{q}_p l} + K_{p\widehat{q}_p l} c_{pl}, p \in \mathcal{S}, l \in \mathcal{P}_M$ and $A_{pl} + K_{ppl} c_{pl},\ p \in \mathcal{P} - \mathcal{S}$ to drop below 1 in norm.

The way in which these output injections are used to establish correctness of a cyclicly switched adaptive control system for the case when $\widehat{p}$ is continuously tuned, is similar to the developed at the end of Chapter 1 of these notes. For the case when $\widehat{p}$ switched in a piecewise constant manner, the output injections are used in a manner similar to that to be discussed in the two lectures which follow this one.

The proof of Theorem 6 exploits the linking which relate the coefficient matrics of E to those of the design models in $\mathcal{D}$. In constructing the $\{K_{pql},\ p \in \mathcal{S}, q \in \mathcal{Q}_S, l \in \mathcal{P}_M\}$, use is made of the observation requirement and the following technical result which will also be used elsewhere in these lectures.

Lemma 11 (Squashing) *Let (C, A) be a constant, observable matrix pair. For each positive number T there exists a positive number λ and a constant output-injection matrix K for which*

$$|e^{(A+KC)t}| \leq e^{-\lambda(t-T)}, \quad t \geq 0 \tag{3.36}$$

The reader is referred to the following publication for more on cyclic switching including the concepts needed to prove Theorem 6 [9, 10, 11, 12].

Proof of Lemma 11 : With n the size of A, pick any set of n distinct, positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\alpha(s, \lambda) = \Pi_{i=1}^{n}(s + \lambda(1 + \lambda_i))$ is a monic polynomial of degree n in s, whose coefficients are polynomials in λ. Since (C, A) is observable, there exists a polynomial matrix $H(\lambda)$ which assigns to $A + HC$, the characteristic polynomial $\alpha(s, \lambda)$ [13]. Thus for $\lambda > 0$, the eigenvalues of $A + HC$, are real, distinct and bounded above by $-\lambda$. Because of this, the matrix M of eigenvectors which diagonalizes $A+HC$, $\{M^{-1}(A+HC)M = \Lambda = \text{diagonal}\}$, is rational and continuous on $(0, \infty)$ as is its inverse M^{-1}. Since for $\lambda > 0$, $e^{(A+HC)t} = M^{-1}e^{\Lambda t}M$ and $|e^{\Lambda t}| \leq ne^{-\lambda t}$ it must be that

$$|e^{(A+HC)t}| \leq \rho e^{-\lambda t}, \quad t \geq 0$$

where $\rho = n|M^{-1}||M|$; in view of the properties of M and its inverse, ρ must be rational and continuous for $\lambda > 0$. Since $e^{\lambda T}$ eventually increases faster than any continuous rational function of λ, (3.36) can be made to hold by picking λ so large that $\rho e^{-\lambda T} \leq 1$ and then setting K to be equal to H at this value. ∎

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